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Emission and Absorption of Langmuir Waves by Anisotropic Unmagnetized Particles

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Abstract

The emission and absorption coefficients for Langmuir waves due to anisotropic unmagnetized particles are reduced to two complementary forms: one involving integrals over momentum p and pitch angle α; the other involving an integral over p and a sum over Legendre polynomials. The quasilinear diffusion coefficients are reduced to the former. It is also shown how the absorption coefficient may be reduced to forms involving neither a p derivative nor an α derivative. The absorption coefficient is evaluated explicitly for five idealized anisotropic distributions, called a 'forward-cone' anisotropy, a 'semi-$\cos^2\alpha$' anisotropy, a loss-cone anisotropy, a P_1 anisotropy and a P_2 anisotropy respectively. All except the P_2 anisotropy can lead to growth of Langmuir waves, but only if the distribution function is an increasing function of p at the resonant phase speed, e.g. only for gap distributions. The results have important implications in connection with the theory of solar radio bursts.

1. Introduction

In most discussions of the interaction between fast particles and Langmuir waves, e.g. in connection with solar radio astronomy, the background magnetic field is either neglected entirely, or it is taken into account both through its effect on the distribution of particles and on the particle–wave interaction. As a consequence, little attention has been paid to the case where the magnetic field affects the motion of the particles but not the particle–wave interaction. However, in practice, it seems that this hybrid case is the appropriate one for fast particles in the solar corona. For, although even the weakest of magnetic fields should guide particle streams, and only modest fields are required to trap significant numbers of particles, the magnetic field affects the particle–wave interaction in only a minor way for $\Omega_e \ll \omega_p$, where Ω_e is the electron gyrofrequency and ω_p is the plasma frequency. Specifically, the magnetic field has only a minor effect on a particle–wave interaction whenever the gyrofrequency is much less than the wave frequency, and the gyroradius is much larger than a wavelength. Both conditions are satisfied for Langmuir waves and fast electrons (and, *a fortiori*, for fast ions) for $\Omega_e \ll \omega_p$, which inequality is generally satisfied in the corona. In other words, in their interaction with Langmuir waves in the corona, fast particles are unmagnetized.

Our purpose in this paper is to develop the theory of the emission and absorption of Langmuir waves by anisotropic distributions of fast particles. Although we do not discuss the detailed implications of the theory here, the applications we have in mind include the following. Firstly, it is widely accepted that type III solar radio bursts are due to plasma emission from Langmuir waves generated by a stream of electrons, but detailed familiar theories of the generation do not take the likely pitch-angle distribution into account. One might expect the predicted spectrum of Langmuir waves to be sufficiently sensitive to the assumed pitch-angle distribution to have observational implications. For example, it has been pointed out by H. Rosenberg (personal communication) and by Melrose (1976) that spontaneous emission perpendicular to the magnetic field lines is possible for a stream with a finite spread in pitch angle, and the resulting Langmuir waves can coalesce directly into a second-harmonic transverse wave, contrary to what one would predict with the conventional one-dimensional treatment. An obvious inadequacy of the existing simple treatments of streaming instabilities occurs for streams near the orbit of the Earth, where the pitch-angle distribution is far from being one-dimensional (Lin 1974) and is roughly of the form of a P_1 anisotropy, as defined in Section 5 below. No existing theory enables one to estimate the growth rate directly for Langmuir waves due to particles with such a distribution.

A second application concerns possible growth of Langmuir waves due to anisotropic nonstreaming distributions. The point is that, although most metre-wave solar radio emissions are thought to be due to some form of plasma emission, the only accepted mechanism for generating adequate Langmuir turbulence is a streaming instability. There is little evidence for adequate streaming motions except in type III bursts. Melrose (1975) suggested that an isotropic 'gap' distribution could produce adequate Langmuir turbulence but Robinson (1977) has pointed out that, when relativistic effects are taken into account, the maximum effective temperature of the resulting Langmuir turbulence is about 3×10^9 K, and this cannot lead to even moderately bright plasma emission. Another suggestion is that Langmuir waves can grow due to a loss-cone anisotropy (Stepanov 1973; Kuijpers 1974). Here we give an explicit expression for the absorption coefficient due to fast particles with a loss-cone anisotropy. Another type of nonstreaming anisotropy we consider in Section 5 is that generated in trapped particles due to a compression or rarefaction of the magnetic field.

The basic equations we use are written down in Section 2, and two complementary ways of reducing them are developed in Sections 3 and 4. The emission and absorption coefficients for specific anisotropic distributions are derived in Section 5.

2. Basic Equations

Let the distribution of fast particles have a number density n_1 and a distribution function $f(p,\alpha)$, where α is the pitch angle. The normalization is defined by

$$\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}\cos\alpha \int_0^{\infty} \mathrm{d}p\, p^2 f(p,\alpha) = n_1\,, \tag{1}$$

where ϕ is the azimuthal angle of $\boldsymbol{p}$ relative to the direction $\alpha = 0$ (the direction of the magnetic field). Let the Langmuir waves be described by their effective temperature $T(k,\theta)$ in units in which Boltzmann's constant is unity, and with θ the angle between $\boldsymbol{k}$ and the magnetic field. Then the normalization is such that the energy density W in the waves is given by

$$W = \int_0^{2\pi} \mathrm{d}\phi' \int_{-1}^{+1} \mathrm{d}\cos\theta \int_0^{\infty} \mathrm{d}k\, k^2 (2\pi)^{-3}\, T(k,\theta)\,, \tag{2}$$

where ϕ' is the azimuthal angle of $\boldsymbol{k}$ relative to the direction $\alpha = 0$.

The quasilinear equations describing the effects of emission and absorption of the Langmuir waves by the particles may be written, for the axially symmetric case, in the generic forms:

$$\mathrm{d}T(k,\theta)/\mathrm{d}t = \alpha(k,\theta) - \gamma(k,\theta)\,T(k,\theta)\,, \tag{3}$$

where $\alpha(k,\theta)$ and $\gamma(k,\theta)$ are the emission and absorption coefficients respectively, and

$$\begin{aligned}\frac{\mathrm{d}f(p,\alpha)}{\mathrm{d}t} = {} & \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\{\sin\alpha\, D_\alpha(p,\alpha)\, f(p,\alpha)\} + \frac{1}{p^2}\frac{\partial}{\partial p}\{p^2 D_p(p,\alpha)\, f(p,\alpha)\} \\ & + \frac{1}{\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\left\{D_{\alpha\alpha}(p,\alpha)\frac{\partial}{\partial\alpha} + D_{\alpha p}(p,\alpha)\frac{\partial}{\partial p}\right\} f(p,\alpha)\right) \\ & + \frac{1}{p^2}\frac{\partial}{\partial p}\left\{p^2\left(D_{p\alpha}(p,\alpha)\frac{\partial}{\partial\alpha} + D_{pp}(p,\alpha)\frac{\partial}{\partial p}\right) f(p,\alpha)\right\},\end{aligned} \tag{4}$$

where D_A is the coefficient describing systematic changes in variable A (due to spontaneous emission) and D_{AB} is the diffusion coefficient for variables A and B (due to induced processes). Thus the effects of spontaneous emission and of induced processes have been included in equations (3) and (4). It is straightforward to write down explicit expressions for the quantities introduced in equations (3) and (4), e.g. using the semiclassical form of the quasilinear equations in the form given by Melrose (1970). The angle χ, say, between $\boldsymbol{k}$ and $\boldsymbol{v}$ appears in $\boldsymbol{k}.\boldsymbol{v} = kv\cos\chi$. With the angles defined above we have

$$\cos\chi = \cos\alpha\cos\theta + \sin\alpha\sin\theta\cos(\phi - \phi')\,. \tag{5}$$

An explicit expression is also required for the differential operator $\boldsymbol{k}.\partial/\boldsymbol{p}$ in terms of the variables used in equations (3) and (4), namely

$$\boldsymbol{k}.\frac{\partial}{\partial\boldsymbol{p}} = k\cos\chi\frac{\partial}{\partial p} - \frac{k(\cos\theta - \cos\alpha\cos\chi)}{p\sin\alpha}\frac{\partial}{\partial\alpha} + \frac{k}{p\sin^2\alpha}\left(\frac{\partial}{\partial\phi}\cos\chi\right)\frac{\partial}{\partial\phi}\,. \tag{6}$$

The ϕ derivative necessarily gives zero in the axially symmetric case.

For simplicity, we set the frequency of the Langmuir waves equal to the plasma frequency ω_{p}. It is also convenient to introduce the following quantities relating to v_ϕ, the phase speed of the waves,

$$v_\phi = \omega_{\mathrm{p}}/k\,, \qquad p_\phi = mv_\phi/(1 - v_\phi^2/c^2)^{\frac{1}{2}}\,, \qquad \cos\chi_0 = v_\phi/v\,. \tag{7}$$

The wave–particle resonance is possible only for $v_\phi < v$, that is, for $0 \leqslant \chi_0 \leqslant \frac{1}{2}\pi$. Actually, Langmuir waves cease to exist for $v_\phi \lesssim V_{\mathrm{e}}$, where V_{e} is the thermal speed of electrons, and for fast particles ($v \gg V_{\mathrm{e}}$) the effective range over which the interaction is possible is $0 \leqslant \chi_0 \lesssim \frac{1}{2}\pi - V_{\mathrm{e}}/v$.

Explicit expressions for the emission and absorption coefficients which appear in equation (3) are

$$\alpha(k,\theta) = 4\pi^2 e^2 \omega_{\mathrm{p}}^2 k^{-3}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{+1}\mathrm{d}\cos\alpha\int_0^\infty \mathrm{d}p\, v^{-1}p^2 f(p,\alpha)\,\delta(\cos\chi - \cos\chi_0)\,, \tag{8}$$

$$\begin{aligned}\gamma(k,\theta) = 4\pi^2 e^2 \omega_{\mathrm{p}}\, k^{-3}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{+1}\mathrm{d}\cos\alpha\int_0^\infty \mathrm{d}p\, v^{-1}p^2\,\delta(\cos\chi - \cos\chi_0) \\ \times\left(\Delta p\frac{\partial}{\partial p} + \Delta\alpha\frac{\partial}{\partial\alpha}\right) f(p,\alpha)\,.\end{aligned} \tag{9}$$

Explicit expressions for the coefficients in equation (4) are

$$\begin{bmatrix} D_A(p,\alpha) \\ D_{AB}(p,\alpha) \end{bmatrix} = \frac{4\pi^2 e^2 \omega_p}{(2\pi)^3 v} \int_0^{2\pi} d\phi' \int_{-1}^{1} d\cos\theta \int_0^{\infty} \frac{dk}{k} \delta(\cos\chi - \cos\chi_0) \begin{bmatrix} \Delta A \\ \Delta A\, \Delta B\, T(k,\theta)/\omega_p \end{bmatrix}, \tag{10}$$

with A, B equal to α, p and with, from equation (6),

$$\Delta p = k\cos\chi, \qquad \Delta\alpha = -k(\cos\theta - \cos\alpha\cos\chi)/p\sin\alpha. \tag{11}$$

The term $\alpha(k,\theta)$ in equation (3) and the terms involving $D_\alpha(p,\alpha)$ and $D_p(p,\alpha)$ in (4) describe the effects of spontaneous emission, while the remaining terms describe the effects of the induced processes. The condition $\cos\chi_0 < 1$ implies that the lower limit of the p integral in equations (8) and (9) may be replaced by p_ϕ, and the lower limit of the k integral in (10) may be replaced by ω_p/v.

3. Direct Integration over Azimuthal Angle

The reduction of the quasilinear equations may be carried out by performing the integrals over azimuthal angle using the δ function. This procedure is useful for sufficiently simple distributions for which the remaining integral over polar angle may be performed explicitly, and several relevant examples are considered in Section 5. The solutions obtained by carrying out the ϕ integrals are also required for formal purposes in the reduction of the equations after expanding in Legendre polynomials in Section 4.

The basic integral we require is

$$\int_0^{2\pi} d\phi\, \delta(\cos\chi - \cos\chi_0) = 2/F(\alpha,\theta,\chi_0) \qquad \cos\alpha_- \leqslant \cos\alpha \leqslant \cos\alpha_+, \tag{12a}$$

$$= 0 \qquad \text{otherwise}; \tag{12b}$$

with

$$F(\alpha,\theta,\chi) = (1 + 2\cos\alpha\cos\theta\cos\chi - \cos^2\alpha - \cos^2\theta - \cos^2\chi)^{\frac{1}{2}} \tag{13}$$

and

$$\cos\alpha_\pm = \cos(\theta \mp \chi_0). \tag{14}$$

The function $F(\alpha,\theta,\chi_0)$ is simply $|\partial\cos\chi/\partial\phi|$ evaluated at $\cos\chi = \cos\chi_0$. The factor of two in equation (12) arises from the fact that there are two solutions for ϕ in the range $0 \leqslant \phi < 2\pi$ for each solution for $\cos\phi$ of $\cos\chi = \cos\chi_0$. Hence we find

$$\alpha(k,\theta) = \frac{8\pi^2 e^2 \omega_p^2}{k^3} \int_{p_\phi}^{\infty} dp\, \frac{p^2}{v} \int_{\cos\alpha_-}^{\cos\alpha_+} d\cos\alpha\, \frac{f(p,\alpha)}{F(\alpha,\theta,\chi_0)}, \tag{15}$$

$$\gamma(k,\theta) = \frac{8\pi^2 e^2 \omega_p^2}{k^3} \int_{p_\phi}^{\infty} dp\, \frac{p}{v^2} \int_{\cos\alpha_-}^{\cos\alpha_+} \frac{d\cos\alpha}{F(\alpha,\theta,\chi_0)} \left(p\frac{\partial}{\partial p} - \frac{\cos\theta - \cos\alpha\cos\chi_0}{\cos\chi_0 \sin\alpha} \frac{\partial}{\partial\alpha} \right) f(p,\alpha), \tag{16}$$

$$\begin{bmatrix} D_A(p,\alpha) \\ D_{AB}(p,\alpha) \end{bmatrix} = \frac{e^2\omega_p}{\pi v} \int_{\omega_p/v} \frac{dk}{k} \int_{\cos\theta_-}^{\cos\theta_+} \frac{d\cos\alpha}{F(\alpha,\theta,\chi_0)} \begin{bmatrix} \Delta A \\ \Delta A\, \Delta B\, T(k,\theta)/\omega_p \end{bmatrix}, \tag{17}$$

with

$$\cos\theta_\pm = \cos(\alpha \mp \chi_0). \tag{18}$$

As in equation (10), A or B are equal to p or α, and Δp and $\Delta\alpha$ are given by (11) with $\chi = \chi_0$.

4. Expansion in Legendre Polynomials

The alternative way of reducing equation (8) to (10) is to expand in Legendre polynomials:

$$f(p,\alpha) = \sum_{n=0}^{\infty} f_n(p)\,\mathrm{P}_n(\cos\alpha), \quad \text{with} \quad f_n(p) = \tfrac{1}{2}(2n+1)\int_{-1}^{+1} \mathrm{d}\cos\alpha\, f(p,\alpha)\,\mathrm{P}_n(\cos\alpha), \tag{19a, b}$$

$$T(k,\theta) = \sum_{n=0}^{\infty} T_n(k)\,\mathrm{P}_n(\cos\theta), \qquad T_n(k) = \tfrac{1}{2}(2n+1)\int_{-1}^{+1} \mathrm{d}\cos\theta\, T(k,\theta)\,\mathrm{P}_n(\cos\theta). \tag{20a, b}$$

Only equations (8) and (9) are discussed explicitly here, and only the expansion (19a) is used. The complete expansion of $f(p,\alpha)$ and $T(k,\theta)$ simultaneously, and the analogous cases for equation (10), can be treated in similar fashion but we do not do so here.

The basic identity required is

$$\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}\cos\alpha\, \mathrm{P}_n(\cos\alpha)\,\delta(\cos\chi - \cos\chi_0) = 2\pi\,\mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(\cos\chi_0)\,. \tag{21}$$

The proof of this result is as follows. We rewrite the integral over solid angle, i.e. the $(\phi, \cos\alpha)$ integral as a $(\phi_1, \cos\chi)$ integral, where χ and ϕ_1 are the polar and azimuthal angles of $\boldsymbol{v}$ relative to $\boldsymbol{k}$. We let θ and ϕ_2 be the corresponding angles of the direction $\alpha = 0$ (the direction of the magnetic field) relative to $\boldsymbol{k}$. Finally, we use the 'addition theorem' for Legendre polynomials (e.g. Gradshteyn and Ryzik 1965, p. 1015),

$$\mathrm{P}_n(\cos\alpha) = \mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(\cos\chi) + 2\sum_{m=1}^{\infty} \frac{(n-m)!}{(n+m)!}\,\mathrm{P}_n^m(\cos\theta)\,\mathrm{P}_n^m(\cos\chi)\cos\big(m(\phi_1-\phi_2)\big). \tag{22}$$

The $\cos\chi$ integral may be trivially performed using the δ function, and the remaining ϕ_1 integral is also trivial.

A second identity we require is

$$\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}\cos\alpha\,(\cos\theta - \cos\alpha\cos\chi)\,\mathrm{P}_n'(\cos\alpha)\,\delta(\cos\chi - \cos\chi_0)$$
$$= -2\pi n(n+1)\,\mathrm{P}_n(\cos\theta)\int_{\cos\chi_0}^{1} \mathrm{d}x\,\mathrm{P}_n(x)\,. \tag{23}$$

The proof of this identity is as follows. We carry out the ϕ integral using equation (12), noting that

$$\frac{\cos\theta - \cos\alpha\cos\chi}{F(\alpha,\theta,\chi)} = -\frac{\partial F(\alpha,\theta,\chi)}{\partial\cos\theta}, \tag{24}$$

and hence obtain

$$\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \mathrm{d}\cos\alpha\,(\cos\theta - \cos\alpha\cos\chi)\,\mathrm{P}_n'(\cos\alpha)\,\delta(\cos\chi - \cos\chi_0)$$
$$= -2\frac{\partial}{\partial\cos\theta}\int_{\cos\alpha_-}^{\cos\alpha_+} \mathrm{d}\cos\alpha\, F(\alpha,\theta,\chi_0)\,\mathrm{P}_n'(\cos\alpha)\,. \tag{25}$$

Here the derivative has been taken outside the integral after noting that $F(\alpha, \theta, \chi_0)$ vanishes for $\cos\alpha = \cos\alpha_\pm$. A partial integration may now be performed, with the integrated parts giving zero for the same reason, and with the $\cos\alpha$ derivative of $F(\alpha, \theta, \chi)$ implied by equation (24) and the obvious symmetry of $F(\alpha, \theta, \chi)$ (see equation 13). The resulting expression may be rewritten in the form of a $(\phi, \cos\alpha)$ integral, giving

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\alpha\, (\cos\theta - \cos\alpha\cos\chi)\, P_n'(\cos\alpha)\, \delta(\cos\chi - \cos\chi_0)$$
$$= -\frac{\partial}{\partial\cos\theta}\int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos\alpha\, (\cos\alpha - \cos\theta\cos\chi)\, P_n(\cos\alpha)\, \delta(\cos\chi - \cos\chi_0). \quad (26)$$

The result (23) then follows by using

$$xP_n(x) = \{(n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)\}/(2n+1) \quad (27)$$

to cast the right-hand side of equation (26) into a sum of terms of the form (21) and, finally, using (27) together with

$$-\int_x^1 dy\, P_n(y) = \{P_{n+1}(x) - P_{n-1}(x)\}/(2n+1) \quad (28)$$

to reduce the result to the stated form. In this way, equations (8) and (9) reduce to

$$\alpha(k,\theta) = \frac{8\pi^3 e^2\omega_p^2}{k^3}\sum_{n=0}^{\infty} P_n(\cos\theta)\int_{p_\phi}^{\infty} dp\, \frac{p^2}{v} f_n(p)\, P_n(\omega_p/kv) \quad (29)$$

and

$$\gamma(k,\theta) = \frac{8\pi^3 e^2\omega_p^2}{k^3}\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{2n+1}\int_{p_\phi}^{\infty} dp\, \frac{p}{v^2}\left\{\left(p\frac{\partial f_n(p)}{\partial p} - n f_n(p)\right)(n+1)\, P_{n+1}(\omega_p/kv)\right.$$
$$\left. + \left(p\frac{\partial f_n(p)}{\partial p} + (n+1)\, f_n(p)\right) n\, P_{n-1}(\omega_p/kv)\right\} \quad (30)$$

respectively.

The result (29) is in a convenient form for direct application to some problems. However, it is usually convenient to reduce equation (30) further by partially integrating over p. The terms involving the p-derivatives in equation (30) are proportional to

$$\sum_{n=0}^{\infty} P_n(\cos\theta)\int_{p_\phi}^{\infty} dp\, \frac{p^2\omega_p}{kv^2}\, P_n(\omega_p/kv)\frac{\partial f_n(p)}{\partial p}$$
$$= \sum_{n=0}^{\infty} P_n(\cos\theta)\int_{p_\phi}^{\infty} dp\, P_n(\omega_p/kv)\left\{\frac{\partial}{\partial p}\left(\frac{\omega_p p^2}{kv^2} f_n(p)\right) - \frac{2\omega_p p f_n(p)}{kc^2}\right\}, \quad (31)$$

where we have used

$$\frac{d}{dp}\left(\frac{p^2}{v^2}\right) = \frac{2p}{c^2}. \quad (32)$$

The partial integration gives

$$\int_{p_\phi}^{\infty} dp\, P_n(\omega_p/kv)\frac{\partial}{\partial p}\left(\frac{\omega_p p^2}{kv^2} f_n(p)\right) = -\frac{\omega_p p_\phi^2}{kv_\phi^2} f_n(p_\phi) + m\int_{p_\phi}^{\infty}\frac{dp}{\gamma} f_n(p)\, P_n'(\omega_p/kv)\frac{\omega_p^2}{k^2 v^2}, \quad (33)$$

where we have used $P_n(1) = 1$ and

$$dv/dp = 1/m\gamma^3, \tag{34}$$

where γ is the Lorentz factor. The other terms in equation (30) may be combined using the identity

$$(x^2-1)P_n'(x) = \{n(n+1)/(2n+1)\}\{P_{n+1}(x)-P_{n-1}(x)\}. \tag{35}$$

The resulting alternative expression for the absorption coefficient is

$$\begin{aligned}\gamma(k,\theta) = \frac{8\pi^3 e^2\omega_p^2}{k^3}\Bigg[&\frac{p_\phi^2}{v_\phi^2} f(p_\phi,\theta) + \sum_{n=0}^{\infty} P_n(\cos\theta)\int_{p\phi}^{\infty} dp\,\frac{2p}{c^2} f_n(p)\,P_n(\omega_p/kv)\\ &- \sum_{n=1}^{\infty} P_n(\cos\theta)\,m^2\int_{p\phi}^{\infty}\frac{dp}{p}\frac{kv}{\omega_p} f_n(p)\,P_n'(\omega_p/kv)\left\{1+(\gamma^2-1)\left(1-\frac{\omega_p^2}{k^2v^2}\right)\right\}\Bigg].\end{aligned} \tag{36}$$

On performing the sums over n, equation (36) reduces to

$$\begin{aligned}\gamma(k,\theta) = \frac{8\pi^3 e^2\omega_p^2}{k^3}\Bigg(&\frac{p_\phi^2}{v_\phi^2} f(p_\phi,\theta) + \frac{1}{\pi}\int_{p\phi}^{\infty} dp\,\frac{2p}{c^2}\int_{\cos\alpha_-}^{\cos\alpha_+}\frac{d\cos\alpha\, f(p,\alpha)}{F(\alpha,\theta,\chi_0)}\\ &-\frac{m^2}{\pi}\int_{p\phi}^{\infty}\frac{dp}{p}\frac{\{1+(\gamma^2-1)\sin^2\chi_0\}}{\cos\chi_0}\frac{\partial}{\partial\cos\chi_0}\int_{\cos\alpha_-}^{\cos\alpha_+}\frac{d\cos\alpha\, f(p,\alpha)}{F(\alpha,\theta,\chi_0)}\Bigg),\end{aligned} \tag{37}$$

where it is to be understood that p is independent of χ_0 for the purpose of carrying out the indicated $\cos\chi_0$ differentiation. In principle, one could derive equation (37) directly from (16), and it is not difficult to do so from hindsight.

5. Specific Anisotropic Distributions

In this section we evaluate the emission coefficient $\alpha(k,\theta)$ and the absorption coefficient $\gamma(k,\theta)$ for specific anisotropic distributions. The distributions considered fall into two classes. The first class consists of separable distributions

$$f(p,\alpha) = f(p)\,\phi(\alpha), \tag{38}$$

with the pitch-angle distribution $\phi(\alpha)$ such that the $\cos\alpha$ integrals in equation (37) can be evaluated explicitly. The second class consists of distributions whose anisotropic parts are proportional to $P_n(\cos\alpha)$ with $n = 1$ and 2, called P_1 and P_2 anisotropies respectively.

The assumption (38) is made here for convenience and it need not be restrictive. The point is that the emission and absorption coefficients may now be written in the forms

$$\alpha(k,\theta) = \frac{8\pi^3 e^2\omega_p^2}{k^3}\int_{p\phi}^{\infty} dp\,\frac{p^2}{v} f(p)\,g(\theta,\chi_0) \tag{39}$$

and

$$\gamma(k,\theta) = \gamma_I(k,\theta)+\gamma_R(k,\theta)+\gamma_A(k,\theta), \tag{40}$$

with

$$\gamma_I(k,\theta) = \frac{8\pi^3 e^2}{k} p_\phi^2 f(p_\phi)\,\phi(\theta), \tag{41a}$$

$$\gamma_{\mathrm{R}}(k,\theta) = \frac{8\pi^3 e^2 \omega_{\mathrm{p}}^2}{k^3} \int_{p\phi}^{\infty} \mathrm{d}p \, \frac{2p}{c^2} g(\theta, \chi_0), \tag{41b}$$

$$\gamma_{\mathrm{A}}(k,\theta) = -\frac{8\pi^3 e^2 \omega_{\mathrm{p}}^2}{k^3} m^2 \int_{p\phi}^{\infty} \frac{\mathrm{d}p}{p} \frac{\{1+(\gamma^2-1)\sin^2\chi_0\}}{\cos\chi_0} f(p) \frac{\partial g(\theta,\chi_0)}{\partial \cos\chi_0}, \tag{41c}$$

that is, in terms of $f(p)$ and a single function

$$g(\theta,\chi_0) \equiv \frac{1}{\pi} \int_{\cos\alpha_-}^{\cos\alpha_+} \mathrm{d}\cos\alpha \, \frac{\phi(\alpha)}{F(\alpha,\theta,\chi_0)}. \tag{42}$$

However, it requires only a change in notation to generalize these results to non-separable distributions.

The three terms of equation (40) have the following interpretations. The first term, $\gamma_{\mathrm{I}}(k,\theta)$ (equation 41a), is a generalization of the only term which remains for isotropic nonrelativistic particles. Specifically, this term is proportional to the distribution function evaluated at the resonant speed $v = v_\phi$ and at the angle of emission $\alpha = \theta$. For a gap distribution (Melrose 1975) this term is negligible. In fact one could define a gap distribution as a distribution of particles which leads to non-negligible emission and negligible $\gamma_{\mathrm{I}}(k,\theta)$ over some range of phase speeds v_ϕ and angles θ, referred to as 'the gap'. The second term, $\gamma_{\mathrm{R}}(k,\theta)$ (equation 41b), may be regarded as a relativistic correction. As was pointed out by Robinson (1976), for a gap distribution this term limits the effective temperature of Langmuir waves in the gap to

$$T(k) < \tfrac{1}{2}\varepsilon_0, \tag{43}$$

where ε_0 is roughly the mean total (kinetic plus rest) energy per particle, i.e. to $T(k) \lesssim 3\times 10^9$ K for a nonrelativistic gap distribution. The third term, $\gamma_{\mathrm{A}}(k,\theta)$ (equation 41c), depends explicitly on the anisotropy, and it is the only term which can be negative. However, growth occurs only if the third term is both negative and sufficient in magnitude to exceed the positive contributions from the other two terms. Growth is possible only for 'extreme anisotropies' (which we do not attempt to define) or for anisotropic gap distributions. In the following discussion we have the latter in mind.

For the separable distributions considered below, the integrals which appear in evaluating equation (42) are of the form

$$I_n = \int \frac{\mathrm{d}\cos\alpha \, (\cos\alpha)^n}{F(\alpha,\theta,\chi)}. \tag{44}$$

For $n = 0, 1, 2, \ldots$, we have

$$I_0 = \arcsin\left(\frac{\cos\alpha - \cos\theta\cos\chi}{\sin\theta\sin\chi}\right), \qquad I_1 = -F(\alpha,\theta,\chi) + (\cos\theta\cos\chi) I_0, \tag{45a, b}$$

$$I_2 = -\tfrac{1}{2}(\cos\alpha + 3\cos\theta\cos\chi) F(\alpha,\theta,\chi) + \tfrac{1}{2}(3\cos^2\theta\cos^2\chi + 1 - \cos^2\theta - \cos^2\chi) I_0, \tag{45c}$$

and so on.

Forward-cone Distribution

The first distribution we consider is a 'forward-cone' distribution,

$$\phi(\alpha) = 2/(1-\cos\alpha_0) \qquad \alpha < \alpha_0, \tag{46a}$$

$$= \quad 0 \qquad \text{otherwise}. \tag{12b}$$

A forward-cone distribution may be regarded as an idealized streaming distribution which takes account of the finite spread in pitch angles expected for streams encountered in practice (e.g. in the solar corona). The effects of this spread on the growth rate of the Langmuir waves, and on the range in which the waves grow are of particular interest in the theory of type III solar radio bursts.

For the forward-cone distribution one finds

$$g(\theta,\chi_0) = 2/(1-\cos\alpha_0) \qquad 0 \leqslant \theta,\ \chi_0 \leqslant \alpha_0; \tag{47a}$$

$$= \frac{2}{1-\cos\alpha_0}\left\{\frac{1}{2}-\frac{1}{\pi}\arcsin\left(\frac{\cos\alpha_0-\cos\theta\cos\chi_0}{\sin\theta\sin\chi_0}\right)\right\} \qquad 0<\theta,\ \ \alpha_0 \leqslant \chi_0 \leqslant \theta+\alpha_0; \tag{47b}$$

$$= 0 \qquad \text{otherwise}; \tag{47c}$$

$$\frac{\partial g(\theta,\chi_0)}{\partial\cos\chi_0} = \frac{2}{1-\cos\alpha_0}\,\frac{(\cos\theta-\cos\alpha_0\cos\chi_0)}{\pi\sin^2(\chi_0)F(\alpha_0,\theta,\chi_0)} \qquad \begin{cases} 0<\theta, & -\alpha_0 \leqslant \chi_0 \leqslant \theta, \\ \alpha_0 < \tfrac{1}{2}\pi; \end{cases} \tag{48a}$$

$$= 0 \qquad \text{otherwise}. \tag{48b}$$

The first two ranges (47a) and (47b) correspond to $v\cos(\alpha_0-\theta) \leqslant \omega_p/k \leqslant v$ for $\theta \leqslant \alpha_0$ and to $v\cos(\alpha_0+\theta) \leqslant \omega_p/k \leqslant v\cos(\theta-\alpha_0)$ for $\theta > \alpha_0$. Recall that χ_0 is necessarily less than $\frac{1}{2}\pi$ and that Landau damping, which is neglected above, will be the dominant effect for $\chi_0-\frac{1}{2}\pi \lesssim V_e/v$, where V_e is the thermal speed of electrons.

One feature of the emission due to a forward-cone distribution is that it extends into the backward hemisphere, that is, $g(\theta,\chi_0)$ is nonzero for $\theta > \frac{1}{2}\pi$. This is not the case for a one-dimensional stream, i.e. for $\alpha_0 = 0$. Emission into the backward hemisphere is of practical interest in that it allows direct coalescence of the Langmuir waves into a second harmonic (H. Rosenberg, personal communication; Melrose 1976).

It is not our intention here to discuss the implications or applications of the resulting expression for the absorption coefficient, but several comments are appropriate. Firstly, $\gamma_A(k,\theta)$ is infinite at the zeros of $F(\alpha_0,\theta,\chi_0)$, and the zero at $|\theta-\chi_0| = \alpha_0$ corresponds to wave growth (and that at $\theta+\chi_0 = \alpha_0$ to absorption). Secondly, the infinity is unrealistic and is due to the assumption of a discontinous pitch-angle distribution. For a continuous pitch-angle distribution, or after integrating over a continuous momentum distribution, the infinity is replaced by a finite peak value. In practice, significant growth is restricted to a range of parameters close to the relevant zero of $F(\alpha_0,\theta,\chi_0)$. Thirdly, $\gamma_A(k,\theta)$ is finite and negative over a range of phase speeds in the gap. The point is that in existing treatments of streaming instabilities the growth rate is proportional to $\partial f/\partial v$ at $v = v_\phi$, and this result is simply not valid in general. Fourthly, for at least a finite range of phase speeds in the gap (for a δ function momentum distribution) growth is possible for a range of angles extending to $\theta = \frac{1}{2}\pi$. However, the maximum growth rate applies only for $\theta \lesssim \alpha_0$.

Semi-$(\cos\alpha)^m$ *Distributions*

A class of pitch-angle distributions which describe idealized streaming motions is the class of 'semi-$(\cos\alpha)^m$ distributions',

$$\phi(\alpha) = (m+1)(\cos\alpha)^m \qquad \theta \leqslant \alpha \leqslant \tfrac{1}{2}\pi, \tag{49a}$$

$$= 0 \qquad \tfrac{1}{2}\pi < \alpha < \pi. \tag{49b}$$

The larger the value of m the more strongly peaked into the forward direction is the distribution. This class of distributions is intermediate between the forward-cone distribution considered above, and the P_1 anisotropy considered below. Unlike the forward-cone distribution, here $\phi(\alpha)$ is a continuous function of α while, unlike the P_1 anisotropy, it includes no particles in the backward hemisphere (i.e. at $\alpha > \frac{1}{2}\pi$).

We merely quote the result for the particular case $m = 2$:

$$g(\theta,\chi_0) = 3(3\cos^2\theta\cos^2\chi_0 + 1 - \cos^2\theta - \cos^2\chi_0) \qquad \theta+\chi_0 \leqslant \tfrac{1}{2}\pi, \quad \theta \leqslant \tfrac{1}{2}\pi; \tag{50a}$$

$$\begin{aligned} &= 3(3\cos^2\theta\cos^2\chi_0 + 1 - \cos^2\theta - \cos^2\chi_0) \\ &\quad \times \{\tfrac{1}{2} + \pi^{-1}\arcsin(\cot\theta\cot\chi_0)\} \\ &\quad + 9\pi^{-1}\cos\theta\cos\chi_0\, F(\tfrac{1}{2}\pi,\theta,\chi_0) \qquad \theta+\chi_0 > \tfrac{1}{2}\pi; \end{aligned} \tag{50b}$$

$$= 0 \qquad \theta+\chi_0 \leqslant \tfrac{1}{2}\pi, \quad \theta > \tfrac{1}{2}\pi; \tag{50c}$$

$$\frac{\partial g(\theta,\chi_0)}{\partial\cos\chi_0} = 6\cos\chi_0(3\cos^2\theta - 1) \qquad \theta+\chi_0 \leqslant \tfrac{1}{2}\pi, \quad \theta \leqslant \tfrac{1}{2}\pi; \tag{51a}$$

$$\begin{aligned} &= 6\cos\chi_0(3\cos^2\theta - 1) \\ &\quad \times \{\tfrac{1}{2} + \pi^{-1}\arcsin(\cot\theta\cot\chi_0)\} \\ &\quad - (6\cos\theta/\pi\sin^2\chi_0)(3\cos^2\chi_0 - 2)\, F(\tfrac{1}{2}\pi,\theta,\chi_0) \quad \theta+\chi_0 > \tfrac{1}{2}\pi; \end{aligned} \tag{51b}$$

$$= 0 \qquad \theta+\chi_0 \leqslant \tfrac{1}{2}\pi, \quad \theta > \tfrac{1}{2}\pi. \tag{51c}$$

P_1 *Anisotropy*

A pure P_1 anisotropy is a distribution which has $f_n(p) \neq 0$ only for $n = 0$ and 1. It may be regarded as yet another type of idealized streaming distribution. The streaming speed U_s, which may be a function of p, may be defined by

$$f_1(p) = 3(U_s/v)\, f_0(p). \tag{52}$$

In this case one finds

$$g(\theta,\chi_0) = 3(U_s/v)\cos\theta\cos\chi_0, \quad \partial g(\theta,\chi_0)/\partial\cos\chi_0 = 3(U_s/v)\cos\theta. \tag{53a, b}$$

An interesting feature of the P_1 anisotropy is that for an idealized gap distribution, i.e. a δ function at $p = p_0$ or $v = v_0$ say, growth occurs for all phase speeds in the gap for $\cos\theta > 0$ provided the streaming speed satisfies

$$U_s > 2v_0^3/3c^2. \tag{54}$$

Of course, for U_s to be negligible it must also exceed the thermal speed of electrons for Landau damping by the thermal electrons.

Qualitatively, as one passes from the forward-cone anisotropy with $\alpha_0 \ll 1$, through the $(\cos\alpha)^m$ distributions to the P_1 distribution, one finds that growth occurs over a wider and wider range of phase speeds in the gap. In contrast, the familiar (and effectively one-dimensional) treatments of streaming instability predict growth only at the resonant phase speeds where $\partial f/\partial v$ is positive, and no growth is predicted in the gap at lower phase speeds.

Loss-cone Distribution

A loss-cone gap distribution is likely to form in the solar corona when fast particles are trapped in a magnetic flux tube. The smaller the pitch angle of the particle, the lower the altitude at which it mirrors, and hence the higher the collision rate it experiences. Thus particles with small pitch angles are preferentially lost. Moreover, the collision rate for Coulomb interactions varies as the inverse cube of the speed of the particle. Hence slower particles should be lost more rapidly than faster particles, leaving a loss-cone gap distribution.

Consider the idealized loss-cone distribution

$$\phi(\alpha) = 1/\cos\alpha_0 \qquad \alpha_0 \leqslant \alpha \leqslant \pi-\alpha_0, \tag{55a}$$

$$= 0 \qquad \text{otherwise}. \tag{55b}$$

In this case one finds

$$g(\theta,\chi_0) = 1/\cos\alpha_0 \qquad \begin{cases} \theta-\chi_0 \geqslant \alpha_0, \quad \theta \geqslant \alpha_0, \\ \chi_0-\theta > \alpha_0; \end{cases} \tag{56a}$$

$$= \frac{1}{\cos\alpha_0}\left\{\tfrac{1}{2} + \pi^{-1}\arcsin\left(\frac{\cos\alpha_0 - \cos\theta\cos\chi_0}{\sin\theta\sin\chi_0}\right)\right\} \qquad \begin{cases} |\theta-\alpha_0| \leqslant \chi_0 \leqslant \theta+\alpha_0, \\ \chi_0 \leqslant \pi-\alpha_0-\theta; \end{cases} \tag{56b}$$

$$= \frac{1}{\cos\alpha_0}\left\{\pi^{-1}\arcsin\left(\frac{\cos\alpha_0 - \cos\theta\cos\chi_0}{\sin\theta\sin\chi_0}\right) + \pi^{-1}\arcsin\left(\frac{\cos\alpha_0 + \cos\theta\cos\chi_0}{\sin\theta\sin\chi_0}\right)\right\} \qquad \chi_0 > \pi-\alpha_0-\theta; \tag{56c}$$

$$= 0 \qquad \chi_0 < \theta-\alpha_0; \tag{56d}$$

$$\frac{\partial g(\theta,\chi_0)}{\partial\cos\chi_0} = 0 \qquad |\theta-\chi_0| \geqslant \alpha_0; \tag{57a}$$

$$= h_+(\theta,\chi_0)/\cos\alpha_0 \qquad \begin{cases} |\theta-\alpha_0| \leqslant \chi_0 \leqslant \theta+\alpha_0, \\ \chi_0 \leqslant \pi-\alpha_0-\theta; \end{cases} \tag{57b}$$

$$= \{h_+(\theta,\chi_0) + h_-(\theta,\chi_0)\}/\cos\alpha_0 \qquad \chi_0 > \pi-\alpha_0-\theta; \tag{57c}$$

with

$$h_\pm(\theta,\chi_0) = \pm\cos\theta + \cos\alpha_0\cos\chi_0/\pi\sin^2\chi_0 F_\pm(\alpha_0,\theta,\chi_0), \tag{58}$$

$$F_\pm(\alpha_0,\theta,\chi_0) \equiv (1 \mp 2\cos\alpha_0\cos\theta\cos\chi_0 - \cos^2\alpha_0 - \cos^2\theta - \cos^2\chi_0)^{\frac{1}{2}}. \tag{59}$$

The results (56) and (57) apply only for $\theta \leqslant \frac{1}{2}\pi$, while the result for $\theta > \frac{1}{2}\pi$ is to be found by appealing to the symmetry (implied by the equations 55)

$$g(\theta,\chi_0) = g(\pi-\theta,\chi_0). \tag{60}$$

Growth of Langmuir waves due to a loss-cone distribution is a plausible source of Langmuir turbulence in the solar corona, as was suggested by Stepanov (1973) and Kuijpers (1974). The implications of the results derived above warrant a detailed investigation which is inappropriate here. Calculations of the growth rate due to loss-cone distributions have been presented by Zaitsev and Stepanov (1975) and Benz and Kuijpers (1976). Suffice it to say for present purposes, growth due to a loss-cone gap distribution is possible and appears favourable for trapped fast particles in the solar corona.

P_2 *Anisotropy*

The only other example of a nonstreaming anisotropy we consider is a P_2 anisotropy, that is, $f_n(p) \neq 0$ only for $n = 0$ and 2. Such a distribution could be generated when trapped particles are subjected to a compression or rarefaction of the magnetic field. Suppose the magnetic induction changes from B to $B+\Delta B$ with $|\Delta B| \ll B$. Then an initially anisotropic distribution develops a P_2 anisotropy with

$$f_2(p) = -(\Delta B/3B)\,p\,\partial f_0(p)/\partial p. \tag{61}$$

In this case the final term in equation (36), which is the only one which can cause the absorption coefficient to be negative, may be integrated (when relativistic effects are ignored) and it gives only a small correction to the first term on the right-hand side of this equation. Consequently, a P_2 anisotropy of the form (61) for nonrelativistic particles cannot lead to growth of Langmuir waves under any circumstances.

6. Discussion and Conclusions

The primary purpose of this paper has been to present a theory for the emission and absorption of Langmuir waves by anisotropic unmagnetized particles. The emission coefficient has been written in the form (8) and reduced to the forms (15) and (29). The absorption coefficient has been written in the form (9) and reduced to the forms (16), (30), (36) and (37), and (40) with (41a)–(41c). We have also written down the quasilinear diffusion coefficients (10) and reduced them to the forms (17).

The emission and absorption coefficients have been evaluated explicitly for several idealized pitch-angle distributions. A common feature of the results, namely equation (40) with (41a)–(41c) and (47), (48), (53), (56) and (57), is that negative absorption would appear to be possible at a given $p_\phi = mv_\phi/(1-v_\phi^2/c^2)^{\frac{1}{2}}$ only if $f(p)$ is an increasing function of $p > p_\phi$. The point is that the common term $\gamma_1(k,\theta)$ (see equation 41a) gives a dominant positive contribution except when $f(p)$ is an increasing function of $p > p_\phi$. Thus, apart from extreme anisotropies, effective growth of Langmuir waves due to anisotropic distributions of particles can occur only when the distributions are also gap distributions (Melrose 1975). A similar conclusion was reached by Robinson (1977).

For gap distributions, growth due to a variety of anisotropies is possible. For streaming anisotropies it is clear from equations (47), (48), (50), (51) and (53) that the magnitude and angular dependence of the absorption coefficient depends on the assumed form of the pitch-angle distribution. Growth at phase speeds much less than the streaming speed and at all angles $\theta \lesssim \frac{1}{2}\pi$ is possible in principle. A loss-cone distribution can lead to negative absorption of Langmuir waves, provided it is also a gap distribution. However, the anisotropy generated by compressing (or rarefying) trapped particles is unfavourable for growth of Langmuir waves.

In conclusion, it is evident that some conventional ideas relating to streaming instabilities in the solar corona are oversimplified, and that other distributions such as a loss-cone anisotropy and P_1 anisotropy could be important in generating intense Langmuir turbulence and hence observable plasma emission. These points need to be explored further.

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