Surface Waves in Viscoelastic Media under the Influence of Gravity

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Abstract

In this paper formulae are derived for surface waves in a viscoelastic medium of Voigt type under the influence of gravity. The wave velocity equations are deduced from Biot's theory of stress by assuming that the effects of gravity are equivalent to a type of initial stress of a hydrostatic nature. The resulting equations are used to briefly investigate the particular surface waves of Rayleigh, Love and Stoneley type. In all cases the final results are in agreement with the corresponding classical results when the effects of gravity and viscosity are neglected.

Introduction

The importance of surface waves in isotropic homogeneous elastic solid media has been long recognized because of the relevance to the study of earthquakes and other phenomena in seismology and geophysics. As a result, the theory of surface waves has been widely developed by many investigators (e.g. Stoneley 1924; Bullen 1965; Jeffreys 1970). However, the effects of gravity, curvature and viscosity in the theory have not been considered to the extent that they deserve. A limited discussion of these effects is given in the monograph by Ewing *et al.* (1957). The influence of gravity on elastic waves and in particular on an elastic globe was first considered by Bromwich (1898). A subsequent investigation of the influence of gravity on superfacial waves was presented by Love (1911) in his standard work, in which he showed that the velocity of Rayleigh waves is increased to a significant extent when the wavelength is large. More recently, Biot (1965) developed a theory of initial stress and used it to investigate the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of a hydrostatic nature and the medium to be incompressible. Following the same theory of initial stress and using the dynamical equations of motion for a homogeneous isotropic elastic solid medium under initial stress, De and Sengupta (1973*a*, 1973*b*) have studied the effects of gravity on elastic waves and vibrations, and also (1974) on the propagation of waves in an elastic layer. However, in all previous investigations the effects of viscosity have been ignored although this property plays an important role in the behaviour of solids. We have thus undertaken here to investigate the combined effects of gravity and viscosity on elastic waves.

In the following sections, Biot's (1965) theory of initial stress taken to represent the influence of gravity on a homogeneous isotropic elastic solid medium is extended to include the effects of viscosity, and a general theory of surface waves is developed. This theory is then applied to the particular examples of Rayleigh waves, Love waves and Stoneley waves. In all cases the wave velocity equation is deduced in a closed form which is in agreement with the corresponding classical result when the effects of gravity and viscosity are ignored. It is found that the presence of gravity and viscosity causes dispersion and absorption of the waves.

General Theory of Surface Waves

Let M_1 and M_2 be two homogeneous semi-infinite viscoelastic media in contact with each other (M_2 above M_1) along a horizontal plane boundary of infinite dimensions. As the reference system we take a set of orthogonal cartesian axes x_1, x_2, x_3 with the origin at any point on the plane boundary and $0x_3$ directed normally into M_1. We consider the possibility of a type of wave moving in the positive x_1 direction and assume that the disturbance is confined to the neighbourhood of the boundary, thus making it a surface wave. We further assume that at any instant all the particles in a line parallel to the x_2 axis have equal displacements, i.e. all partial derivatives with respect to x_2 vanish.

Let u_1, u_2, u_3 be the components of a displacement at any point x_1, x_2, x_3 at time t. We may separate out the purely dilatational and purely rotational disturbances associated with the components u_1 and u_3 by introducing the two displacement potentials ϕ and ψ, which are functions of x_1, x_3 and t, in the form

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}, \qquad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1}, \tag{1}$$

whence

$$\nabla^2\phi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \equiv \Delta, \qquad \nabla^2\psi = \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}.$$

The component u_2 is associated with purely distortional movement. We note that ϕ, ψ and u_2 are respectively associated with P waves, SV waves and SH waves, the symbols having their usual significance (Bullen 1965).

The dynamical equations of motion for the three-dimensional problem under the influence of gravity are (Biot 1965)

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho g \frac{\partial u_3}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{2a}$$

$$\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho g \frac{\partial u_3}{\partial x_2} = \rho \frac{\partial^2 u_2}{\partial t^2}, \tag{2b}$$

$$\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} - \rho g \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{2c}$$

where ρ is the density of the medium, g is the acceleration due to gravity and the σ_{ij} are the stress components. According to Voigt's definition, the stress–strain relations in a viscoelastic medium are

$$\sigma_{ij} = \delta_{ij}\left(\lambda + \lambda' \frac{\partial}{\partial t}\right)\Delta + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) e_{ij}, \tag{3}$$

where λ, μ and λ', μ' are respectively the elastic constants of Lamé and the constants representing the effect of viscosity. Substituting the definitions (3) into equations (2), and remembering that all derivatives with respect to x_2 vanish, we obtain

$$\left(\lambda+\mu+(\lambda'+\mu')\frac{\partial}{\partial t}\right)\frac{\partial \Delta}{\partial x_1}+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2 u_1+\rho g\frac{\partial u_3}{\partial x_1} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{4a}$$

$$\left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2 u_2 = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{4b}$$

$$\left(\lambda+\mu+(\lambda'+\mu')\frac{\partial}{\partial t}\right)\frac{\partial \Delta}{\partial x_3}+\left(\mu+\mu'\frac{\partial}{\partial t}\right)\nabla^2 u_3-\rho g\frac{\partial u_1}{\partial x_1} = \rho\frac{\partial^2 u_3}{\partial t^2}. \tag{4c}$$

On introduction of the functions ϕ and ψ and after some manipulation of the expressions (4), we may then write down the following set of equations

$$\left(\alpha_j^2+\alpha_j'^2\frac{\partial}{\partial t}\right)\nabla^2\phi_j+g\frac{\partial\psi_j}{\partial x_1} = \frac{\partial^2\phi_j}{\partial t^2}, \tag{5a}$$

$$\left(\beta_j^2+\beta_j'^2\frac{\partial}{\partial t}\right)\nabla^2\psi_j-g\frac{\partial\phi_j}{\partial x_1} = \frac{\partial^2\psi_j}{\partial t^2}, \tag{5b}$$

$$\left(\beta_j^2+\beta_j'^2\frac{\partial}{\partial t}\right)\nabla^2(u_2)_j = \frac{\partial^2(u_2)_j}{\partial t^2}, \tag{5c}$$

where the suffixes $j = 1,2$ have been used to designate quantities for the media M_1 and M_2 and

$$\alpha_j^2 = (\lambda_j+2\mu_j)/\rho_j, \qquad \beta_j^2 = \mu_j/\rho_j,$$

with corresponding definitions of α_j' and β_j' in terms of the constants of viscosity. It may be noted that the equations (5) are particular solutions of (4).

We shall now solve equations (5) for the medium M_1 and then write down the corresponding solutions for M_2. In order to do this we take the functions ϕ_1, ψ_1 and $(u_2)_1$ to be of the form

$$\phi_1 = F(x_3)\exp\{\mathrm{i}\omega(x_1-ct)\}, \tag{6a}$$

$$\psi_1 = G(x_3)\exp\{\mathrm{i}\omega(x_1-ct)\}, \tag{6b}$$

$$(u_2)_1 = H(x_3)\exp\{\mathrm{i}\omega(x_1-ct)\}. \tag{6c}$$

Substituting for ϕ_1 and ψ_1 into equations (5a) and (5b), we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}+h_1^2\right)F+\frac{\mathrm{i}g\omega G}{\alpha_1^2-\mathrm{i}\omega c\alpha_1'^2} = 0, \tag{7a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}+k_1^2\right)G-\frac{\mathrm{i}g\omega F}{\beta_1^2-\mathrm{i}\omega c\beta_1'^2} = 0, \tag{7b}$$

where

$$h_1^2 = \frac{\omega^2c^2}{\alpha_1^2-\mathrm{i}\omega c\alpha_1'^2}-\omega^2, \qquad k_1^2 = \frac{\omega^2c^2}{\beta_1^2-\mathrm{i}\omega c\beta_1'^2}-\omega^2.$$

From equations (7) we find that F and G satisfy the ordinary differential equation

$$\left\{\left(\frac{d^2}{dx_3^2}+p_1^2\omega^2\right)\left(\frac{d^2}{dx_3^2}+q_1^2\omega^2\right)\right\}\{F,G\} = 0, \tag{8}$$

where

$$p_1^2+q_1^2 = (h_1^2+k_1^2)/\omega^2, \qquad p_1^2 q_1^2 = (h_1^2 k_1^2 - m_1^2)/\omega^4,$$

with

$$m_1^2 = \omega^2 g^2/(\alpha_1^2 - i\omega\alpha_1'^2 c)(\beta_1^2 - i\omega\beta_1'^2 c).$$

A solution for F from equation (8) is

$$F = A_1 \exp(i\omega p_1 x_3) + B_1 \exp(i\omega q_1 x_3) + L_1 \exp(-i\omega p_1 x_3) + N_1 \exp(-i\omega q_1 x_3), \tag{9}$$

where A_1, B_1, L_1 and N_1 are constants. In order that the wave may be a surface wave, the function F must tend to zero as the distance from the boundary approaches infinity. Remembering our choice of axes, we see that the above requirement is fulfilled if the real part of the argument of the exponential function is negative, which in turn demands that the imaginary parts of p_1 and q_1 should be positive. In view of this condition, with the form (9) for F we note that the constants L_1 and N_1 must vanish in the medium M_1. The solution then for ϕ_1 in M_1 is

$$\phi_1 = \{A_1 \exp(i\omega p_1 x_3) + B_1 \exp(i\omega q_1 x_3)\} \exp\{i\omega(x_1 - ct)\}. \tag{10a}$$

We may proceed in exactly the same manner and obtain ψ_1 and $(u_2)_1$ as

$$\psi_1 = \{C_1 \exp(i\omega p_1 x_3) + D_1 \exp(i\omega q_1 x_3)\} \exp\{i\omega(x_1 - ct)\}, \tag{10b}$$

$$(u_2)_1 = E_1 \exp\{i\omega(s_1 x_3 + x_1 - ct)\}, \tag{10c}$$

where

$$s_1 = \{\rho_1 c^2/(\mu_1 - i\omega c\mu'_1) - 1\}^{\frac{1}{2}},$$

of which the imaginary part is positive, arguing as before. From equation (7a) we observe that the constants C_1 and D_1 are respectively related to the constants A_1 and B_1 by

$$C_1 = n_1 A_1, \qquad D_1 = r_1 B_1,$$

where

$$n_1 = (\omega^2 p_1^2 - h_1^2)(\alpha_1^2 - i\omega c\alpha_1'^2)/i\omega g, \qquad r_1 = (\omega^2 q_1^2 - h_1^2)(\alpha_1^2 - i\omega c\alpha_1'^2)/i\omega g.$$

The corresponding quantities in the medium M_2 may be determined by the method outlined above, with due caution only for the vanishing of the particular constants of integration in the defined system of coordinates.

Let us now formulate two boundary conditions which must be satisfied for our problem:

I. The components of displacement at the boundary surface between the media M_1 and M_2 must be continuous at all points and times.

II. The stress components σ_{31}, σ_{32} and σ_{33} must also be continuous at all points and times across the boundary surface.

Using the boundary condition I, we obtain from the values of ϕ and ψ in the two media, after use of the relations (1) in each case,

$$A_1(1-n_1p_1)+B_1(1-r_1q_1) = A_2(1+n_2p_2)+B_2(1+r_2q_2), \tag{11a}$$

$$E_1 = E_2, \tag{11b}$$

$$A_1(p_1+n_1)+B_1(q_1+r_1) = A_2(-p_2+n_2)+B_2(-q_2+r_2). \tag{11c}$$

The components of stress in the viscoelastic media of Voigt type are given by

$$(\sigma_{31})_j = \left(\mu_j+\mu_j'\frac{\partial}{\partial t}\right)\left(2\frac{\partial^2\phi_j}{\partial x_1\,\partial x_3}+\frac{\partial^2\psi_j}{\partial x_1^2}-\frac{\partial^2\psi_j}{\partial x_3^2}\right), \tag{12a}$$

$$(\sigma_{32})_j = \left(\mu_j+\mu_j'\frac{\partial}{\partial t}\right)\frac{\partial(u_2)_j}{\partial x_3}, \tag{12b}$$

$$(\sigma_{33})_j = \left(\lambda_j+\lambda_j'\frac{\partial}{\partial t}\right)\nabla^2\phi_j+2\left(\mu_j+\mu_j'\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\phi_j}{\partial x_3^2}+\frac{\partial^2\psi_j}{\partial x_1\,\partial x_3}\right), \tag{12c}$$

where, as before, $j = 1,2$ for the media M_1, M_2. Applying now the second boundary condition, we obtain, using equations (12),

$$\begin{aligned}\mu_1^*\{A_1(n_1p_1^2-2p_1-n_1)+B_1(r_1q_1^2-2q_1-r_1)\}\\ = \mu_2^*\{A_2(n_2p_2^2+2p_2-n_2)+B_2(r_2q_2^2+2q_2-r_2)\},\end{aligned} \tag{13a}$$

$$s_1\mu_1^*E_1 = -s_2\mu_2^*E_2, \tag{13b}$$

$$\begin{aligned}A_1\{\lambda_1^*(1+p_1^2)+2\mu_1^*p_1(p_1+n_1)\}+B_1\{\lambda_1^*(1+q_1^2)+2\mu_1^*q_1(q_1+r_1)\}\\ = A_2\{\lambda_2^*(1+p_2^2)+2\mu_2^*p_2(p_2-n_2)\}+B_2\{\lambda_2^*(1+q_2^2)+2\mu_2^*q_2(q_2-r_2)\},\end{aligned} \tag{13c}$$

where the asterisks indicate the complex quantities as

$$\theta^* = \theta-\mathrm{i}\omega c\theta'.$$

It follows from equations (11b) and (13b) that both E_1 and E_2 vanish and hence there is no displacement in the x_2 direction, i.e. there is no transverse component of displacement. Thus no SH waves occur in this case.

The wave velocity equation may finally be obtained by eliminating the constants A_1, B_1, A_2 and B_2 from the equations (11a), (11c), (13a) and (13c) to give, in determinant form, the result

$$\begin{vmatrix} 1-n_1p_1 & 1-r_1q_1 & -(1+n_2p_2) & -(1+r_2q_2) \\ p_1+n_1 & q_1+r_1 & p_2-n_2 & q_2-r_2 \\ F_1(p_1,n_1) & F_1(q_1,r_1) & F_2(p_2,n_2) & F_2(q_2,r_2) \\ H_1(p_1,n_1) & H_1(q_1,r_1) & H_2(p_2,n_2) & H_2(q_2,r_2) \end{vmatrix} = 0, \tag{14}$$

where

$$F_1(x,y) = \mu_1^*(x^2y-2x-y), \qquad F_2(x,y) = \mu_2^*(y-yx^2-2x),$$
$$H_1(x,y) = -\lambda_1^*(1+x^2)-2\mu_1^* x(x+y), \qquad H_2(x,y) = \lambda_2^*(1+x^2)-2\mu_2^* x(y-x).$$

The roots of equation (14) determine the wave velocity of general surface waves propagating, in a gravitational field, along a common boundary between two viscoelastic solid media of Voigt type. Although the influences of viscosity and gravity are small, the present analysis should prove to be useful in circumstances where these effects cannot be neglected.

Examples of Surface Waves

Rayleigh Waves

For the case of Rayleigh waves the medium M_2 is replaced by a vacuum, so that the plane boundary now becomes a free surface. As found above, we note here also that there can be no SH waves. Then, in equations (13a) and (13c) we have $A_2 = B_2 = 0$, and these equations take the modified forms

$$A_1(n_1 p_1^2-2p_1-n_1)+B_1(r_1 q_1^2-2q_1-r_1) = 0, \tag{15a}$$

$$A_1\{\lambda_1^*(1+p_1^2)+2\mu_1^* p_1(p_1+n_1)\}+B_1\{\lambda_1^*(1+q_1^2)+2\mu_1^* q_1(q_1+r_1)\} = 0. \tag{15b}$$

Elimination of the constants A_1 and B_1 from equations (15) yields the following result

$$\begin{aligned}(n_1 p_1^2-2p_1-n_1)\{(1+q_1^2)\lambda_1^*+2\mu_1^* q_1(q_1+r_1)\}\\ -(r_1 q_1^2-2q_1-r_1)\{(1+p_1^2)\lambda_1^*+2\mu_1^* p_1(p_1+n_1)\} = 0.\end{aligned} \tag{16}$$

This is the required wave velocity equation for Rayleigh waves in a viscoelastic solid medium under the influence of gravity. When the effects of gravity and viscosity are ignored, equation (16) reduces to the corresponding classical result (Bullen 1965).

Love Waves

For Love waves to exist we consider a layered semi-infinite medium in which M_2 is bounded by two horizontal plane surfaces at a finite distance H apart, while M_1 remains infinite as before. We now have to investigate only the displacement component u_2 in the direction of the x_2 axis.

For the medium M_1 we proceed exactly as in the general case, and thus $(u_2)_1$ is given by equation (10c) with the imaginary part of s_1 positive. However, for the medium M_2 we must retain the full solution, since the displacement no longer diminishes with increasing distance from the bounding surface of the two media. Hence

$$(u_2)_2 = A' \exp\{i\omega(s_2 x_3+x_1-ct)\}+B' \exp\{i\omega(-s_2 x_3+x_1-ct)\}, \tag{17}$$

where the imaginary part of the complex quantity s_2 is not now positive.

Because the displacement component u_2 and the stress component σ_{32} must be continuous across the plane of contact, we have

$$(u_2)_1 = (u_2)_2, \quad (\sigma_{32})_1 = (\sigma_{32})_2 \qquad \text{at} \quad x_3 = 0. \tag{18}$$

From equations (10c) and (17) with these conditions, we get

$$E_1 = A' + B', \qquad \mu_1^* s_1 E_1 = \mu_2^* s_2 (A' - B'). \tag{19}$$

Elimination of E_1 between equations (19) yields

$$A'(s_2 \mu_2^* - s_1 \mu_1^*) = B'(s_2 \mu_2^* + s_1 \mu_1^*). \tag{20}$$

Also making use of the boundary condition that there is no stress across the free surface,

$$(\sigma_{32})_2 = 0 \qquad \text{at} \quad x_3 = -H, \tag{21}$$

we have from equation (17)

$$A' \exp(-\mathrm{i}\omega s_2 H) = B' \exp(\mathrm{i}\omega s_2 H), \tag{22}$$

and, on eliminating A' and B' between equations (20) and (22), we obtain the result

$$s_2 \mu_2^* \tan(\omega s_2 H) + \mathrm{i} s_1 \mu_1^* = 0. \tag{23}$$

This is the required wave velocity equation for Love waves in a viscoelastic medium under the influence of gravity. It is seen from this equation that Love waves are not affected by the presence of a gravitational field. For perfectly elastic media, $\mu_1' = \mu_2' = 0$ and equation (23) also reduces to the corresponding classical result (Bullen 1965).

Stoneley Waves

In classical theory Stoneley waves are a generalized form of Rayleigh waves propagating along the common boundary of two elastic media. In the presence of viscous and gravity effects therefore, the wave velocity of Stoneley waves is determined by the roots of equation (14), which, of course, reduces once more to the classical result in the absence of these effects.

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