The Effect of Plate Oscillations on Horizontal Free Convection Flow

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Abstract

The free convection flow along a semi-infinite horizontal plate oscillating in its own plane is analysed. The basic flow is purely buoyancy induced, while the oscillations in the plate cause a time-dependent boundary layer flow and heat transfer. The boundary layer equations are linearized and the first two approximations are considered. Two separate solutions valid for high and low frequency ranges are obtained by a series expansion in terms of frequency parameters. The skin friction and the rate of heat transfer are studied for both frequency ranges. For very high frequencies, the oscillatory flow pattern is of a 'shear-wave' type, unaffected by the mean flow. It is found that the phase of the skin friction at the plate lags that of the plate oscillations by $\frac{1}{4}\pi$ and the rate of heat transfer has a phase lag of $\frac{1}{2}\pi$.

Introduction

In the study of unsteady boundary layer flows, one aspect that has received much attention in recent years is concerned with boundary layer responses to imposed oscillations. The theory of this was initiated by Lighthill (1954) to study the effect of free stream oscillations on flow and heat transfer along plates and cylinders. The extension of the theory for free convection boundary layers along a semi-infinite vertical plate was carried out by Nanda and Sharma (1963), Eshghy *et al.* (1965) and Kelleher and Yang (1968), but the case when the plate is horizontal was not considered. Recently, however, Muhuri and Maiti (1967) have studied free convection flow and heat transfer along a semi-infinite horizontal plate when the plate temperature oscillates about a constant mean, and they have obtained separate solutions for low and high frequency ranges. These oscillatory flow and heat transfer problems are important in engineering because such flows occur often in practice. To an observer standing on the rotating element of a turbomachine the flow appears as an oscillation superimposed on a mean flow. The effect of fluid or surface oscillations on the heat transfer from a surface to the surrounding flow is of particular interest to the engineer.

The main aim of the present investigation is to study the effect of plate oscillations on the free convection flow and heat transfer along a semi-infinite horizontal plate. We consider a thin flat plate extending from the origin to infinity in the x direction, where x measures the distance along the plate lying horizontally in quiescent fluid and oscillating in its plane at right angles to its edge. The plate is heated to a uniform temperature T_w and placed in an ambient fluid at T_∞. Thus the basic flow is entirely due to buoyancy forces over a horizontal plate whose temperature differs from that of the free stream. The effect of the buoyancy forces is to induce a longitudinal pressure gradient which causes flow. It is an interesting flow in its own right, yielding a steady outer streaming for the boundary layer equations as a result of free convection alone; moreover, this problem should be easily amenable to experiment in a laboratory. The steady free convection flow over the plate is perturbed due to its harmonic oscillations. By assuming that the magnitude of the oscillations is small (of order $\varepsilon \ll 1$) compared with the mean velocity induced by the convection, we are able to employ techniques of linearization for this perturbation.

In the present treatment the basic steady flow is considered using the Kármán–Pohlhausen method, and an approximate solution to be used in the subsequent study of unsteady flow is obtained. Two different solutions for low and high frequency ranges are developed from the perturbation equations. The method of solving the problem is essentially the same as that developed by Lighthill (1954), except that it is extended to obtain series solutions for low and high frequencies. The matching between the two solutions is found to be quite satisfactory. For very high frequencies the phase of the rate of heat transfer lags behind that of the plate oscillations by $\frac{1}{2}\pi$ and the skin friction has a phase lag of $\frac{1}{4}\pi$.

Basic Equations

The boundary layer equations for two-dimensional free-convection incompressible unsteady flow past a semi-infinite horizontal plate are (Verma 1970; see also Sparrow and Minkowycz 1962)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta\frac{\partial}{\partial x}\left(\int_y^\infty \theta \,\mathrm{d}y\right)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1b}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}, \tag{1c}$$

where $\theta = T-T_\infty$, with T and T_∞ being the temperature of the boundary layer and free stream respectively; u and v are the velocity components along the x and y directions respectively, x being the coordinate along the plate and y normal to it; g is the acceleration due to gravity; α is the thermal diffusivity of the fluid; and β and ν are the coefficients of thermal expansion and kinematic viscosity respectively. In accordance with the usual practice for free convection flows, we restrict the effect of the density variations to the formation of the 'buoyant force', which is the first term on the right-hand side of equation (1a).

The boundary conditions for equations (1) are

$$y = 0: \qquad u = \varepsilon U(t), \quad v = 0, \quad \theta = \theta_\mathrm{w} \quad (\varepsilon \ll 1); \tag{2a}$$

$$y \to \infty: \qquad u \to 0, \qquad \theta \to 0; \tag{2b}$$

where $U(t) = U_0 \exp(\mathrm{i}\omega t)$, U_0 being assumed to be independent of the frequency of oscillations ω, and $\theta_\mathrm{w} = T_\mathrm{w}-T_\infty$, T_w being the plate temperature. It is assumed that the plate oscillation velocity is small compared with the mean horizontal velocity induced by the steady state convection.

The solution of equations (1) with (2) are obtained in terms of complex functions, the real parts of which have physical significance. We take u, v and θ each to be the sum of a steady component and a small oscillating component:

$$u = u_0 + \varepsilon u_1 \exp(\mathrm{i}\omega t), \quad v = v_0 + \varepsilon v_1 \exp(\mathrm{i}\omega t), \quad \theta = \theta_0 + \varepsilon \theta_1 \exp(\mathrm{i}\omega t), \tag{3}$$

where the components u_0, v_0 and θ_0 for the steady mean flow satisfy the equations

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = g\beta \frac{\partial}{\partial x}\left(\int_y^\infty \theta_0 \, \mathrm{d}y\right) + \nu \frac{\partial^2 u_0}{\partial y^2}, \tag{4a}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \tag{4b}$$

$$u_0 \frac{\partial \theta_0}{\partial x} + v_0 \frac{\partial \theta_0}{\partial y} = \alpha \frac{\partial^2 \theta_0}{\partial y^2}, \tag{4c}$$

with the boundary conditions

$$y = 0: \qquad u_0 = 0, \quad v_0 = 0, \quad \theta_0 = \theta_w; \tag{5a}$$

$$y \to \infty: \qquad u_0 \to 0, \qquad \theta_0 \to 0. \tag{5b}$$

Neglecting squares of ε and dividing by $\exp(\mathrm{i}\omega t)$, we find that u_1, v_1 and θ_1 satisfy the following set of differential equations

$$\mathrm{i}\omega u_1 + u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + u_1 \frac{\partial u_0}{\partial x} + v_1 \frac{\partial u_0}{\partial y} = g\beta \frac{\partial}{\partial x}\left(\int_y^\infty \theta_1 \, \mathrm{d}y\right) + \nu \frac{\partial^2 u_1}{\partial y^2}, \tag{6a}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{6b}$$

$$\mathrm{i}\omega \theta_1 + u_0 \frac{\partial \theta_1}{\partial x} + v_0 \frac{\partial \theta_1}{\partial y} + u_1 \frac{\partial \theta_0}{\partial x} + v_1 \frac{\partial \theta_0}{\partial y} = \alpha \frac{\partial^2 \theta_1}{\partial y^2}, \tag{6c}$$

with the boundary conditions

$$y = 0: \qquad u_1 = U_0, \quad v_1 = 0, \quad \theta_1 = 0; \tag{7a}$$

$$y \to \infty: \qquad u_1 \to 0, \qquad \theta_1 \to 0. \tag{7b}$$

Steady State Solution

Let us consider the set of equations (4) and (5) which describe the steady free convection boundary layer flow along a horizontal plate. We shall employ the Kármán–Pohlhausen method of integration to solve these equations. Integrating equations (4a) and (4c) over the width of the boundary layer, we obtain

$$\frac{\partial}{\partial x}\left(\int_0^\infty u_0^2 \, \mathrm{d}y\right) + \nu \left(\frac{\partial u_0}{\partial y}\right)_{y=0} = g\beta \frac{\partial}{\partial x}\left(\int_0^\infty \int_y^\infty \theta_0 \, \mathrm{d}y \, \mathrm{d}y\right), \tag{8a}$$

$$\frac{\partial}{\partial x}\left(\int_0^\infty u_0 \theta_0 \, \mathrm{d}y\right) = -\alpha \left(\frac{\partial \theta_0}{\partial y}\right)_{y=0}. \tag{8b}$$

We assume the velocity and temperature profiles

$$u_0 = V_0(x)(\eta - 2\eta^2 + 2\eta^4 - \eta^5), \qquad \theta_0 = \tfrac{1}{3}\theta_w(3 - 5\eta_\theta + 5\eta_\theta^4 - 3\eta_\theta^5), \tag{9}$$

where

$$V_0(x) = \tfrac{1}{12} g\beta\theta_w \delta^2 \, d\delta_\theta/dx, \qquad \eta = y/\delta, \qquad \eta_\theta = y/\delta_\theta,$$

δ and δ_θ being the viscous and thermal boundary layer thicknesses. The expressions (9) satisfy the conditions (5a) at $y = 0$, together with

$$y \to \infty: \qquad \frac{\partial u_0}{\partial y} \to 0, \quad \frac{\partial^2 u_0}{\partial y^2} \to 0, \quad \frac{\partial \theta_0}{\partial y} \to 0, \quad \frac{\partial^2 \theta_0}{\partial y^2} \to 0;$$

$$y = 0: \qquad \nu\left(\frac{\partial^2 u_0}{\partial y^2}\right)_{y=0} + g\beta\frac{\partial}{\partial x}\left(\int_0^\infty \theta_0 \, dy\right) = 0, \qquad \left(\frac{\partial^2 \theta_0}{\partial y^2}\right)_{y=0} = 0,$$

$$\nu\left(\frac{\partial^3 u_0}{\partial y^3}\right)_{y=0} = 0, \qquad \left(\frac{\partial^3 \theta_0}{\partial y^3}\right)_{y=0} = 0.$$

Substituting the profiles (9) into equations (8a) and (8b) and considering only the similarity cases, we obtain

$$94\sigma^{-1} = 99\Delta^2(40\Delta - 21)G(\Delta), \qquad A^5 = 12375(40\Delta - 21)/47\Delta, \tag{10}$$

where

$$\Delta = \delta_\theta/\delta, \qquad \delta = Ax^{2/5}(\nu^2/g\beta\theta_w)^{1/5},$$

$$\sigma = \nu/\alpha, \qquad V_0(x) = \tfrac{1}{30}\Delta A^5 x^{1/5}(g\beta\theta_w \nu^3)^{2/5},$$

and

$$G(\Delta) = \tfrac{1}{5} - \tfrac{5}{42}\Delta^{-1} + \tfrac{1}{63}\Delta^{-4} - \tfrac{9}{1540}\Delta^{-5}, \qquad \Delta \geqslant 1,$$

$$= \tfrac{5}{21}\Delta - \tfrac{5}{28}\Delta^2 + \tfrac{2}{45}\Delta^4 - \tfrac{1}{77}\Delta^5, \qquad \Delta \leqslant 1.$$

It should be noted that the mean horizontal velocity induced by the steady convection is zero at the edge of the plate, and the analysis is invalid for small values of x. However, the mean velocity increases as $x^{1/5}$ and so the results are valid far downstream.

The results of practical interest here are the rate of heat transfer, or Nusselt number Nu, and the skin-friction characteristics of the problem, and these can now be obtained easily. The steady-state rate of heat transfer Nu_0 in nondimensional form is given by

$$Nu_0 = -\left(\frac{\partial \theta_0}{\partial y}\right)_{y=0} \frac{x^{2/5}}{\theta_w}\left(\frac{\nu^2}{g\beta\theta_w}\right)^{1/5} = \frac{5}{3\Delta A}, \tag{11}$$

while the corresponding nondimensional skin friction at the wall τ_0^* is

$$\tau_0^* = \left(\frac{\nu x}{(g\beta\theta_w)^3}\right)^{1/5}\left(\frac{\partial u_0}{\partial y}\right)_{y=0} = \frac{\Delta A^2}{30}. \tag{12}$$

Specific values of Nu_0 and τ_0^* for three values of the ratio σ are set out below.

σ	Nu_0	τ_0^*
0·72	0·328	0·901
1·0	0·361	0·785
10	0·652	0·361

It is obvious that the skin friction decreases and the heat transfer increases as σ increases. These results are a reliable guide to the physical situation since the Kármán–Pohlhausen method has been proved to be a successful tool in predicting the effects of an increase in the Prandtl number on the physical properties of a flow (Lighthill 1954).

We now proceed to investigate the nature of the flow and the temperature fields due to the oscillations of the plate. As noted previously, the solutions for low and high frequency ranges are developed separately.

Oscillating Plate Solutions

(a) Low Frequency Range

We make use of the Kármán–Pohlhausen method again here to solve equations (6) and (7). Integrating (6a) and (6c) from $y = 0$ to ∞, we obtain the averaging conditions

$$\nu\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = g\beta\frac{\partial}{\partial x}\left(\int_0^\infty \int_y^\infty \theta_1 \,\mathrm{d}y\,\mathrm{d}y\right) - 2\frac{\partial}{\partial x}\left(\int_0^\infty u_0 u_1 \,\mathrm{d}y\right) - \mathrm{i}\omega\int_0^\infty u_1 \,\mathrm{d}y\,, \tag{13a}$$

$$-\alpha\left(\frac{\partial \theta_1}{\partial y}\right)_{y=0} = \mathrm{i}\omega\int_0^\infty \theta_1 \,\mathrm{d}y + \frac{\partial}{\partial x}\left(\int_0^\infty (u_0\theta_1 + u_1\theta_0)\,\mathrm{d}y\right). \tag{13b}$$

Also, equations (6a) and (6c) and their first differentials at $y = 0$ are

$$\mathrm{i}\omega U_0 = g\beta\frac{\partial}{\partial x}\left(\int_0^\infty \theta_1 \,\mathrm{d}y\right) + \nu\left(\frac{\partial^2 u_1}{\partial y^2}\right)_{y=0}, \tag{14a}$$

$$\mathrm{i}\omega\left(\frac{\partial u_1}{\partial y}\right)_{y=0} + U_0\left(\frac{\partial^2 u_0}{\partial y\,\partial x}\right)_{y=0} = \nu\left(\frac{\partial^3 u_1}{\partial y^3}\right)_{y=0}, \tag{14b}$$

$$\left(\frac{\partial^2 \theta_1}{\partial y^2}\right)_{y=0} = 0, \tag{14c}$$

$$\mathrm{i}\omega\left(\frac{\partial \theta_1}{\partial y}\right)_{y=0} = \alpha\left(\frac{\partial^3 \theta_1}{\partial y^3}\right)_{y=0} - U_0\left(\frac{\partial^2 \theta_1}{\partial x\,\partial y}\right)_{y=0}. \tag{14d}$$

Consistent with the conditions (7) and (14), we assume the profiles for u_1 and θ_1

$$u_1/U_0 = (1-5\eta^4+4\eta^5)+A_1(\eta-4\eta^4+5\eta^5) + A_2(\eta^2-3\eta^4+2\eta^5)+A_3(\eta^3-2\eta^4+\eta^5)\,, \tag{15a}$$

$$\theta_1/\theta_\mathrm{w} = (U_0\,\delta^2/\nu x)\{B_1(\eta_\theta-4\eta_\theta^4+3\eta_\theta^5)+B_2(\eta_\theta^2-2\eta_\theta^4+\eta_\theta^5)\}\,. \tag{15b}$$

In these equations the A_i and B_j are functions of ω to be determined. They may be obtained by substituting the profiles (15) into equations (13) and (14), which leads to the following expressions.

$$A_1 = \tfrac{1}{175}A^5\Delta^2(10B_1+B_2) - \mathrm{i}\omega^*\left(\tfrac{2}{3}+\tfrac{1}{5}A_1+\tfrac{1}{15}A_2+\tfrac{1}{60}A_3\right) - \tfrac{1}{25}A^5\Delta\left(\tfrac{29}{495}+\tfrac{47}{2772}A_1+\tfrac{47}{9240}A_2+\tfrac{4}{3465}A_3\right), \tag{16a}$$

$$A_2 = \tfrac{1}{2}\mathrm{i}\omega^* - \tfrac{1}{100}A^5(12B_1+B_2), \qquad A_3 = \tfrac{1}{6}\mathrm{i}\omega^* A_1 - \tfrac{1}{900}A^5\Delta, \tag{16b}$$

$$B_1 = -\tfrac{1}{5}\sigma\Delta^2\left\{\tfrac{1}{24}\mathrm{i}\omega^*(12B_1+B_2) + \tfrac{1}{20}A^5\left(B_1 H(\Delta) + B_2 G_1(\Delta)\right) + F(\Delta) + A_1 E(\Delta) + A_2 I(\Delta) + A_3 J(\Delta)\right\}, \tag{16c}$$

$$B_2 = \tfrac{1}{6}\mathrm{i}\omega^*\sigma\Delta^2 B_1 + \tfrac{1}{9}\sigma\Delta^2, \tag{16d}$$

where

$$\begin{aligned}
H(\Delta) &= \tfrac{2}{21}\Delta - \tfrac{3}{28}\Delta^2 + \tfrac{2}{45}\Delta^4 - \tfrac{6}{385}\Delta^5, && \Delta \leqslant 1,\\
&= \tfrac{1}{42}\Delta^{-1} - \tfrac{4}{315}\Delta^{-4} + \tfrac{475}{1155}\Delta^{-5}, && \Delta \geqslant 1;\\
G_1(\Delta) &= \tfrac{1}{105}\Delta - \tfrac{1}{84}\Delta^2 + \tfrac{1}{180}\Delta^4 - \tfrac{1}{495}\Delta^5, && \Delta \leqslant 1,\\
&= \tfrac{3}{280}\Delta^{-1} - \tfrac{1}{105}\Delta^{-4} + \tfrac{121}{1540}\Delta^{-5}, && \Delta \geqslant 1;\\
F(\Delta) &= \tfrac{1}{3} - \tfrac{1}{27}\Delta^4 + \tfrac{2}{77}\Delta^5, && \Delta \leqslant 1,\\
&= 2 - \tfrac{25}{21}\Delta^{-1} + \tfrac{2}{9}\Delta^{-4} - \tfrac{1}{11}\Delta^{-5}, && \Delta \geqslant 1;\\
E(\Delta) &= \tfrac{5}{63}\Delta - \tfrac{4}{135}\Delta^4 + \tfrac{2}{77}\Delta^5, && \Delta \leqslant 1,\\
&= \tfrac{3}{5} - \tfrac{10}{11}\Delta^{-1} + \tfrac{1}{9}\Delta^{-4} - \tfrac{87}{385}\Delta^{-5}, && \Delta \geqslant 1;\\
I(\Delta) &= \tfrac{5}{168}\Delta^2 - \tfrac{4}{135}\Delta^4 + \tfrac{2}{231}\Delta^5, && \Delta \leqslant 1,\\
&= \tfrac{1}{5} - \tfrac{1}{3}\Delta^{-1} + \tfrac{1}{2}\Delta^{-4} - \tfrac{9}{440}\Delta^{-5}, && \Delta \geqslant 1;\\
J(\Delta) &= \tfrac{1}{72}\Delta^3 - \tfrac{2}{135}\Delta^4 + \tfrac{1}{231}\Delta^5, && \Delta \leqslant 1,\\
&= \tfrac{1}{20} - \tfrac{1}{21}\Delta^{-1} + \tfrac{1}{72}\Delta^{-4} - \tfrac{1}{165}\Delta^{-5}, && \Delta \geqslant 1;
\end{aligned}$$

and the nondimensional frequency parameter $\omega^* = \omega\delta^2/\nu$. Equations (16) may be solved by expanding the A_i and B_j in series of the form

$$A_i = \sum_{n=1}^{\infty} A_{in}(\mathrm{i}\omega^*)^n, \qquad i = 1,2,3, \qquad B_j = \sum_{n=1}^{\infty} B_{jn}(\mathrm{i}\omega^*)^n, \qquad j = 1,2. \tag{17}$$

Substituting these expressions in equations (16) and comparing like powers of $\mathrm{i}\omega^*$ on both sides, we get:

for $n = 0$,

$$A_{10} = \tfrac{1}{25}\Delta A^5\left\{\tfrac{1}{7}\Delta(10B_{10}+B_{20}) - \left(\tfrac{29}{495}+\tfrac{47}{2772}A_{10}+\tfrac{47}{9240}A_{20}+\tfrac{4}{3465}A_{30}\right)\right\}, \tag{18a}$$

$$B_{10} = -\tfrac{1}{5}\sigma\Delta^2\left\{\tfrac{1}{20}A^5\left(B_{10}H(\Delta) + B_{20}G_1(\Delta)\right) + F(\Delta) + A_{10}E(\Delta) + A_{20}I(\Delta) + A_{30}J(\Delta)\right\}, \tag{18b}$$

$$A_{20} = -\tfrac{1}{600}\Delta A^5(12B_{10}+B_{20}), \qquad A_{30} = -\tfrac{1}{900}\Delta A^5, \qquad B_{20} = \tfrac{1}{9}\sigma A^2; \tag{18c}$$

for $n = 1$,

$$A_{11} = \tfrac{1}{25}\Delta A^5\{\tfrac{1}{7}\Delta(10B_{11}+B_{21}) - (\tfrac{47}{2772}A_{11} + \tfrac{47}{1294}A_{21} + \tfrac{4}{3465}A_{31})\} - (\tfrac{2}{3} + \tfrac{1}{5}A_{10} + \tfrac{1}{15}A_{20} + \tfrac{1}{60}A_{30}), \tag{19a}$$

$$B_{11} = -\tfrac{1}{5}\sigma\Delta^2\{\tfrac{1}{24}\Delta(12B_{10}+B_{20}) + \tfrac{1}{20}A^5(B_{11}H(\Delta) + B_{21}G_1(\Delta)) + A_{11}E(\Delta) + A_{21}I(\Delta) + A_{31}J(\Delta)\}, \tag{19b}$$

$$A_{21} = \tfrac{1}{2} - \tfrac{1}{600}\Delta A^5(12B_{11}+B_{21}), \qquad A_{31} = \tfrac{1}{6}A_{10}, \qquad B_{21} = \tfrac{1}{6}\sigma\Delta^2 B_{10}; \tag{19c}$$

and, for $n \geqslant 2$,

$$A_{1n} = \tfrac{1}{25}\Delta A^5\{\tfrac{1}{7}\Delta(10B_{1n}+B_{2n}) - (\tfrac{47}{2772}A_{1n} + \tfrac{47}{9240}A_{2n} + \tfrac{4}{3465}A_{3n})\} - (\tfrac{2}{3} + \tfrac{1}{5}A_{1,n-1} + \tfrac{1}{15}A_{2,n-1} + \tfrac{1}{60}A_{3,n-1}), \tag{20a}$$

$$B_{1n} = -\tfrac{1}{5}\sigma\Delta^5\{\tfrac{1}{24}\Delta(12B_{1,n-1}+B_{2,n-1}) + \tfrac{1}{20}A^5(B_{1n}H(\Delta) + B_{2n}G_1(\Delta)) + A_{1n}E(\Delta) + A_{2n}I(\Delta) + A_{3n}J(\Delta)\}, \tag{20b}$$

$$A_{2n} = -\tfrac{1}{600}\Delta A^5(12B_{1n}+B_{2n}), \qquad A_{3n} = \tfrac{1}{6}A_{1,n-1}, \qquad B_{2n} = \tfrac{1}{6}\sigma\Delta^2 B_{1,n-1}. \tag{20c}$$

The solution of equations (6) and (7) in the limiting case $\omega \to 0$ is the quasi-steady solution, to be denoted by u_s, v_s, θ_s. These quantities are the coefficients of ε in the velocity and temperature field distributions for steady flow with an imposed oscillation εU_0 of the plate. Hence

$$u_s = U_0\frac{\partial u_0}{\partial U_0}, \qquad v_s = U_0\frac{\partial v_0}{\partial U_0}, \qquad \theta_s = U_0\frac{\partial \theta_0}{\partial U_0}. \tag{21}$$

That this solves equations (6) and (7) where $\omega = 0$ can be verified by direct substitution. It can also be verified easily that this quasi-steady solution corresponds to A_{10}, A_{20}, A_{30}, B_{10} and B_{20} given by equations (18).

The profiles (15) for u_1 and θ_1 can now be expressed as the sum of the in-phase and out-of-phase components as

$$u_1 = u_r + \mathrm{i}u_2, \qquad \theta_1 = \theta_r + \mathrm{i}\theta_2. \tag{22}$$

The longitudinal components of the velocity and temperature fields may thus be written in the form

$$u = u_0 + \varepsilon R_u\cos(\omega t + \phi_u), \qquad \theta = \theta_0 + \varepsilon R_\theta\cos(\omega t + \phi_\theta), \tag{23}$$

where

$$R_u = (u_r^2 + u_2^2)^{\frac{1}{2}}, \qquad R_\theta = (\theta_r^2 + \theta_2^2)^{\frac{1}{2}},$$

$$\phi_u = \arctan(u_2/u_r), \qquad \phi_\theta = \arctan(\theta_2/\theta_r).$$

The rate of heat transfer for low frequency fluctuations is finally obtained in dimensionless form as

$$Nu_l^* = -\varepsilon \exp(\mathrm{i}\omega t)\left(\frac{\partial \theta_1}{\partial y}\right)_{y=0} \frac{x^{2/5}}{\theta_\mathrm{w}}\left(\frac{\nu^2}{g\beta\theta_\mathrm{w}}\right)^{1/5}$$
$$= -\varepsilon A\Delta^{-1}\,\mathrm{Re}\left(\exp(\mathrm{i}\omega t)\sum_{n=0}^{\infty} B_{1n}(\mathrm{i}\omega^*)^n\right) = \varepsilon Nu_1 \cos(\omega t + \phi_{Nu}), \tag{24}$$

while the dimensionless skin friction is

$$\tau_l^* = \varepsilon \exp(\mathrm{i}\omega t)\left(\frac{\partial u_1}{\partial y}\right)_{y=0}\left(\frac{\nu x}{g^3\beta^3\theta_\mathrm{w}^3}\right)^{1/5}$$
$$= \varepsilon A^{-1}\,\mathrm{Re}\left(\exp(\mathrm{i}\omega t)\sum_{n=0}^{\infty} A_{1n}(\mathrm{i}\omega^*)^n\right) = \varepsilon\tau_1\cos(\omega t + \phi_\tau), \tag{25}$$

where the amplitudes Nu_1, τ_1 and phases ϕ_{Nu}, ϕ_τ are defined by these expressions.

(*b*) *High Frequency Range*

For high frequencies of oscillation Lighthill (1954) has shown that the oscillating flow is to a close approximation an ordinary 'shear wave' unaffected by the mean flow. Following Lighthill, we have the oscillatory horizontal component u_1 of the velocity given by

$$u_1 = U_0 \exp\{-(\mathrm{i}\omega/\nu)^{\frac{1}{2}} y\},$$

which is obtained by retaining the terms with the factor ω together with the derivatives of highest order in equation (6a). This relation shows that for very high frequencies the thickness of the oscillatory boundary layer is of order $(\nu/\omega)^{\frac{1}{2}}$, that is, it is small in comparison with the thickness of the steady boundary layer, which is of order $(\nu x/U_0)^{\frac{1}{2}}$. Thus one can expect the entire oscillatory flow to be contained within the steady boundary layer. Because of this, in order to solve equations (6) and (7) for large ω, we expand u_1, v_1 and θ_1 in inverse powers of $\omega^{\frac{1}{2}}$:

$$u_1 = u_{10} + \omega^{-1/2}u_{11} + \omega^{-1}u_{12} + \omega^{-3/2}u_{13} + \ldots, \tag{26a}$$

$$v_1 = \omega^{-1/2}v_{10} + \omega^{-1}v_{11} + \omega^{-3/2}v_{12} + \ldots, \tag{26b}$$

$$\theta_1 = \theta_{10} + \omega^{-1/2}\theta_{11} + \omega^{-1}\theta_{12} + \omega^{-3/2}\theta_{13} + \ldots. \tag{26c}$$

In terms of a new variable $z = y\omega^{\frac{1}{2}}$, equations (6) take the form

$$\nu\frac{\partial^2 u_1}{\partial z^2} - \mathrm{i}u_1 = \omega^{-1/2}\left(v_0\frac{\partial u_1}{\partial z} + v_1\frac{\partial u_0}{\partial z}\right) + \omega^{-1}\left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right)$$
$$-\omega^{-3/2}g\beta\frac{\partial}{\partial x}\left(\int_{z/\omega^{1/2}}^{\infty}\theta_1\,\mathrm{d}z\right), \tag{27a}$$

$$\frac{\partial u_1}{\partial x} + \omega^{\frac{1}{2}}\frac{\partial v_1}{\partial z} = 0, \tag{27b}$$

$$\alpha\frac{\partial^2\theta_1}{\partial z^2} - \mathrm{i}\theta_1 = \omega^{-\frac{1}{2}}\left(v_0\frac{\partial\theta_1}{\partial z} + v_1\frac{\partial\theta_0}{\partial z}\right) + \omega^{-1}\left(u_0\frac{\partial\theta_1}{\partial x} + u_1\frac{\partial\theta_0}{\partial x}\right). \tag{27c}$$

Within the oscillatory boundary layer, the steady flow can be approximated (in terms of $y = z\omega^{-\frac{1}{2}}$) as

$$u_0 = u_0(0) + y\left(\frac{\partial u_0}{\partial y}\right)_{y=0} + \frac{y^2}{2!}\left(\frac{\partial^2 u_0}{\partial y^2}\right)_{y=0} + \ldots$$
$$= (V_0/\delta)z\omega^{-\frac{1}{2}} - (2V_0/\delta^2)z^2\omega^{-1} + \ldots, \tag{28a}$$

$$v_0 = v_0(0) + y\left(\frac{\partial v_0}{\partial y}\right)_{y=0} + \frac{y^2}{2!}\left(\frac{\partial^2 v_0}{\partial y^2}\right)_{y=0} + \ldots$$
$$= (V_0/5\delta x)z^2\omega^{-1} + \ldots, \tag{28b}$$

$$\theta_0 = \theta_0(0) + y\left(\frac{\partial \theta_0}{\partial y}\right)_{y=0} + \frac{y^2}{2!}\left(\frac{\partial^2 \theta_0}{\partial y^2}\right)_{y=0} + \ldots$$
$$= \theta_w - (5\theta_w/3\Delta\delta)z\omega^{-\frac{1}{2}} + \ldots. \tag{28c}$$

Substituting equations (26) into (27) and using (28), we obtain for u_{10} and θ_{10} the differential set

$$\nu\frac{\partial^2 u_{10}}{\partial z^2} - \mathrm{i}u_{10} = 0, \qquad \alpha\frac{\partial^2 \theta_{10}}{\partial z^2} - \mathrm{i}\theta_{10} = 0, \tag{29}$$

with the conditions

$$z = 0: \quad u_{10} = U_0, \quad \theta_{10} = 0; \qquad z \to \infty: \quad u_{10} \to 0, \quad \theta_{10} \to 0.$$

The solution of equations (29) is

$$u_{10} = U_0 \exp\{-(\mathrm{i}\omega/\nu)^{\frac{1}{2}}y\}, \qquad \theta_{10} = 0, \tag{30}$$

which is unaffected by the steady mean flow. Interaction terms, however, appear in the subsequent higher approximations. The first nonzero term in θ_1 is θ_{13} which satisfies the equation

$$\alpha\frac{\partial^2 \theta_{13}}{\partial z^2} - \mathrm{i}\theta_{13} = -\frac{5\theta_w U_0}{3\Delta}\frac{\partial}{\partial x}\left(\frac{1}{\delta}\right) z\exp\{-(\mathrm{i}/\nu)^{\frac{1}{2}}z\}, \tag{31}$$

with the boundary conditions

$$z = 0: \quad \theta_{13} = 0; \qquad z \to \infty: \quad \theta_{13} \to 0.$$

From equation (31) we get

$$\theta_{13} = \frac{2U_0\theta_w}{3x\alpha\Delta\delta}\left\{\frac{\nu z\exp\{-(\mathrm{i}/\nu)^{\frac{1}{2}}z\}}{\mathrm{i}(1-\sigma)} + \frac{2\nu^2(\mathrm{i}/\nu)^{\frac{1}{2}}}{(1-\sigma)}\left(\exp\{-(\mathrm{i}/\alpha)^{\frac{1}{2}}z\} - \exp\{-(\mathrm{i}/\nu)^{\frac{1}{2}}z\}\right)\right\}. \tag{32}$$

We thus obtain the velocity component u and the temperature field θ as

$$u = u_0 + \varepsilon\exp(\mathrm{i}\omega t)\,U_0\exp\{-(\mathrm{i}\omega/\nu)^{\frac{1}{2}}y\}, \tag{33a}$$

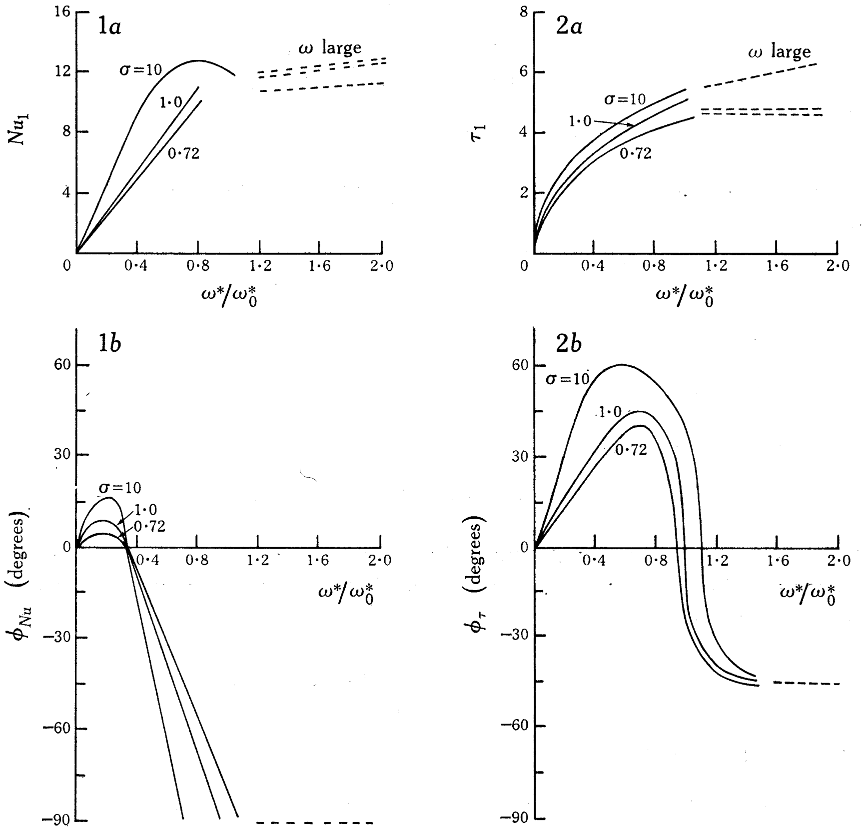
$$\theta = \theta_0 + \varepsilon\exp(\mathrm{i}\omega t)\frac{2U_0\theta_w\sigma}{\omega^{3/2}\Delta x\delta}\left\{\frac{y\omega^{\frac{1}{2}}\exp\{-(\mathrm{i}\omega/\nu)^{\frac{1}{2}}y\}}{\mathrm{i}(1-\sigma)}\right.$$
$$\left. + \frac{2(\mathrm{i}\nu)^{\frac{1}{2}}}{(1-\sigma)^2}\left(\exp\{-(\mathrm{i}\omega/\alpha)^{\frac{1}{2}}y\} - \exp\{-(\mathrm{i}\omega/\nu)^{\frac{1}{2}}y\}\right)\right\}. \tag{33b}$$

We finally have then the rate of heat transfer for high frequency oscillations given in dimensionless form by

$$Nu_h^* = -\varepsilon\exp(\mathrm{i}\omega t)\left(\frac{\partial\theta_1}{\partial y}\right)_{y=0}\frac{x^{2/5}}{\theta_w}\left(\frac{\nu^2}{g\beta\theta_w}\right)^{1/5} = \frac{2\varepsilon\sigma\exp\{\mathrm{i}(\omega t-\frac{1}{2}\pi)\}}{3\Delta\omega^* A(1+\sigma^{\frac{1}{2}})^2}, \tag{34}$$

while the nondimensional skin friction is

$$\tau_h^* = \left(\frac{\partial u_1}{\partial y}\right)_{y=0}\varepsilon\exp(\mathrm{i}\omega t)\left(\frac{\nu x}{g^3\beta^3\theta_w^3}\right)^{1/5} = \varepsilon A^{-1}(\omega)^{*\frac{1}{2}}\exp\{\mathrm{i}(\omega t-\tfrac{1}{4}\pi)\}. \tag{35}$$



Figs 1 and 2. Frequency responses of the amplitudes (a) and phases (b) of (1) the rate of heat transfer Nu and (2) the skin friction τ. Each characteristic is shown as a function of the dimensionless frequency parameter ω^*/ω_0^* for three values of σ. In all cases the dashed curves represent the high frequency solutions.

For sufficiently large values of ω^* we find that the amplitude of the rate of heat transfer increases with frequency and its phase lags behind that of the plate oscillations by 90°, while the skin friction increases with frequency and has a phase lag of 45°.

(c) *Discussion of Results*

The frequency responses of the amplitude and phase angle of the rate of heat transfer and of the skin friction as functions of ω^*/ω_0^* for values of σ of 0·72, 1·0 and 10 are shown in Figs 1 and 2 respectively. The low and high frequency solutions were matched on the basis of the skin-friction oscillations, taking the matching point as the critical frequency ω_0^* (which depends on σ) at which the low frequency solution predicts a phase lag equal to that of the shear-wave solution (Fig. 2*b*). It can be seen that the amplitude and phase of the fluctuating components of the rate of heat transfer and the skin friction all increase initially at low frequencies. For higher frequencies the phase lead of Nu and τ reaches a maximum and then decreases to become a phase lag and approaches an asymptotic value at very high frequencies. It is found that, while the amplitude of the rate of heat transfer changes significantly as σ increases from $0 \cdot 72$ to 10, the amplitude of the skin friction is not greatly affected.

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