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On Space-like Geodesics in Two Special Gravitational Fields

A. Banerjee[A] *and S. B. Dutta Choudhury*[B]

[A] Department of Physics, Jadavpur University, Calcutta 700032, India.
[B] Department of Physics, St Anthony's College, Shillong 793003, India.

Abstract

Studies are presented of space-like geodesics in the fields of a charged mass point and an uncharged static cylinder. For the first case, it is shown that circular space-like trajectories are unstable in character under radial perturbations for a physically plausible situation ($\varepsilon^2 < m^2$). For the second case, it is shown that no trajectories in the r–ϕ plane satisfy the conditions of stability.

Introduction

Recently the role of tachyons, or particles which move faster than light, has received attention owing to the possible involvement of tachyons in astrophysical and cosmological phenomena (Davies 1975; Narlikar and Sudarshan 1976). Studies of the trajectories of such particles in the Schwarzschild manifold (Hettel and Helliwell 1973; Raychaudhuri 1974; Honig *et al.* 1974) have yielded certain interesting results which differ considerably from those for the usual time-like or null trajectories. In particular, the phenomenon of bounce from below the event horizon for a tachyon has interesting possibilities, since such particles could be used as probes to the interior of a black hole. As this investigation is worth pursuing, we consider in the present note the radial and circular space-like trajectories in the fields of: (1) a charged mass point; (2) an infinitely long static uncharged cylinder. Our study leads to the following results which may be compared with those corresponding to the Schwarzschild case:

(i) Depending on the energy, a radially falling tachyon in the field of a charged mass point, i.e. in a Reissner–Nordström field, either has a turning point in the region between the two horizons or may proceed right up to the singularity $r = 0$.

(ii) Circular space-like trajectories in the above field are unstable in character under radial perturbation in a physically plausible situation ($\varepsilon^2 < m^2$).

(iii) There are no turning points in the inbound or outbound radial trajectories of tachyons in the gravitational field of an infinitely long static cylinder.

(iv) Circular trajectories in the r–ϕ plane outside such a static cylinder are time-like, null or space-like depending on the magnitude of the mass per unit length. This is in contrast with the Schwarzschild or Reissner–Nordström field, where the three types coexist in different zones.

In the present discussion, stability considerations for off-plane perturbations (Liang 1974) are omitted because these do not alter the basic conclusions that the space-like circular trajectories are unstable in character in both cases under consideration.

Space-like Geodesics in Field of Charged Mass Point

The geodesic equations in the field of a charged mass point, i.e. a Reissner–Nordström field, are represented by the equations ($\theta = \frac{1}{2}\pi$)

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + \tfrac{1}{2}\frac{\mathrm{d}\lambda}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r\exp(-\lambda)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \tfrac{1}{2}\exp(\nu-\lambda)\frac{\mathrm{d}\nu}{\mathrm{d}r}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0, \tag{1}$$

$$\exp(\lambda)\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 - \exp(\nu)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 + \Omega = 0, \tag{2}$$

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right) = \frac{h}{r^2}, \qquad \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right) = k\exp(-\nu), \tag{3,4}$$

where h and k are constants of integration, while

$$\exp(\nu) = \exp(-\lambda) = 1 - 2m/r + \varepsilon^2/r^2, \tag{5}$$

and $\Omega = -1$, 0 or $+1$ for tachyons, photons or tardyons (particles moving slower than light).

Radial Trajectories

For a radial trajectory $\mathrm{d}\phi/\mathrm{d}s = 0$, and from equations (2) and (4) we obtain

$$(\mathrm{d}r/\mathrm{d}s)^2 = k^2 - \Omega(1 - 2m/r + \varepsilon^2/r^2). \tag{6}$$

Equation (6) reads for tachyons ($\Omega = -1$)

$$(\mathrm{d}r/\mathrm{d}s)^2 = k^2 + (1 - 2m/r + \varepsilon^2/r^2). \tag{7}$$

If we have $m > |\varepsilon|$, which is the physical case (Regge and Wheeler 1957; Novikov 1967; Bardeen 1968), then the quantity $1 - 2m/r + \varepsilon^2/r^2$ is positive for $r < r_-$ and for $r > r_+$, where r_+ and r_- are the greater and smaller roots of the equation

$$1 - 2m/r + \varepsilon^2/r^2 = 0. \tag{8}$$

The derivative $\mathrm{d}r/\mathrm{d}s$ can vanish only if the left-hand side of equation (8) is negative, which occurs for $r_- < r < r_+$. However, this expression has a minimum value of $1 - m^2/\varepsilon^2$ at $r = \varepsilon^2/m$, so that there will be no turning point if we have $k^2 > m^2/\varepsilon^2 - 1$. Thus, depending on the energy, the tachyon will turn back or proceed right up to the singularity $r = 0$. By solving equation (7) for $\mathrm{d}r/\mathrm{d}s = 0$, it is not difficult to show that, if there is a turning point at all, it will be at $r = r_1$ for an ingoing tachyon and at $r = r_2$ for an outgoing one, where

$$r_1 = \frac{m}{k^2+1} + \frac{1}{(k^2+1)^{\frac{1}{2}}}\left(\frac{m^2}{k^2+1} - \varepsilon^2\right)^{\frac{1}{2}},$$

$$r_2 = \frac{m}{k^2+1} - \frac{1}{(k^2+1)^{\frac{1}{2}}}\left(\frac{m^2}{k^2+1} - \varepsilon^2\right)^{\frac{1}{2}}.$$

Circular Trajectories

For circular orbits $dr/ds = d^2r/ds^2 = 0$, and hence from equation (1), using (5), we can write

$$r\left(\frac{d\phi}{ds}\right)^2 = \left(\frac{m}{r^2} - \frac{\varepsilon^2}{r^3}\right)\left(\frac{dt}{ds}\right)^2. \tag{9}$$

Again, from equation (2), we obtain

$$r^2\left(\frac{d\phi}{ds}\right)^2 - \exp(\nu)\left(\frac{dt}{ds}\right)^2 + \Omega = 0, \tag{10}$$

and thus from equations (9) and (10) we have

$$\left(\frac{d\phi}{ds}\right)^2 = -\Omega\frac{(m/r^2 - \varepsilon^2/r^3)}{3m - (r + 2\varepsilon^2/r)}. \tag{11}$$

Now for real circular orbits it follows from equations (9) and (11) that:

$$m/r^2 - \varepsilon^2/r^3 > 0, \tag{12}$$

$$\frac{\Omega}{3m - (r + 2\varepsilon^2/r)} < 0. \tag{13}$$

We can write now

$$r^2 + 2\varepsilon^2 - 3mr = (r - r_3)(r - r_4), \tag{14}$$

where

$$r_3 = \tfrac{1}{2}\{3m - (9m^2 - 8\varepsilon^2)^{\frac{1}{2}}\} \qquad \text{and} \qquad r_4 = \tfrac{1}{2}\{3m + (9m^2 - 8\varepsilon^2)^{\frac{1}{2}}\}.$$

Now from the inequality (12) we have $r > \varepsilon^2/m$ and, since the relation $r_4 > \frac{3}{2}m > \varepsilon^2/m$ holds for real values of r_3 and r_4, we have two different situations depending on the relative magnitudes of the charge and mass parameters:

Case I. When $\varepsilon^2/m \leqslant r_3$, the circular tachyon orbits can exist only in the region where $r_3 < r < r_4$. Tardyons can move in circular trajectories in the region between $r = \varepsilon^2/m$ and r_3 and also for $r > r_4$. The condition $\varepsilon^2/m \leqslant r_3$ is, however, equivalent to the condition $\varepsilon^2 \geqslant m^2$ and thus in this particular case we have $\frac{9}{8} > \varepsilon^2/m^2 \geqslant 1$. The situation $\varepsilon^2 > m^2$ is, of course, apparently unphysical, as mentioned earlier.

Case II. When $r_3 < \varepsilon^2/m < r_4$, there can be circular tachyon trajectories for $\varepsilon^2/m < r < r_4$. Orbits at $r > r_4$ are those for tardyons. This is the situation when $\varepsilon^2 < m^2$ holds, in view of the condition $r_3 < \varepsilon^2/m$. It may be remarked here that circular photon orbits may be obtained by putting $\Omega = 0$ in equation (10) and combining this with equation (9). This gives the relation

$$1 - 3m/r + 2\varepsilon^2/r^2 = 0,$$

which has two roots

$$r^{-1} = \{3m \pm (9m^2 - 8\varepsilon^2)^{\frac{1}{2}}\}/4\varepsilon^2.$$

In other words, the magnitudes of r for photon orbits are the same as r_3 and r_4 of equation (14) above.

Stability of Circular Orbits

It has been shown by Hettel and Helliwell (1973) that circular orbits for tachyons, i.e. space-like circular geodesics in the field of a Schwarzschild black hole, are not stable. It can be proved also that in a physically meaningful situation circular orbits are not stable under radial perturbations when they move in the field of a charged mass point. Thus for purely circular trajectories it follows from equations (3), (10) and (11) that

$$h^2 = \Omega\left(\frac{mr^2-\varepsilon^2 r}{3m-(r+2\varepsilon^2/r)}\right) \tag{15}$$

and

$$\exp(\nu)\left(\frac{dt}{ds}\right)^2 = \left(\frac{h^2}{r^2}+\Omega\right). \tag{16}$$

We consider a very small perturbation Δr in the radius of the orbit and use the relation (16) in the geodesic equation (1) with $dr/ds = 0$ for circular orbits to get finally

$$\frac{d^2\Delta r}{ds^2} - h^2\frac{d}{dr}\left(\frac{1-2m/r+\varepsilon^2/r^2}{r^3}\right)\Delta r + h^2\frac{d}{dr}\left(\frac{m-\varepsilon^2/r}{r^4}\right)\Delta r + \Omega\frac{d}{dr}\left(\frac{m-\varepsilon^2/r}{r^2}\right)\Delta r = 0. \tag{17}$$

Equation (17) gives in turn

$$\frac{d^2\Delta r}{ds^2} = \frac{\Omega}{r^4}\left(\frac{mr^2+9m\varepsilon^2-4\varepsilon^4/r-6m^2 r}{3m-(r+2\varepsilon^2/r)}\right)\Delta r. \tag{18}$$

From elementary arguments it can be shown that the circular orbits are stable under radial perturbations when the coefficient of Δr on the right-hand side of equation (18) is negative, and this is true in view of the inequality (13) only when we have

$$mr^2+9m\varepsilon^2-4\varepsilon^4/r-6m^2 r > 0. \tag{19}$$

The relation (19) agrees exactly with the result of Hettel and Helliwell (1973) for $\varepsilon = 0$, which yields the condition $r > 6m$ in the Schwarzschild case. But in the Schwarzschild case, tachyon circular orbits can exist only in the region $r < 3m$ and hence they cannot be stable.

For $\varepsilon \neq 0$, we may write

$$F(r) = mr^2+9m\varepsilon^2-4\varepsilon^4/r-6m^2 r.$$

We compute the values of $F(r)$ for two limiting cases of r, namely $r = r_3$ and r_4. We can write

$$r_3 = \tfrac{1}{2}(3m-\alpha) \quad \text{and} \quad r_4 = \tfrac{1}{2}(3m+\alpha),$$

with

$$\alpha = (9m^2-8\varepsilon^2)^{\frac{1}{2}} > 0.$$

So for $r = r_3$, we have

$$F(r_3) = -\tfrac{1}{2}\alpha^2 m + \tfrac{3}{8}m^2\alpha + \alpha^3/8,$$

and for $r = r_4$,

$$F(r_4) = -\tfrac{1}{2}\alpha^2 m - \tfrac{3}{8}m^2\alpha - \alpha^3/8.$$

Now $F(r)$ is negative for all values of α when $r = r_4$ while, for $r = r_3$, $F(r)$ is positive only when we have $\varepsilon^2 > m^2$. On the other hand, when we have $\varepsilon^2 < m^2$, $F(r)$ is not positive in the region $r_3 < r < r_4$. Hence we can conclude that, in a physically realistic situation ($\varepsilon^2 < m^2$), circular tachyon orbits in the field of a charged mass point are not stable. However, we may note that, in the field of a source with $\varepsilon^2 > m^2$ (naked singularity), there may be some stable circular tachyon orbits.

Space-like Geodesics in Field of Infinitely Long Static Cylinder

The well-known metric (Marder 1958) in vacuum outside an infinitely long static cylinder is given by

$$\mathrm{d}s^2 = r^{2C}\mathrm{d}t^2 - r^{2(1-C)}\mathrm{d}\phi^2 - A^2 r^{-2C(1-C)}(\mathrm{d}r^2+\mathrm{d}z^2), \tag{20}$$

where C and A are constants. The quantity $\frac{1}{2}C$ is interpreted as the mass parameter per unit length of the cylinder. The equations of geodesics in this field are

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} - \frac{C(1-C)}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - \frac{(1-C)}{A^2}r^{1-2C^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \left(\frac{C}{A^2}\right)r^{4C-2C^2-1}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = 0, \tag{21}$$

$$r^{2C}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - r^{2(1-C)}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 - A^2 r^{-2C(1-C)}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \Omega, \tag{22}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \beta r^{-2C(1-C)}, \qquad \frac{\mathrm{d}t}{\mathrm{d}s} = \alpha_1 r^{-2C}, \tag{23, 24}$$

where β and α_1 are integration constants, and Ω takes the values $+1$, 0 or -1 according as the geodesics are time-like, null or space-like.

Radial Trajectories

For a radial trajectory we have $\mathrm{d}\phi/\mathrm{d}s = 0$, and from equations (22) and (24) we get

$$r^{2C^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \frac{\alpha_1^2 - \Omega r^{2C}}{A^2}. \tag{25}$$

For tachyons we have $\Omega = -1$, and equation (25) reduces to

$$r^{2C^2}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = \frac{\alpha_1^2 + r^{2C}}{A^2}. \tag{26}$$

It is evident from equation (26) that tachyons which move radially inwards or outwards do not bounce at any stage, while a tardyon ($\Omega = +1$) travelling in the outward direction along a radial trajectory may turn back towards the source under certain circumstances. In other words, tachyons can in no case be trapped in the gravitational field of an infinitely long cylinder.

Circular Trajectories

For circular orbits we have both $\mathrm{d}r/\mathrm{d}s = 0$ and $\mathrm{d}^2r/\mathrm{d}s^2 = 0$, and so from equations (21) and (22) we obtain the relation

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = -\Omega\left(\frac{C}{2C-1}\right)r^{2(C-1)}. \tag{27}$$

It is evident from equation (27) that circular trajectories for all values of r are possible for tachyons, photons or tardyons depending on the magnitude of the mass per unit length of the cylinder. When $\frac{1}{2} < C < 1$, all of the possible trajectories are for tachyons, while all of them are for tardyons when $0 < C < \frac{1}{2}$. Only circular null orbits are possible when $C = \frac{1}{2}$ exactly. Here the mass per unit length is the factor which determines whether the orbits are space-like, null or time-like. This is determined by the magnitude of the radius of the circular orbit in the spherically symmetric field.

Stability of Circular Orbits

Proceeding exactly in the same way as for a charged mass point (above) we can obtain in the present case, for a small radial perturbation, from equations (21), (22) and (23) the relations:

$$\frac{\mathrm{d}^2\Delta r}{\mathrm{d}s^2} = \left(\frac{(1-2C)}{A^2}\beta^2(4C-2C^2-3)\,r^{4C-2C^2-4} - \frac{C}{A^2}\Omega(2C-2C^2-1)\,r^{2C-2C^2-3}\right)\Delta r \tag{28}$$

and

$$\beta^2 = \{C\Omega/(1-2C)\}\,r^{2(1-C)}. \tag{29}$$

Using these relations we finally obtain the result

$$\mathrm{d}^2\Delta r/\mathrm{d}s^2 = (2C/A^2)\,r^{2C-2C^2-2}\,\Omega(C-1)\,\Delta r. \tag{30}$$

Since $C < 1$, it follows from the relation (30) that the right-hand side is positive for tachyons ($\Omega = -1$) and negative for tardyons ($\Omega = +1$). Thus we may conclude from the previous arguments that the space-like circular geodesics in this case are not stable, while ordinary particles or tardyons move along stable circular trajectories.

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