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Generalization of Planck's Law of Radiation to Anisotropic Dispersive Media

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Abstract

The distribution of radiant energy in media which are not only dispersive but also anisotropic is derived. The distribution is found to depend on the phase velocity V_s, the group velocity V_{gs} and the angle α_s between them for each mode s. Planck's law is a limiting case when $V_s = V_{gs} = c$ and $\alpha_s = 0$ and applies strictly only to a vacuum.

Consider momentum space with usual coordinates (p, θ, ϕ). The volume element in momentum space between p and $p+\mathrm{d}p$, θ and $\theta+\mathrm{d}\theta$, and ϕ and $\phi+\mathrm{d}\phi$ is given by

$$p^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \,\mathrm{d}p\,. \tag{1}$$

We express this element in terms of the wave propagation speeds and frequency.

In a general medium the phase speed V_s of mode s is anisotropic and a function of frequency ν and also temperature T, that is,

$$V_s = V_s(T, \theta, \phi, \nu) = \omega/k_s\,, \tag{2}$$

where ω is the angular frequency $(2\pi\nu)$ and k the wave number. Likewise for the group velocity $\boldsymbol{V}_{gs}$

$$\begin{aligned} \boldsymbol{V}_{gs} &= \boldsymbol{V}_{gs}(T, \theta, \phi, \nu) = \partial\omega/\partial\boldsymbol{k}_s \\ &= \hat{k}_s \frac{\partial\omega}{\partial|k_s|} + \hat{\theta}\frac{1}{|k_s|}\frac{\partial\omega}{\partial\theta} + \hat{\phi}\frac{1}{|k_s|\sin\theta}\frac{\partial\omega}{\partial\phi}\,, \end{aligned} \tag{3}$$

where $\hat{k}_s$ is a unit vector in the direction of the wave normal.

The energy E and momentum p of a photon are given respectively by

$$E = \hbar\omega \qquad \text{and} \qquad p = \hbar k \tag{4}$$

and it follows that $p^2 = h^2\nu^2/V_s^2$ for waves of mode s. Further, the directional derivative of ν with respect to p for mode s is given by

$$\frac{\mathrm{d}\nu}{\mathrm{d}p} = \frac{\mathrm{d}\boldsymbol{p}}{|\mathrm{d}p|}\cdot\frac{\partial\nu}{\partial\boldsymbol{p}} = \frac{1}{h}\frac{\mathrm{d}\boldsymbol{p}}{|\mathrm{d}p|}\cdot\frac{\partial\omega}{\partial\boldsymbol{k}_s}\,.$$

Since the volume element (1) is constructed with d$\boldsymbol{p}$ in the direction of $\boldsymbol{p}$ (i.e. the same direction as the phase velocity V_s), it is given by

$$\frac{h^3 \nu^2 \sin\theta}{V_s^2 V_{gs} |\cos\alpha_s|} \, d\theta \, d\phi \, d\nu , \tag{5}$$

where α_s is the angle between the phase velocity and the group velocity, i.e. the angle between $\boldsymbol{k}_s$ and $\partial\omega/\partial\boldsymbol{k}_s$, so that

$$V_{gs} \cos\alpha_s = \partial\omega/\partial |k_s| .$$

The absolute value sign is put around $\cos\alpha_s$ in the expression (5) to allow for situations in which V_{gs} may be in negative directions with regard to V_s and because the volume element is a positive quantity.

The number of cells in the same momentum space given by the volume element (1), each of size h^3 for mode s of oscillations, is represented by $C_s(\theta, d\theta, d\phi)$. The number of photons N_s of mode s is related to the number of cells C_s by (e.g. Joos 1958)

$$N_s = C_s(e^{\beta h\nu} - 1)^{-1} , \tag{6}$$

where $\beta = (k_B T)^{-1}$. It follows that the density of photons of mode s in volume V in the frequency range ν to $\nu + d\nu$ is

$$N_s/V = (e^{\beta h\nu} - 1)^{-1} \int_0^\pi \int_0^{2\pi} \frac{\nu^2 \sin\theta}{V^2 V_{gs} |\cos\alpha_s|} d\theta \, d\phi \, d\nu . \tag{7}$$

To find the total density of photons in volume V we must make the summation $N = \Sigma_s N_s$. The energy density of photons in the frequency range ν to $\nu + d\nu$ is then given by

$$U(\nu) \, d\nu = (e^{\beta h\nu} - 1)^{-1} \sum_s \int_0^\pi \int_0^{2\pi} \frac{h\nu^3 \sin\theta}{V_s^2 V_{gs} |\cos\alpha_s|} \, d\theta \, d\phi \, d\nu . \tag{8}$$

In the special case in which the medium is dispersive but isotropic, $\alpha_s = 0$ and

$$U(\nu) = (e^{\beta h\nu} - 1)^{-1} \sum_s (4\pi h\nu^3 / V_s^2 V_{gs}) . \tag{9}$$

An expression equivalent to (9) was found earlier by van Roosbroech and Shockley (1954) who discussed dispersive isotropic media. The generalization given by equation (8) is apparently new. Planck's law follows for the special case where $V_s = c = V_{gs}$ and isotropy exists.

The above theory is of general interest since anisotropic media occur widespread in nature. Expression (8) provides the basic thermal radiation distribution in such media. This expression could also be written as

$$U(\nu) = (e^{\beta h\nu} - 1)^{-1} \sum_s \int_0^\pi \int_0^{2\pi} \frac{h\nu^3 \sin\theta}{V_s^2 |\partial\omega/\partial |k_s||} \, d\theta \, d\phi .$$

Applications of the foregoing theory will be presented in future publications.

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References

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