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Mode Coupling in the Solar Corona. V* Reduction of the Coupled Equations

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Abstract

A simplified version of the mode-coupling theory of Clemmow and Heading is developed by reducing the set of coupled equations to two for the magnetoionic theory and three for the MHD theory. The simplified theory reproduces known results for coupling in the neighbourhood of coupling points. It is used to treat coupling between the MHD waves, and it is found that coupling between the fast mode and the Alfvén mode for $v_A \gtrsim c_s$ is stronger than the coupling between any other pair of modes. The strongest coupling of all is between the Alfvén and slow (magneto-acoustic) modes for $v_A \ll c_s$.

Introduction

Mode-coupling theory is a semiquantitative method of patching up a weakness in geometric optics when applied to anisotropic media. The basic concept in geometric optics is that of a ray and, for an anisotropic medium, rays can be labelled with the wave mode involved. The weakness is that, when dealing with an inhomogeneous medium, waves in a single initial mode can be partially converted into waves in two or more modes. In mode-coupling theory one attempts to rectify this weakness by defining coupling ratios between any two modes. If all coupling ratios involving the ray in question are small everywhere along the ray path, then the ray always corresponds to the given initial mode. However, if the coupling ratio is not small anywhere along the ray path then a partial conversion into other modes occurs and the magnitude of the coupling ratio enables one to treat this semiquantitatively. (The fully quantitative theory would be a full wave theory in which the concept of a ray was not invoked.)

In the discussion of mode coupling in Parts I–IV of this series (respectively Melrose 1974*a*, 1974*b*, 1977; Melrose and Simpson 1977) the calculations have become increasingly cumbersome. Since a cumbersome method is particularly undesirable in a theory which is intended to be only semiquantitative, it would be useful to have a simpler method for calculating the coupling ratios. It is my purpose in this paper to propose such a method, to show that it reproduces the important features of the results obtained in Parts I–IV and to use it to treat a problem which was bypassed in Part IV as too involved to consider analytically.

There are two steps in the proposed simplification of the general theory, which is taken to be that formulated by Clemmow and Heading (1954). The first step has already been made in Part II:

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Coupling is to be considered only in the neighbourhood of coupling points where a perturbation approach (the 'coupling approximation') may be used to calculate the wave properties using the known properties for a homogeneous medium.

The second step is made here:

The set of coupled equations is to be reduced from the original set of $n = 4$ or 6 to a set of $m = 2$ or 3 which involve only the modes of direct interest.

The idea behind the second step may be illustrated as follows. When considering coupling between the magnetoionic modes, one is interested only in the two upgoing modes o↑ and x↑ (in the notation of Part II). However, to treat the coupling using the general theory, one must find the properties of the downgoing modes o↓ and x↓ and use a set of four coupled equations which includes the various couplings between all four modes. Only after having calculated coupling coefficients does one assume the amplitudes of the modes o↓ and x↓ to be zero and calculate the desired coupling ratio. On physical grounds it seems that the properties of the modes o↓ and x↓ cannot be of significance and one should be able to develop a theory in which only the modes of interest, namely o↑ and x↑ here, appear. Put another way, one would like to set the amplitudes of the other two modes to zero at the outset. The situation is worse for the magnetohydrodynamic (MHD) waves where one has six modes, e.g. A↑, F↑, S↑ and A↓, F↓, S↓ (A, Alfvén; F, fast; S, slow). In some situations one would wish to consider coupling between only two of these (between A↑ and F↑ for $v_{\rm A}^2 \gg c_{\rm s}^2$) or alternatively one might wish to consider coupling only between the three upgoing modes.

In the general theory of mode coupling, one introduces a column vector $\boldsymbol{e}$ with $n = 4$ wave variables in the magnetoionic theory and with $n = 6$ wave variables in the MHD theory. The n eigenvalues q_i and eigenfunctions $\boldsymbol{e}^{[i]}$ of the coupled equations are then to be found. (It is at this stage that the coupling approximation leads to great simplification.) The coupling coefficients are calculated as the off-diagonal elements of the $n \times n$ matrix $-\mathbf{R}^{-1}\mathbf{R}'$, where the n rows of $\mathbf{R}$ consist of the n eigenfunctions $\boldsymbol{e}^{[i]}$ with $i = 1, ..., n$. The objective in reducing the set of coupled equations is to replace the $n \times n$ matrix $\mathbf{R}$ by an $m \times m$ matrix with $m < n$. Suppose the modes are ordered such that $i = 1, ..., m$ are the modes of interest. One step in the reduction is simply to ignore columns $m+1$ to n. The other step is to ignore an equal number of rows. Some physical assumption is required to determine which rows to exclude and which to retain. There are physically obvious choices: these are the electric components E_x and E_y for the magnetoionic waves, and the components ξ_x, ξ_y and ξ_z of the fluid displacement in the MHD theory. If one were to consider coupling between A↑ and F↑ for $v_{\rm A} \gg c^2$, a plausible choice would be the components B_x and B_y of the magnetic field.

(A reduced set of coupled equations was used by Simpson (1976) to treat a particular case of coupling between magnetoionic waves. He also assumed vertical incidence, in which case the properties of the modes o↓ and x↓ are related simply to those of o↑ and x↑ respectively and this enables one to reduce the set of equations from four to two in a more rigorous way.)

The crucial step in reducing the set of coupled equations is this choice of wave variables. From a mathematical viewpoint, one should exclude all choices which would lead to the $m \times m$ matrix $\mathbf{R}$ being singular at the coupling point. It seems that, subject to the exclusion of such unacceptable choices, different sets of variables generally give roughly the same results, and this is all that is required from a semi-quantitative viewpoint. On the other hand, if otherwise acceptable choices led to substantially different results one would have to conclude that the initial physical assumption that only the coupling between the m selected modes is important was unjustified. For example, one might expect the reduced 2×2 matrix $\mathbf{R}$ for coupling between o↑ and x↑ to give an inadequate description of the coupling in the neighbourhood of the reflection point for the x-mode, where coupling between x↑ and x↓ becomes important. The only test of a particular choice of the reduced set of wave variables is whether or not it gives an adequate approximation to the results of the general theory.

Reduction of Coupled Equations

Consider a general theory which allows n independent modes ($n = 4$ for the magnetoionic theory and the cold plasma theory and $n = 6$ for the MHD theory) and suppose that simultaneous coupling between only m of these is of interest ($m = 2$ or 3 in practice). Let the equations for small amplitude disturbances be written in the generic form

$$\boldsymbol{e}' + \mathrm{i}(\omega/c)\mathbf{T}\,\boldsymbol{e} = 0\,, \tag{1}$$

which is equation (2) of Part I (hereinafter abbreviated (I.2) etc.), or equation (IV.1) with $\mathbf{T} \equiv \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{C} = 0$. The matrix $\mathbf{T}$ is assumed to have eigenvalues q_i with $i = 1,\dots,n$ and corresponding eigenfunctions (column vectors) $\boldsymbol{e}^{[i]}$. The $n\times n$ matrix $\mathbf{R}$ (defined in Parts I or IV) may be defined by

$$R_{ji} \equiv e_j^{[i]}\,, \tag{2}$$

where $e_j^{[i]}$ denotes the jth row of the column vector $\boldsymbol{e}^{[i]}$.

Simpson (1976, personal communication) has suggested that to treat coupling between modes 1 to m one may seek an approximate solution of equation (1) of the form

$$\boldsymbol{e} = \sum_{i=1}^{m} f_i\,\boldsymbol{e}^{[i]}\,, \tag{3}$$

where the $f_1,\dots,f_m$ are coefficients to be found. On inserting equation (3) in (1), one finds

$$\sum_{i=1}^{m} f_i'\,\boldsymbol{e}^{[i]} + \mathrm{i}(\omega/c)q_i f_i\,\boldsymbol{e}^{[i]} = -\sum_{i=1}^{m} f_i(\boldsymbol{e}^{[i]})'\,, \tag{4}$$

which is a set of n first-order differential equations for m variables. The redundancy is due to the extra information introduced by the assumed form (3) of the solution.

Suppose only the first m of equations (4) are retained, i.e. only the equations involving components $e_1^{[i]}$ to $e_m^{[i]}$ are retained and those involving $e_{m+1}^{[i]}$ to $e_n^{[i]}$ are simply ignored. Then, provided the $m\times m$ matrix R_{ij} with $i,j = 1,\dots,m$ can be inverted, i.e. provided its determinant is nonzero, one obtains from the result (4) the set of equations

$$f_i' + \mathrm{i}(\omega/c)q_i f_i = -\sum_{r,s=1}^{m} (R^{-1})_{ir} R'_{rs} f_s\,, \tag{5}$$

where $\mathbf{R}^{-1}$ denotes the $m\times m$ matrix which is the inverse of the leading $m\times m$ submatrix of $\mathbf{R}$. Equation (5) is of the same form as equation (I.5), and the calculation of the coupling coefficients proceeds as in Part I, but now involving matrices of order $m = 2$ or 3 rather than $n = 4$ or 6.

The choice of the first m equations is arbitrary: the ordering of the components of $\boldsymbol{e}$ is arbitrary and 'the first m' means any m which are then relabelled 1 to m for convenience. In fact different choices lead to different results and the simplified theory is incomplete in the absence of any prescription for choosing the m rows to be retained.

There is one mathematical requirement which affects the choice of rows to be retained. This is that the resulting $m \times m$ matrix $\mathbf{R}$ must have an inverse at the coupling point.

Magnetoionic Modes

For the magnetoionic waves the proposed choice of variables is E_x and E_y. In this section it is shown how, with this choice, the simplified theory reproduces existing results. Firstly, however, the coupling coefficient Q is calculated for any choice of the two variables (this is relevant to a subsequent discussion of the effect of other choices of variables).

For $m = 2$, the derivation of the coupling ratio Q is elementary. After having chosen the two wave variables, let their ratio be denoted by J. Then one has

$$\mathbf{R} = \begin{bmatrix} J_1 & J_2 \\ 1 & 1 \end{bmatrix}, \tag{6}$$

and hence

$$\mathbf{R}^{-1} = (J_1 - J_2)^{-1} \begin{bmatrix} 1 & -J_2 \\ -1 & J_1 \end{bmatrix}. \tag{7}$$

The coupling coefficients Γ_{12} and Γ_{21} are given by the off-diagonal components of $-\mathbf{R}^{-1}\mathbf{R}'$ (cf. equation (I.6)). Hence one finds

$$(\Gamma_{12}\Gamma_{21})^{\frac{1}{2}} = (J_1' J_2')^{\frac{1}{2}} / | J_1 - J_2 | . \tag{8}$$

Finally, using equations (II.32a, b), (II.36) and (II.37), one has

$$Q = (2c/\omega) |\cos\rho| (J_1' J_2')^{\frac{1}{2}} / | (\mu_1 - \mu_2)(J_1 - J_2) | . \tag{9}$$

For the magnetoionic waves the choice $J = E_x/E_y$ is an obvious one. This choice was made by Simpson (1976). For vertical incidence this choice leads to $J = R$ with R given by equation (I.13). The result (8) then coincides with that implied by equations (I.15a, b) in the neighbourhood of the coupling point, i.e. when corrections of order $(\mu_o - \mu_x)/(\mu_o + \mu_x)$ are neglected. For oblique incidence the actual value of J is given by equation (II.14). However, in Part II the term involving P (see equations (II.15)) was neglected and this corresponds to making the approximation $J = R$. The result (8) and the results (II.28a$_{1,2}$) then agree. Consequently, the simplified method reproduces the existing results with the choice $J = E_x/E_y$.

Suppose some other choice of J were made. It is evident from equation (II.16) that the choices E_x/B_y and E_y/B_x are unacceptable; they lead to singular matrices $\mathbf{R}$ at the coupling point. The other choices are $E_x/B_x = (E_x/E_y)/q$, $B_y/E_y = \mu R$ and $B_y/B_z = -\mu R/q$. Consequently, provided the term involving P is ignored and provided the gradient in μ is unimportant, these choices lead to results which are equivalent to the result obtained for E_x/E_y.

For the hydromagnetic waves in a cold plasma, the choice E_x/E_y is not an obvious one because the important amplitudes of the waves are the magnetic amplitudes. With the choice $J = -B_y/B_x$, in the neighbourhood of the coupling points equations (III.34–36) with (III.9) imply

$$J_1 = \frac{\psi-\rho}{\phi\sin\psi\cos\psi}, \qquad J_2 = -\frac{\phi\sin\psi}{(\psi-\rho)\cos\psi}, \tag{10}$$

that is,

$$J_1 = a_1\sec^2\psi, \qquad J_2 = a_2\sec^2\psi. \tag{11}$$

Consequently, providing the variation of $c/v_A\cos^2\psi$ is slow compared with that of $(\psi-\rho)/\phi$, as was assumed in equations (III.39a, b), the result (8) reproduces equations (III.38a,b) and the result (9) reproduces equation (III.40). It is also clear from equations (III.34–36) that choosing $J = E_x/E_y$ would lead to the same results to within the approximations made.

Finally, for coupling between the Alfvén (A) and fast (F) mode for $v_A \gtrsim c_s$ in the MHD theory, the choice $J = \xi_x/\xi_y$ for $\sin\theta \approx 0$ leads to $J_A = -q\beta/\mu\alpha$ and $J_F = q\alpha/\mu\beta$ (cf. equations (IV.21a, b)). The product $\Gamma_{12}\Gamma_{21}$ from equation (8) then reproduces that from equation (IV.29). Alternatively, if one chose $J = B_x/B_y$ or $J = E_x/E_y$ then using equations (IV.21a, b) and (IV.22a, b) one would find that the final result for the coupling ratio was unchanged.

It may be concluded that the reduced equations reproduce all the known results, at least to within unimportant terms, for coupling in the neighbourhood of coupling points.

MHD Modes

In this section simultaneous coupling between three upgoing MHD waves is treated using the reduced equations with $m = 3$ and with the choices of the three components of the fluid displacement as the three wave variables.

From equations (IV.14a, b), the 3×3 matrix $\mathbf{R}$ constructed from the normalized fluid displacements has the explicit representation

$$\tilde{\mathbf{R}} = \begin{bmatrix} 0 & \sin\Psi^+ & \sin\Psi^- \\ 1 & 0 & 0 \\ 0 & \cos\Psi^+ & \cos\Psi^- \end{bmatrix}, \tag{12}$$

where the coordinate axes are such that $\boldsymbol{b}$ is along the 3-axis and $\boldsymbol{\kappa}$ is in the 1–3 plane. Let $\mathbf{S}$ be a rotation matrix such that

$$\mathbf{R} = \mathbf{S}\tilde{\mathbf{R}}\mathbf{S}^{-1} \tag{13}$$

is the representation in the coordinate axes used in mode coupling theory (cf. equations (II.4), (II.11a, b, c) or (IV.16a, b, c)). The coupling matrix $\boldsymbol{\Gamma}$ is given by the off-diagonal components of $-\mathbf{R}^{-1}\mathbf{R}'$. One has

$$\mathbf{R}^{-1}\mathbf{R}' = \mathbf{S}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{R}}'\mathbf{S}^{-1} + \mathbf{S}(\mathbf{S}^{-1})' + \mathbf{S}\tilde{\mathbf{R}}^{-1}\mathbf{S}^{-1}\mathbf{S}'\tilde{\mathbf{R}}\mathbf{S}^{-1}. \tag{14}$$

Also, because $\mathbf{S}\mathbf{S}^{-1}$ is equal to the unit matrix, one has

$$\mathbf{S}(\mathbf{S}^{-1})' = -\mathbf{S}^{-1}\mathbf{S}'. \tag{15}$$

The first term on the right-hand side of equation (14) is easily evaluated using the representation (12). Note that the orthogonality of the modes requires

$$\Psi^{+} - \Psi^{-} = \tfrac{1}{2}\pi, \tag{16}$$

which identity may be established directly from equations (IV.14a, b). Using the result (16), one finds

$$\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{R}}' = \Psi' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{17}$$

where Ψ can be either Ψ^{+} or Ψ^{-}.

It is rather tedious to calculate the terms in equation (14) in general. However, if only coupling in the neighbourhood of the coupling points is of interest, one may expand in the small quantities

$$\theta, \qquad \alpha \approx \psi - \rho, \qquad \beta \approx \phi \sin\psi.$$

To lowest order one finds

$$\mathbf{S} = \theta^{-1} \begin{bmatrix} -\alpha\cos\psi & \beta\cos\psi & \theta\sin\psi \\ -\beta & -\alpha & 0 \\ \alpha\sin\psi & -\beta\sin\psi & \theta\cos\psi \end{bmatrix}. \tag{18}$$

Hence the first term in equation (14) reduces to

$$\mathbf{S}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{R}}'\mathbf{S}^{-1} = \theta^{-1}\Psi' \begin{bmatrix} 0 & -\alpha\sin\psi & -\beta \\ \alpha\sin\psi & 0 & \cos\psi \\ \beta & -\alpha\cos\psi & 0 \end{bmatrix}. \tag{19}$$

Also, explicit evaluation gives

$$\mathbf{S}^{-1}\mathbf{S}' = -\frac{2\theta'}{\theta}\mathbf{1} + \frac{\alpha\beta' - \alpha'\beta}{\theta^2}\mathbf{\Delta}^{(1)}, \tag{20}$$

where $\mathbf{1}$ is the unit matrix and

$$\mathbf{\Delta}^{(1)} \equiv \begin{bmatrix} 0 & \cos\psi & 0 \\ -\cos\psi & 0 & \sin\psi \\ 0 & -\sin\psi & 0 \end{bmatrix}. \tag{21}$$

The diagonal components in equation (20) cancel between the final two terms in equation (14), in view of the result (15) (cf. equation (23) below). The remaining contribution from the middle term on the right-hand side of equation (14) is just minus the final term of (20). The last term in equation (14) is given by postmultiplying the final term in (20) by

$$\tilde{\mathbf{R}}\mathbf{S}^{-1} = \theta^{-1}\begin{bmatrix} \beta\cos\psi\sin\Psi-\theta\sin\psi\cos\Psi & -\alpha\sin\Psi & -\beta\sin\psi\sin\Psi-\theta\cos\psi\cos\Psi \\ -\alpha\cos\psi & -\beta & \alpha\sin\psi \\ \beta\cos\psi\cos\Psi+\theta\sin\psi\sin\Psi & -\alpha\cos\Psi & -\beta\sin\psi\cos\Psi+\theta\cos\psi\sin\Psi \end{bmatrix} \tag{22}$$

and by premultiplying by the inverse of (22), which is its transpose. (In equation (22) and below Ψ denotes Ψ^{+}.) This gives

$$\mathbf{S}\tilde{\mathbf{R}}^{-1}\mathbf{S}^{-1}\mathbf{S}'\tilde{\mathbf{R}}\mathbf{S}^{-1} = \frac{2\theta'}{\theta}\mathbf{1} + \frac{\alpha\beta'-\alpha'\beta}{\theta^2}\mathbf{\Delta}^{(2)}, \tag{23}$$

where

$$\mathbf{\Delta}^{(2)} \equiv \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \tag{24}$$

with

$$a \equiv -\cos\psi\sin(\Psi-\psi) + (\beta/\theta)\sin\psi\cos(\Psi-\psi),$$

$$b \equiv -(\alpha/\theta)\cos(\Psi-\psi),$$

$$c \equiv -\sin\psi\sin(\Psi-\psi) - (\beta/\theta)\cos\psi\cos(\Psi-\psi).$$

Hence the coupling coefficients reduce to

$$\Gamma_{12} = -\Gamma_{21} = \frac{\Psi'\alpha\sin\psi}{\theta} - \frac{\alpha\beta'-\alpha'\beta}{\theta^2}\left(-\cos\psi\,\{1+\sin(\Psi-\psi)\} + \frac{\beta\sin\psi\cos(\Psi-\psi)}{\theta}\right), \tag{25a}$$

$$\Gamma_{13} = -\Gamma_{31} = \frac{\Psi'\beta}{\theta} + \frac{\alpha\beta'-\alpha'\beta}{\theta^2}\left(\frac{\alpha\cos(\Psi-\psi)}{\theta}\right), \tag{25b}$$

$$\Gamma_{23} = -\Gamma_{32} = -\frac{\Psi'\alpha\cos\psi}{\theta} - \frac{\alpha\beta'-\alpha'\beta}{\theta^2}\left(-\sin\psi\{1+\sin(\Psi-\psi)\} - \frac{\beta\cos\psi\cos(\Psi-\psi)}{\theta}\right). \tag{25c}$$

The coupling ratio (25a) between the Alfvén mode and the magnetoacoustic mode reproduces the result (IV.29) apart from unimportant angular factors. It also follows from equations (25) that the coupling coefficients between the various modes are roughly equal. Consequently, the strongest coupling occurs between the two modes whose q's are most nearly equal. Except for $v_A \approx c_s$, it follows that the strongest coupling is between the Alfvén mode and the magnetoacoustic mode (the fast mode for $v_A > c_s$ and the slow mode for $v_A < c_s$).

The coupling coefficients (25) contain two terms, and the term proportional to $\alpha\beta'-\alpha'\beta$ appears even in the cold plasma approximation (Part III). The other term involves the derivative of Ψ and is unique to the MHD treatment. The angle Ψ is roughly constant for $v_A \gg c_s$ and for $v_A \ll c_s$ (being equal to $\frac{1}{2}\pi$ and θ in the two limits respectively). Consequently, the terms involving Ψ' can be important only for $v_A \approx c_s$. For the moment let us consider only those terms proportional to Ψ'. For

$c_s \approx v_A$ and $\theta \approx 0$ equation (IV.15) gives

$$\tan \Psi^{\pm} \approx \{-t \pm (1+t^2)^{\frac{1}{2}}\}^{-1} = t \pm (1+t^2)^{\frac{1}{2}}, \tag{26}$$

with

$$t \equiv x/\theta, \qquad x \equiv (v_A^2 - c_s^2)/(v_A^2 + c_s^2). \tag{27}$$

Hence one has

$$|\Psi'| = |t'|/2(1+t^2)^{\frac{1}{2}}. \tag{28}$$

Now let us compare the magnitudes of the two terms in each of equations (25a, b, c), that is, compare $|\Psi'|$ with $|\theta'/\theta|$. It follows from equation (28) that the favourable case for the former term to dominate is $|t| \lesssim 1$, that is, $|x| \lesssim \theta$. In this case one finds that the only condition under which the former term dominates is

$$|x'| > |\theta'|, \qquad |x| \lesssim \theta, \tag{29}$$

where θ' describes any characteristic rate of change of the angles. However, changes in x should cause refraction and lead to changes in the angles. In fact, refraction should cause θ' to be at least as large as x'. In other words, the terms proportional to Ψ' in equations (25) probably never exceed the other terms in magnitude. For semiquantitative purposes the coupling coefficients (25) may be approximated by θ'/θ.

Finally let us compare the differences between the three q values for $v_A \approx c_s$ and $\theta \approx 0$, that is, under the same conditions as apply in equation (28). One finds

$$|q_1 - q_2| : |q_2 - q_3| : |q_1 - q_3| = |(1+t^2)^{\frac{1}{2}} - t| : 2(1+t^2)^{\frac{1}{2}} : |(1+t^2)^{\frac{1}{2}} + t|. \tag{30}$$

Now, because the coupling coefficient is of order $|\theta'/\theta|$ divided by the difference between the two relevant q's, it follows that for $|t| \lesssim 1$ the couplings between the three modes (taken in pairs) are roughly equal. Thus, the strongest coupling is that between the Alfvén mode and the magnetoacoustic mode except for $|x| \lesssim \theta$ when the couplings between all modes are comparable.

Discussion

It may be concluded that the proposed simplification of mode-coupling theory gives an adequate approximation to the more general theory, at least for coupling in the neighbourhood of coupling points. For the magnetoionic modes 'the neighbourhood of the coupling points' includes virtually all parameter ranges of interest.

The simplified theory has been used here to treat coupling between the three MHD modes. The treatment given amounts to a major departure from earlier treatments, notably that of Frisch (1964). In particular Frisch assumed 'vertical' incidence and a fixed finite angle θ of propagation relative to the magnetic field, which effectively excludes the case of most interest, namely for oblique incidence and nearly parallel propagation. The qualitative results found here are that each of the three coupling ratios for $\sin\theta \approx 0$ and $c_s \approx v_A$ may be approximately

$$Q_{ij} \approx \frac{2c}{\omega} \frac{|\theta'|}{\theta} \frac{1}{|q_i - q_j|}, \tag{31}$$

with

$$|q_i - q_j| \approx \frac{c}{2|\cos\psi|} \frac{|v_{\phi i}^2 - v_{\phi j}^2|}{v_A^3}, \tag{32}$$

where $v_{\phi i}$ denotes the phase speed of the ith mode (i = A, F or S). In particular, coupling between Alfvén and fast modes ($v_A \gg c_s$) is strongest due to the fact that it is for this case that the phase speeds are most nearly equal. In particular this implies that a flux of fast-mode waves from the region $c_s > v_A$ incident on the region with $c_s = v_A$ will produce a transmitted flux in which the secondary components are such that the slow-mode component is necessarily less than the Alfvén-mode component.

One coupling which has not been considered so far is that between the Alfvén mode and the slow mode for $v_A \ll c_s$. In this case, for nearly parallel propagation, one finds

$$|q_1 - q_3| \approx \frac{c}{2v_A|\cos\psi|}\left(\frac{v_A^2 \sin^2\theta}{c_s^2}\right) \tag{33}$$

and then equation (25b) implies

$$Q_{13} \approx \frac{4v_A|\cos\psi|}{\omega}\,\frac{c_s^2|\theta'|}{v_A^2\theta^3} \qquad (v_A \ll c_s). \tag{34}$$

This coupling is quite strong and implies that any inhomogeneities in the magnetic field in regions with $c_s \gg v_A$ can lead to a strong coupling between the slow (magnetoacoustic) mode and the Alfvén mode.

Finally it should be noted that a practical limitation on the range of validity of the simplified theory as developed here is that any reflection points must be far from the coupling points of interest. For example, coupling between o↑ and x↑ in the neighbourhood of the reflection point for the x mode (Part I) should not be treated using the simplified theory. Similarly, the simplified theory should not be used to treat coupling between the three MHD modes when the reflection point for the fast mode is near the 'double' coupling point (Section 5 of Part IV). However, it is reasonable to speculate that a generalized version of the simplified theory could be used to treat these cases; one would need to retain the extra mode (i.e. include the relevant downgoing mode) and to retain an additional wave variable in constructing the matrix **R**.

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