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Mode Coupling in the Solar Corona. III*
Alfvén and Magnetoacoustic Waves

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Abstract

Coupling between Alfvén waves and fast mode waves obliquely incident on a stratified medium is treated using the method of Clemmow and Heading (1954) within the framework of the cold plasma approximation. A result due to Frisch (1964) is rederived in the special case of vertical incidence. The coupling is strongest for nearly parallel (to the magnetic field lines) propagation, and the coupling ratio may be approximated by $Q = (\theta_0/\theta)^3$, where θ is the angle between the wave vector and the magnetic field lines, while $\theta_0^3 = \lambda/L$, with λ the wavelength and L the scalelength of the inhomogeneity. This result may be of significance in connection with the heating of the solar corona by the dissipation of waves generated initially as acoustic waves in the photosphere, and perhaps with the propagation of hydromagnetic waves in the interplanetary medium.

1. Introduction

In Parts I and II (Melrose 1974*a*, 1974*b*) the method of Clemmow and Heading (1954) was used to treat coupling between magnetoionic waves in a stratified medium, and the results were applied to the interpretation of the polarization of solar radio bursts (Melrose 1973, 1975*a*). In the present paper and in a forthcoming Part IV of this series (Melrose and Simpson 1977) the same method is used to treat coupling between hydromagnetic waves in a stratified medium, with the waves regarded as cold plasma waves in the present paper and as MHD (magnetohydrodynamic) waves in the following part. (Magnetohydrodynamics here implies an approach based on fluid equations, as opposed to the cold plasma approach which is based on the use of a dielectric tensor; cf. Melrose 1975*b*). The results are applied to a particular proposed mechanism for the heating of the solar corona, and may also be of interest in connection with hydromagnetic waves in the solar wind.

The generalization of the theory of Parts I and II (for the magnetoionic modes) to any modes in a cold plasma is straightforward and is carried out explicitly in the Appendix. The two cold plasma modes for $\omega \ll \Omega_i$, where Ω_i is the ion gyrofrequency, are identified as the Alfvén mode and the magnetoacoustic mode. Consequently, the theory developed in Parts I and II may be applied with only minor modification to treat coupling between Alfvén and magnetoacoustic waves. This coupling is treated in detail in the present paper, while it will be shown in Part IV (Melrose and Simpson 1977) that the MHD theory leads to the same results in detail in the limit $c_s^2/v_A^2 \to 0$, where c_s is the sound speed and v_A is the Alfvén speed.

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The calculations are relevant to one model for the heating of the solar corona. It should be emphasized at the outset that the heating of the corona is not understood. There is wide agreement that the heating is due to mechanical energy generated in some form of wave motion in the lower atmosphere, e.g. in the photospheric regions, and that these waves propagate to the chromosphere–corona transition region where they dissipate into heat. However, the generation mechanism, the frequency and mode of the waves generated, the details of the propagation and the nature of the dissipation mechanism remain poorly understood and the subjects of controversy (see e.g. the reviews by Lighthill 1967; Schatzman and Souffrin 1967; Kuperus 1969; Stein and Leibacher 1974). One detailed model, which was the first proposed (Biermann 1948; Schwarzschild 1948), involves generation of acoustic waves in the photosphere, where $v_A \ll c_s$ implies that the waves are in the fast mode, and their propagation through a transition region with $v_A \approx c_s$ before dissipation in the corona with $v_A \gg c_s$ (see e.g. the detailed discussion by Osterbrock 1961). One outstanding problem with this model is that fast mode waves propagating into a direction where v_A is increasing (for $v_A > c_s$) tend to refract away from that direction. As a result one expects only a small fraction of the initial energy flux to reach the corona. One suggestion for overcoming this has been emphasized by E. Schatzman and was explored quantitatively by Frisch (1964): if the fast mode is coupled to the Alfvén mode then part of the energy flux is converted into the Alfvén mode which is unaffected by refraction.

A semantic difficulty arises in discussing refraction of Alfvén waves. The semantic point is that it is not clear whether 'refraction' refers to a change in the direction of the ray or of the wave normal. For Alfvén waves, the wave normal direction changes, e.g. in accord with Snell's law, but the ray direction is independent of the wave normal direction. The ray direction, i.e. the direction of the energy flux, is always along the magnetic field lines.

Frisch (1964) treated coupling between the MHD waves in a stratified medium with the foregoing application in mind. The method he used is closely related to that used here, but he restricted his discussion to the case of vertical incidence, i.e. for $\boldsymbol{\kappa}$ and $\boldsymbol{n}$ parallel (where the notation is defined in the next paragraph), and this virtually precludes consideration of what turns out to be the most effective coupling. In particular, for oblique incidence and nearly parallel propagation, coupling between magnetoacoustic and Alfvén waves is much stronger than one would infer on the basis of Frisch's results.

The notation used here for the various directions and angles is:

> $\boldsymbol{n}$ is the 'vertical' (defined below), $\boldsymbol{\kappa}$ is the 'wave-normal' direction or the direction of 'wave propagation', $\boldsymbol{b}$ is the direction of the ambient magnetic field, θ is the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{b}$, ρ is the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{n}$, while ψ and ϕ are the polar and the azimuthal angles of $\boldsymbol{b}$ relative to $\boldsymbol{n}$.

> 'Vertical incidence' implies $\boldsymbol{\kappa}$ parallel (or antiparallel) to $\boldsymbol{n}$, that is, $\rho = 0$ and $\theta = \psi$. 'Parallel propagation' implies $\boldsymbol{\kappa}$ parallel (or antiparallel) to $\boldsymbol{b}$, that is, $\theta = 0$, $\psi = \rho$ and $\phi = 0$. Further, when solving for the properties of any specific mode in the stratified medium ψ, ϕ, $\boldsymbol{b}$ and $\boldsymbol{n}$ are fixed by the choice of independent variabiles, and then θ, ρ, $\boldsymbol{\kappa}$ and numerous other variables are each described by different functions (of the independent variables) for different modes. To indicate this, where it is relevant, a subscript i is used to label the relevant mode (for example, θ_i, ρ_i, $\boldsymbol{\kappa}_i$, ...).

The use here of 'vertical', meaning normal to the strata, arises from ionospheric applications where the only quantity which varies significantly is the electron density, whose gradient is in the vertical direction. For hydromagnetic waves, gradients associated with the ambient magnetic induction $\boldsymbol{B}$ tend to have a dominant effect in causing mode coupling, and the definition of the 'vertical' is then not an obvious one. Let $\boldsymbol{B}$ be described by its magnitude B and by the angles ψ and ϕ. By hypothesis, the only variation is along $\boldsymbol{n}$, and $\operatorname{div}\boldsymbol{B} = 0$ implies $\boldsymbol{B}.\boldsymbol{n} = \text{const.}$, that is, $B\cos\psi = \text{const.}$ Consequently, B and ψ cannot vary independently of each other. Let the $\boldsymbol{B}$ variations be separated into two classes: ϕ variations and (B,ψ) variations. The ϕ variations correspond to 'twists', e.g. to a helical magnetic field, and the vertical may be identified as the average (over many twists) direction of $\boldsymbol{B}$. The (B,ψ) variations include the two opposite limiting cases of pure B variations and pure ψ variations. In pure B variations the field lines are parallel with a gradient in B across the field lines, that is, $\boldsymbol{n}$ is orthogonal to $\boldsymbol{b}$ and one has $\psi = \frac{1}{2}\pi$. In pure ψ variations the field lines are bent and the flux tubes have constant cross section, and then $\boldsymbol{n}$ is parallel to $\boldsymbol{b}$ and one has $\psi = 0$. In general, for (B,ψ) variations the 'vertical' direction $\boldsymbol{n}$ is at some intermediate angle, that is, $0 < \psi < \frac{1}{2}\pi$. (To be specific ψ is equal to $\arctan B'/B\psi'$.) Finally, it is possible for coupling to occur due to a gradient in v_A (where $v_A'/v_A = B'/B - n'/2n$, with n the number density of ions), and if this gradient is due entirely to the gradient in the plasma density ($B' = 0$) then the vertical is in the direction of the gradient in the plasma density, as for the ionospheric application.

2. Mode Coupling for Hydromagnetic Waves

To use the theory of Parts I and II we start from the wave properties of the particular wave modes of interest. From the Appendix, or from Stix (1962), the relevant wave properties in the case where ω and θ are the independent variables are: Alfvén mode,

$$\mu_A^2 = c^2/v_A^2\cos^2\theta, \qquad T_A = \infty, \qquad K_A/T_A = \tan\theta. \tag{1}$$

Magnetoacoustic mode,

$$\mu_M^2 = c^2/v_A^2, \qquad T_M = 0, \qquad K_M = 0. \tag{2}$$

(The notation is as in Parts I and II.) In using the method of Clemmow and Heading (1954), one requires the wave properties not as functions of ω and θ, but rather in terms of ω, r $(= c|\boldsymbol{k}\times\boldsymbol{n}|/\omega)$, ψ and ϕ as the independent variables. In principle the wave properties can be found by firstly solving a quartic equation (for $q = c\boldsymbol{k}.\boldsymbol{n}/\omega$). The quartic is a generalization of the Booker quartic (e.g. Budden 1961, Section 8.17) which applies only to magnetoionic waves. Now it was pointed out in Part II that implicit solutions to the quartic could be written down directly from the known solutions for μ^2 as a function of ω and θ. In the present case these implicit solutions are sufficiently simple to be used to find explicit solutions.

The implicit solutions follow from equations (1) and (2) by writing

$$q_{A\pm} = \pm(\mu_A^2 - r^2)^{\frac{1}{2}}, \qquad q_{M\pm} = \pm(\mu_M^2 - r^2)^{\frac{1}{2}}, \tag{3a, b}$$

with

$$\cos\theta_i = \frac{q_i\cos\psi + r\sin\psi\cos\phi}{(q_i^2 + r^2)^{\frac{1}{2}}}. \tag{4}$$

Here the four modes are labelled A+, M+, A− and M− (corresponding to $i = 1, ..., 4$ below), while in the equations (3a, b) μ_A^2 and μ_M^2 are to be regarded as functions of (ω and) $\theta_{A\pm}$ and $\theta_{M\pm}$ respectively, with $\theta_{A\pm}$ and $\theta_{M\pm}$ given by (4) in terms of the new independent variables. For both the Alfvén and magnetoacoustic modes, the implicit solution may be reduced to quadratic equations for the q_i. In other words, the quartic factorizes into two quadratic equations. The relevant solutions are

$$q_{A\pm} = (1/\cos\psi)(\pm c/v_A - r\sin\psi\cos\phi), \qquad q_{M\pm} = \pm(c^2/v_A^2 - r^2)^{\frac{1}{2}}, \tag{5a b,}$$

with

$$\cos\theta_{A\pm} = \pm(c/v_A)(q_{A\pm}^2 + r^2)^{-\frac{1}{2}}, \qquad \cos\theta_{M\pm} = (c/v_A)(q_{M\pm}\cos\psi + r\sin\psi\cos\phi). \tag{6}$$

The four modes are upgoing Alfvén (A+ or 1), upgoing magnetoacoustic (M+ or 2), downgoing Alfven (A− or 3) and downgoing magnetoacoustic (M− or 4). Here 'upgoing' and 'downgoing' do not necessarily refer to the sign of q, but rather to the angle between the group velocity and $\boldsymbol{n}$ (acute for upgoing and obtuse for downgoing). For the Alfvén modes the group velocity is along $\boldsymbol{b}$ or $-\boldsymbol{b}$ and for the magnetoacoustic modes it is along $\boldsymbol{\kappa}$.

A 'reflection point' is a point at which the upgoing and downgoing solutions for q are equal ($q_{A+} = q_{A-}$ or $q_{M+} = q_{M-}$), and then the vertical component of the group velocity vanishes. For the Alfvén modes the reflection points must satisfy $\cos\psi = 0$, and this is not possible in general for $\boldsymbol{B}$ variations. In other words, there are no reflection points for the Alfvén mode in general. (A gradient in v_A with $\boldsymbol{B}$ = const. would allow reflection points when the magnetic field is in the horizontal plane.) The reflection points for the magnetoacoustic modes are for horizontal propagation ($\boldsymbol{\kappa}.\boldsymbol{n} = 0$). Reflection is of no interest in the following discussion because these reflection points do not coincide with 'coupling points' in cases of interest. A coupling point is a point at which the upgoing (or downgoing) solutions for q are equal for the two different modes, e.g. for $q_{A+} = q_{M+}$. The coupling points in the present case occur for parallel propagation ($\theta = 0$ or π).

Coupling Ratio

Only coupling between modes 1 and 2 need be considered (the 'upgoing' and 'downgoing' labels are arbitrary). Following the approach used in Parts I and II, the coupling may be described by a coupling ratio

$$Q = \frac{|\Gamma_{12}\Gamma_{21}|^{\frac{1}{2}}}{(\omega/2c)|q_1 - q_2|}, \tag{7}$$

with the coupling coefficients Γ_{ij} given by

$$\Gamma_{12} = -(\mathbf{R}^{-1})_{1i}R'_{i2}, \qquad \Gamma_{21} = -(\mathbf{R}^{-1})_{2i}R'_{i1}, \tag{8}$$

where summation over i from 1 to 4 is implied. The matrix $\mathbf{R}$ may be written

$$\mathbf{R} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ -1 & -1 & -1 & -1 \\ -q_1 & -q_2 & -q_3 & -q_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}, \tag{9}$$

with

$$a_i = (q_i R_i + rP_i)/\mu_i, \qquad b_i = \mu_i R_i, \tag{10a, b}$$

$$R_i = (\alpha_i T_i - \mathrm{i}\beta)/(\beta T_i + \mathrm{i}\alpha_i), \qquad P_i = -K_i \sin\theta_i/(\beta T_i + \mathrm{i}\alpha_i), \tag{10c, d}$$

where $\mathrm{i} = \sqrt{-1}$, and with

$$\mu_i = (q_i^2 + r^2)^{\frac{1}{2}}, \qquad \alpha_i = (q_i \sin\psi \cos\phi - r\cos\psi)/\mu_i, \qquad \beta = \sin\psi \sin\phi. \tag{11a, b, c}$$

After some algebraic manipulation, the following explicit expressions for the a_i and b_i may be derived for the hydromagnetic waves:

$$a_{\mathrm{A}\pm} = \frac{\mp\mathscr{R} + \sin\psi\cos\phi}{\sin\psi\sin\phi}, \qquad a_{\mathrm{M}\pm} = \frac{(1-\mathscr{R}^2)^{\frac{1}{2}}\sin\psi\sin\phi}{\pm\mathscr{R}\cos\psi - (1-\mathscr{R}^2)^{\frac{1}{2}}\sin\psi\cos\phi}; \tag{12a, b}$$

$$\mathrm{b}_{\mathrm{A}\pm} = \frac{c}{v_{\mathrm{A}}} \frac{\pm\sin\psi\cos\phi - \mathscr{R}(\cos^2\psi + \sin^2\psi\cos^2\phi)}{\cos\psi\sin\psi\sin\phi}, \tag{13a}$$

$$b_{\mathrm{M}\pm} = \frac{c}{v_{\mathrm{A}}} \frac{\sin\psi\sin\phi}{\mathscr{R}\cos\psi \mp (1-\mathscr{R}^2)^{\frac{1}{2}}\sin\psi\cos\theta}; \tag{13b}$$

with

$$\mathscr{R} = rv_{\mathrm{A}}/c. \tag{14}$$

Explicit formulae for the coupling coefficients follow from

$$\Gamma_{12} = -\left(a_2' \frac{\partial}{\partial a_1} + q_2' \frac{\partial}{\partial q_1} + b_2' \frac{\partial}{\partial b_1}\right) \det\mathbf{R}, \tag{15a}$$

$$\Gamma_{21} = -\left(a_1' \frac{\partial}{\partial a_2} + q_1' \frac{\partial}{\partial q_2} + b_1' \frac{\partial}{\partial b_2}\right) \det\mathbf{R}, \tag{15b}$$

with

$$\begin{aligned}\det\mathbf{R} = & -(a_1 - a_2)(q_3 b_4 - q_4 b_3) + (a_1 - a_3)(q_2 b_4 - q_4 b_2) \\ & -(a_1 - a_4)(q_2 b_3 - q_3 b_2) - (a_2 - a_3)(q_1 b_4 - q_4 b_1) \\ & +(a_2 - a_4)(q_1 b_3 - q_3 b_1) - (a_3 - a_4)(q_1 b_2 - q_2 b_1).\end{aligned} \tag{16}$$

3. Vertical Incidence

The special case of vertical incidence is particularly simple, and it is relevant to consider this case first in order to compare the results of the present method with Frisch's (1964) results. In particular, Frisch found that, for vertical incidence, coupling between Alfvén and fast mode waves results only from twists in the magnetic field, i.e. from $\phi' \neq 0$, and that the coupling ratio is given (roughly) by $Q \approx (v_{\mathrm{A}}/\omega)\,|\phi'|$. The fact that Q vanishes for $\phi' = 0$ is also consistent with the work of Poeverlein (1964), who assumed ϕ = const. and vertical incidence, and found that the Alfvén mode was decoupled from the other two MHD modes.

Vertical incidence corresponds to $r = 0$, and then the equations (3) become

$$q_{\mathrm{A}+} = -q_{\mathrm{A}-} = c/v_{\mathrm{A}}\,|\cos\psi|, \qquad q_{\mathrm{M}+} = -q_{\mathrm{M}-} = c/v_{\mathrm{A}}; \tag{17a, b}$$

the equations (12) become

$$a_1 = a_3 = \cot\phi, \qquad a_2 = a_4 = -\tan\phi; \tag{18a, b}$$

and the equations (13) become

$$b_1 = -b_3 = (c/v_A|\cos\psi|)\cot\phi, \qquad b_2 = -b_4 = -(c/v_A)\tan\phi. \quad (19a, b)$$

The coupling coefficients (15a) and (15b) reduce to

$$\Gamma_{12} = -\frac{a_2'}{a_1-a_2}\frac{q_1+q_2}{2q_1}, \qquad \Gamma_{21} = -\frac{a_1'}{a_2-a_1}\frac{q_1+q_2}{2q_2}, \quad (20a, b)$$

and the coupling ratio (7) becomes

$$Q = \frac{v_A}{\omega}\frac{1+|\cos\theta|}{1-|\cos\theta|}|\cos\theta|^{\frac{1}{2}}|\phi'|. \quad (21)$$

(The angles θ and ψ coincide for vertical incidence.)

Frisch's (1964) result differs from equation (21) only in the dependence on θ, and this difference is important only for $|\cos\theta| \approx 1$. However, $|\cos\theta| \approx 1$ corresponds to parallel propagation, which is the coupling point for the Alfvén and magneto-acoustic modes. For nearly parallel propagation, equation (21) implies $Q \propto \theta^{-2}$, and the fact that Q diverges for $\theta \to 0$ may be attributed to the fact that $|q_1-q_2|$ in the denominator in equation (7) becomes proportional to θ^2 for small θ. Thus the dependence on θ in equation (21) is associated with an important qualitative effect which was overlooked by Frisch.

It should be noted that the coupling described by equation (21) (for $|\cos\theta|$ not ~ 1) is rather ineffective. Specifically, the coupling ratio is of the order of the ratio of the wavelength λ $(= 2\pi v_A/\omega)$ of the hydromagnetic waves to the scalelength L_ϕ $(= |\phi'|^{-1})$ of the twists in the magnetic field. Consequently, moderately strong coupling ($Q \approx 1$) occurs only at the very limit of applicability of geometric optics. One could speculate that, in the limit $\lambda \approx L_\phi$, one could replace the slowly varying stratified medium by a set of discrete strata and interpret the mode coupling in terms of the transmission and reflection characteristics at each boundary (e.g. Simon (1958) and Stein (1971) found that, in general, waves in one MHD mode incident on a boundary lead to transmitted components and reflected components in all three MHD modes). Strictly, mode-coupling theory is valid only for $\lambda \ll |\phi'|^{-1}$ or $|\psi'|^{-1}, \ldots$.

Nearly Vertical Incidence

The result (21) has been derived for strictly vertical incidence, and it is of interest to determine the range of parameters over which Q may be approximated by (21). Now, vertical incidence corresponds to $r = 0$ (or $\mathcal{R} = 0$) in equations (3), (12) and (13), and it is evident that the corrections to the q_i, a_i and b_i for finite $\mathcal{R}$ remain small for

$$\mathcal{R} \ll \sin\psi\cos\phi. \quad (22)$$

The range of parameters (22) may be said to define 'nearly vertical incidence'. It then follows that the coupling ratio may be approximated by equation (21) for nearly vertical incidence.

The range (22) excludes the important case of nearly parallel propagation ($\mathcal{R} \approx \sin\psi$ and $\phi = 0$) which is discussed in Section 5. The range (22) also excludes both propagation nearly normal to the plane containing $\boldsymbol{n}$ and $\boldsymbol{b}$ (that is, $\cos\phi \approx 0$) and the case of a nearly vertical magnetic field ($\sin\psi \approx 0$). It is straightforward to treat these two as special cases, and this is done in Section 4. Finally, the range (22) also excludes nearly horizontal propagation, but this case is of no conceivable interest.

4. Special Cases ($\mathscr{R} \gg \sin\psi\cos\phi$)

The case $\mathscr{R} \ll \sin\psi\cos\phi$ corresponds to nearly vertical incidence, and it is of interest to evaluate the coupling ratio when the opposite inequality, namely

$$\mathscr{R} \gg \sin\psi\cos\phi\,, \tag{23}$$

obtains.

Case $\cos\phi \approx 0$

One way in which the condition (23) can be satisfied is for $\cos\phi$ to be small. All quantities including the coupling coefficients and the coupling ratio approach finite values as $\cos\phi$ approaches zero and, consequently, it is reasonable to approximate them by their values for $\cos\phi = 0$ provided the condition (23) applies.

For $\cos\phi = 0$ one finds

$$q_1 = -q_3 = \frac{c}{v_A}\frac{1}{\cos\psi}, \qquad q_2 = -q_4 = \frac{c}{v_A}(1-\mathscr{R}^2)^{\frac{1}{2}}\,; \tag{24a, b}$$

$$a_1 = -a_3 = \frac{-\mathscr{R}}{\sin\psi}, \qquad a_2 = -a_4 = \frac{(1-\mathscr{R}^2)^{\frac{1}{2}}\sin\psi}{\mathscr{R}\cos\psi}\,; \tag{25a, b}$$

$$b_1 = b_3 = -\frac{c}{v_A}\frac{\mathscr{R}\cos\psi}{\sin\psi}, \qquad b_2 = b_4 = \frac{c}{v_A}\frac{\sin\psi}{\mathscr{R}\cos\psi}. \tag{26a, b}$$

It is then straightforward to find

$$\Gamma_{12} = \frac{\sin\psi}{2(\sin^2\psi + \mathscr{R}^2\cos^2\psi)}\big[-2(v_A'/v_A)\sin\psi + \{\sec\psi + (1-\mathscr{R}^2)^{\frac{1}{2}}\}\psi'\big], \tag{27a}$$

$$\Gamma_{21} = \frac{\mathscr{R}^2\cot\psi}{2(1-\mathscr{R}^2)^{\frac{1}{2}}(\sin^2\psi + \mathscr{R}^2\cos^2\psi)}\big[-2(v_A'/v_A)\sin\psi + \{\sec\psi + (1-\mathscr{R}^2)^{\frac{1}{2}}\}\psi'\big], \tag{27b}$$

and hence

$$Q = \frac{v_A}{\omega}\frac{\mathscr{R}}{\sin^2\psi + \mathscr{R}^2\cos^2\psi}\left(\frac{|\cos\psi|}{(1-\mathscr{R}^2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\frac{|2(v_A'/v_A)\sin\psi - \{\sec\psi + (1-\mathscr{R}^2)^{\frac{1}{2}}\}\psi'|}{|\sec\psi - (1-\mathscr{R}^2)^{\frac{1}{2}}|}. \tag{28}$$

Case $\psi \approx 0$

In the limit $\psi = 0$, some quantities become infinite and others vanish. Consequently, it is necessary to allow ψ to be finite. For arbitrarily small ψ one finds

$$q_1 = -q_3 = c/v_A\,, \qquad q_2 = -q_4 = (c/v_A)(1-\mathscr{R}^2)^{\frac{1}{2}}; \tag{29a, b}$$

$$a_1 = -a_3 = \frac{\mathscr{R}}{\psi\sin\phi}, \qquad a_2 = -a_4 = -\frac{(1-\mathscr{R}^2)^{\frac{1}{2}}}{\mathscr{R}}\psi\sin\phi; \tag{30a, b}$$

$$b_1 = b_3 = -\frac{c}{v_A}\frac{\mathscr{R}}{\psi \sin\phi}, \qquad b_2 = b_4 = \frac{c}{v_A}\frac{\psi \sin\phi}{\mathscr{R}}. \tag{31a, b}$$

One then finds

$$\Gamma_{12} = \psi \sin\phi[2(v_A'/v_A)\psi \sin\phi - \{1+(1-\mathscr{R}^2)^{\frac{1}{2}}\}(\psi \sin\phi)']/2\mathscr{R}^2, \tag{32a}$$

$$\Gamma_{21} = -[2(v_A'/v_A)\psi \sin\phi - \{1+(1-\mathscr{R}^2)^{\frac{1}{2}}\}(\psi \sin\phi)']/2(1-\mathscr{R}^2)^{\frac{1}{2}}\psi \sin\phi. \tag{32b}$$

Hence one finds

$$Q = \frac{v_A}{\omega}\frac{1}{\mathscr{R}(1-\mathscr{R}^2)^{\frac{1}{4}}}\frac{|2(v_A'/v_A)\psi \sin\phi - \{1+(1-\mathscr{R}^2)^{\frac{1}{2}}\}(\psi \sin\phi)'|}{1-(1-\mathscr{R}^2)^{\frac{1}{2}}}, \tag{33}$$

which is clearly compatible with equation (28).

Comparing equations (21), (28) and (33), one concludes that in each case Q is of order λ/L, where λ is the wavelength of the hydromagnetic wave and L is a characteristic distance associated with the inhomogeneity. Qualitatively, different types of inhomogeneity (twists or bends in the magnetic field lines or changes in v_A) are important in the different cases. However, in all cases the coupling is weak except for $\lambda \lesssim L$, when geometric optics starts to break down. Thus, no important new features are introduced when inequality (22) is replaced by its opposite (23).

5. Nearly Parallel Propagation

The coupling points for Alfvén waves with magnetoacoustic waves correspond to parallel propagation. In the neighbourhood of a coupling point, coupling is necessarily strong ($Q \gg 1$), and consequently there must be a range for nearly parallel propagation for which coupling is strong. This range can be estimated by expanding in powers of θ (and other small quantities) when deriving an expression for Q for nearly parallel propagation.

For strictly parallel propagation one must have $\mathscr{R} = \sin\psi$ and $\phi = 0$. For nearly parallel propagation one may set $\mathscr{R} = \sin\rho$ and $\phi \ll 1$, and expand in $\psi-\rho$ and ϕ. (It is not necessary to distinguish between ρ_1 and ρ_2, and θ_1 and θ_2, because the differences $\rho_1-\rho_2$ and $\theta_1-\theta_2$ are of higher order in $\psi-\rho$ and ϕ.) One finds, to lowest order in small quantities,

$$q_1 = (c/v_A)\cos\psi = q_2, \qquad q_3 = -(c/v_A)(2\sec\psi - \cos\psi), \qquad q_4 = -(c/v_A)\cos\psi; \tag{34a–34c}$$

$$a_1 = \{(\psi-\rho)/\phi\}\cot\psi, \quad a_2 = -\{\phi/(\psi-\rho)\}\sin\psi\cos\psi, \quad a_3 = 2/\phi, \quad a_4 = -\tfrac{1}{2}\phi; \tag{35a–35d}$$

$$b_1 = \frac{c}{v_A}\frac{\psi-\rho}{\phi\sin\psi}, \quad b_2 = -\frac{c}{v_A}\frac{\phi\sin\psi}{\psi-\rho}, \quad b_3 = -2\frac{c}{v_A}\frac{1}{\phi\cos\psi}, \quad b_4 = \frac{c}{v_A}\frac{\phi}{2\cos\psi}. \tag{36a–36d}$$

One also requires

$$\theta^2 = (\psi-\rho)^2 + (\phi\sin\psi)^2 \qquad \text{and} \qquad |q_1 - q_2| = c\theta^2/2v_A|\cos\psi|. \tag{37a, b}$$

With the approximations (34)–(36), the coupling coefficients reduce to

$$\Gamma_{12} = \frac{-\{(c/v_{\mathrm{A}})a_2\}'}{(c/v_{\mathrm{A}})(a_1-a_2)}, \qquad \Gamma_{21} = \frac{-\{(c/v_{\mathrm{A}})a_1\}'}{(c/v_{\mathrm{A}})(a_2-a_1)}. \tag{38a, b}$$

To lowest order in the small quantities one has

$$\{(c/v_{\mathrm{A}})a_1\}' \approx (c/v_{\mathrm{A}})\sin\psi\cos\psi\{(\psi'-\rho')\phi-(\psi-\rho)\phi'\}(\psi-\rho)^{-2} \tag{39a}$$

and

$$\{(c/v_{\mathrm{A}})a_2\}' \approx (c/v_{\mathrm{A}})\cot\psi\{(\psi'-\rho')\phi-(\psi-\rho)\phi'\}\phi^{-2}. \tag{39b}$$

Hence the coupling ratio becomes

$$Q = 4\frac{v_{\mathrm{A}}}{\omega}\frac{\sin\psi\,|\cos\psi|}{\theta^4}\,|(\psi'-\rho')\phi-(\psi-\rho)\phi'|. \tag{40}$$

An unexpected feature of equation (40) is that it implies that Q diverges as θ^{-3} for small θ (where ϕ and $\psi-\rho$ are taken to be of order θ). By contrast equation (21) implies $Q \propto \theta^{-2}$ for small θ. In other words, on the basis of the result for vertical incidence one would predict $Q \propto \theta^{-2}$, whereas the actual result for oblique incidence corresponds to $Q \propto \theta^{-3}$. There is no incompatibility, because vertical incidence and parallel propagation require $\psi = \theta$ and $\phi = 0$, and equation (40) also predicts $Q \propto \theta^{-2}$ in this special case. Thus, the assumption of vertical incidence is too restrictive to allow adequate treatment of nearly parallel propagation.

Elliptical Polarization

There is a minor inconsistency in the foregoing discussion: one cannot assume that the wave properties are given by equations (1) and (2) for very small θ. Amongst the approximations made in deriving those equations is $\omega \ll \Omega_i$ (see the Appendix) and, for

$$\theta \lesssim (2\omega/\Omega_i)^{\frac{1}{2}}, \tag{41}$$

the corrections due to the finiteness of ω/Ω_i are more important than the corrections due to the finiteness of θ. The waves are significantly elliptically polarized in the range (41), and for $\theta \ll (2\omega/\Omega_i)^{\frac{1}{2}}$ the polarization approaches circular, that is, $|T_{\mathrm{A}}| \approx 1 \approx |T_{\mathrm{M}}|$.

It may be shown that, when the inequality (41) is a strong one, the coupling ratio may be approximated by, in place of equation (40),

$$Q \approx \frac{v_{\mathrm{A}}}{\omega}|\cos\psi|\left(\frac{\Omega_i}{\omega}\right)^2\theta\theta'.$$

Thus Q actually tends to zero as θ tends to zero. From a semiquantitative viewpoint, the significant result is that Q cannot exceed the value given by setting $\theta \approx (\omega/\Omega_i)^{\frac{1}{2}}$ in equation (40).

The circular polarization in the limit $\theta \to 0$, which is absent in the MHD theory, may be attributed to the effect of the spiralling motion of the ions. In a collision-dominated plasma, to which the MHD theory may be applied, ions experience many collisions per gyroperiod, and spiralling motion cannot be said to occur.

6. Discussion

The main result of this paper is that coupling between Alfvén waves and magnetoacoustic waves can be much stronger than Frisch's (1964) results imply. Specifically, Frisch found the coupling ratio to be roughly $Q \approx \lambda/L$, where λ is the wavelength of the wave and L is the characteristic distance over which the magnetic field is twisted. The present investigation confirms Frisch's result and shows that the same semiquantitative expression applies when Frisch's assumption of vertical incidence is relaxed (with L now interpreted as the characteristic distance over which any relevant plasma parameter changes). However, the important exception is that for nearly parallel propagation the coupling ratio can be greatly enhanced:

$$Q \approx (\theta_0/\theta)^3, \qquad \theta_0^3 \approx \lambda/L. \tag{42}$$

Consequently, for waves propagating at sufficiently small θ the coupling is necessarily strong. The result (40) may be of significance in connection with the heating of the solar corona, and with the propagation of hydromagnetic waves in the interplanetary medium.

The implications of this result on the heating of the solar corona are discussed only briefly here. One implication concerns the frequency (or wavelength) of the waves involved. In recent years there has been a tendency to favour heating by longer period waves, with periods up to five minutes. However, to some extent this is based on no more than observational evidence that such longer period waves are present (see e.g. the review by Stein and Leibacher 1974). There is no evidence against shorter period (for example, $\lesssim 1$ s) waves being important, and there is radio evidence for their presence (see e.g. the review by Wild and Smerd 1972). Frisch's (1964) coupling ratio for waves with a period of around a second (implying λ of order several hundred kilometres) would require a very inhomogeneous corona for the coupling to be effective. In fact, the corona may be inhomogeneous on a relative fine scale (Melrose 1975*a*), and the more effective coupling found in the present investigation may well allow a substantial transfer of an energy flux (in relatively short period waves) from the fast mode to the Alfvén mode.

The original suggestion that the corona might be heated by acoustically generated waves (Biermann 1948; Schwarzschild 1948; Osterbrock 1961) should be kept in mind as a possible alternative to more recent suggestions for the heating of the corona (Zhugzhda 1972; Piddington 1973). The point made by E. Schatzman (Frisch 1964) and Pikel'ner (1961, p. 209) that refraction would prevent the waves from reaching the corona can be overcome by mode coupling to the Alfvén mode. The important contribution of the present paper to the discussion of this point is that the mode coupling can be much more effective than earlier calculations might indicate. In particular, the restriction of the earlier calculations (Frisch 1964; Poeverlein 1964) to 'vertical incidence' excludes the most effective case: that of propagation nearly along the magnetic field lines. Such coupling is particularly effective because it is 'in the neighbourhood of the coupling point'.

The propagation of hydromagnetic waves in the interplanetary medium has been reviewed recently by Hollweg (1975). Mode coupling is of significance in connection with the partial conversion of the flux, known to be predominantly in the Alfvén mode, into fast mode waves. Indeed Belcher and Davis (1971) appealed to such coupling to explain one feature of their observations, namely the fact that the power spectrum of the magnetic fluctuations did not coincide with that expected for pure Alfvén waves. Barnes and Hollweg (1974) proposed an alternative explanation of this observation in terms of finite amplitude disturbances propagating nearly along the field lines.

The strong mode coupling for nearly parallel propagation implies that mixed-mode disturbances can exist. In other words, any disturbance composed of components in the two modes can propagate without significant decomposition into its two components. Thus, for $\theta \ll \theta_0$ one could have a circularly polarized disturbance despite the fact that formally the two modes are linearly polarized. This allows a further possible explanation of the observations of Belcher and Davis (1971). As in the mechanism proposed by Barnes and Hollweg (1974), the waves are partially circularly polarized but, unlike their mechanism, it is not necessary to appeal to a detailed finite amplitude structure. This alternative mechanism could be effective for waves with, say, $\lambda \gtrsim 10^{-2} L$ (corresponding to $\theta_0 \lesssim 10°$). Even with $L \approx 0 \cdot 1$ A.U., this range corresponds to periods longer than about an hour, and most of the observed power is in such waves.

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Appendix

The generalization of the results of Parts I and II to apply to any waves in a cold plasma involves simply inserting the relevant wave properties. We consider a cold plasma consisting of various species α of particle, with plasma frequency $\psi_{p\alpha}$, gyrofrequency Ω_α and sign of charge ε_α. Following Stix (1962, Sections 1 and 2) we may write the dielectric tensor in the cold plasma approximation in terms of three quantities S, D and P defined by

$$S = \tfrac{1}{2}(R_+ + R_-), \qquad D = \tfrac{1}{2}(R_+ - R_-), \tag{A1a, b}$$

with

$$R_\pm = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left(\frac{\omega}{\omega \pm \varepsilon_\alpha \Omega_\alpha}\right) \tag{A1c}$$

and

$$P = 1 - \sum_\alpha \omega_{p\alpha}^2/\omega^2, \tag{A1d}$$

where the sums are over all species. The properties of the two modes are usually found by choosing ω and θ as dependent variables and solving for the refractive index μ. We can either solve

$$P\{\mu^4 - 2S\mu^2 + S^2 - D^2\} - \mu^2 \sin^2\theta (P-S)\mu^2 + S^2 - D^2 - PS = 0 \tag{A2}$$

for μ^2, or we can solve

$$T^2 - \frac{(PS - S^2 + D^2)\sin^2\theta}{PD\cos\theta} T - 1 = 0 \tag{A3}$$

for the axial ratios ($T = T_\pm$, say) for the two modes, and then substitute for T in

$$\mu^2 = \frac{P(S - T^{-1}D\cos\theta)}{P\cos^2\theta + S\sin^2\theta} = \frac{S^2 - D^2}{S - TD\cos\theta}. \tag{A4}$$

The other quantity K used to describe the polarization (cf. equation 10 of Part II) is given by substituting for T in

$$K = \frac{\sin\theta\{(P-S)T\cos\theta - D\}}{P\cos^2\theta + S\sin^2\theta} = \frac{\sin\theta}{P}\frac{(PS - S^2 + D^2)T\cos\theta - PD}{S - TD\cos\theta}. \tag{A5}$$

In principle, solutions for q (and T, K, ...) are required in terms of the independent variables r, ψ and ϕ. The quartic equation (which is the appropriate generalization of the Booker quartic) for q is given by making the substitution

$$\mu^2 = q^2 + r^2, \qquad \mu^2\sin^2\theta = q^2 + r^2 - (q\cos\psi + r\sin\psi\cos\phi)^2 \tag{A6}$$

in equation (A2).

For a charge neutral plasma consisting of electrons and various positively charged ionic species, the low frequency limit corresponds to

$$S \approx 1 + \frac{c^2}{v_A^2} \qquad \text{and} \qquad D \approx -\sum_i \frac{\omega_{pi}^2}{\Omega_i^2}\frac{\omega}{\Omega_i}, \tag{A7a, b}$$

where

$$\frac{c^2}{v_A^2} \approx \sum_i \frac{\omega_{pi}^2}{\Omega_i^2} \tag{A8}$$

has been used, and where the correction terms are of order ω^2/Ω_i^2 smaller than the terms retained. The approximation used in the text corresponds to setting $P = -\infty$, $S = c^2/v_A^2$ and $D = 0$, and the effect discussed at the end of Section 5 (cf. the equations 38) emerges where the finiteness of D ($\approx -(c^2/v_A^2)\omega/\Omega_i$ for a plasma with only one ionic species) is taken into account.

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