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A Note on the Abraham–Lorentz Equation

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Abstract

A generalization of the usual nonrelativistic energy integral is found for the Abraham–Lorentz equation with a constant external force term.

The equations of motion for a classical charged particle with radiation reaction, which have been discussed in detail by Rohrlich (1965), are still a subject of controversy. Some philosophical aspects of this problem have been dealt with recently by Grünbaum (1976). Kasher (1976) has solved the relativistic Lorentz–Dirac equation numerically for one-dimensional motion of a charge in a Coulomb field with a wide range of initial conditions, and has shown that unphysical results are obtained. In particular, the total energy of the moving charge may increase with time. If anything, the energy should decrease because of the radiative loss of energy. In view of this difficulty, it is of interest to consider the possible existence of an energy-like integral for at least some special cases of the Lorentz–Dirac equation. I shall consider here the nonrelativistic version of this, the Abraham–Lorentz equation

$$m\,\mathrm{D}^2\boldsymbol{r} - m\tau\,\mathrm{D}^3\boldsymbol{r} = \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{F}$ is the external force, $\tau = 2e^2/3mc^3$ and D is used here to denote $\mathrm{d}/\mathrm{d}t$.

If $\boldsymbol{F} = -\nabla V$, scalar multiplication of equation (1) by $\mathrm{D}\boldsymbol{r}$ yields

$$\mathrm{D}\{\tfrac{1}{2}m(\mathrm{D}\boldsymbol{r})^2 + V\} = m\tau\,\mathrm{D}\boldsymbol{r}\,.\,\mathrm{D}^3\boldsymbol{r} = m\tau\{\mathrm{D}(\mathrm{D}\boldsymbol{r}\,.\,\mathrm{D}^2\boldsymbol{r}) - (\mathrm{D}^2\boldsymbol{r})^2\}$$

or

$$\mathrm{D}\{\tfrac{1}{2}m(\mathrm{D}\boldsymbol{r})^2 + V - m\tau\,\mathrm{D}\boldsymbol{r}\,.\,\mathrm{D}^2\boldsymbol{r}\} = -m\tau(\mathrm{D}^2\boldsymbol{r})^2\,. \tag{2}$$

If equation (1) is now used to replace one $\mathrm{D}^2\boldsymbol{r}$ factor on the right-hand side of (2), we find

$$\mathrm{D}\{\tfrac{1}{2}m(\mathrm{D}\boldsymbol{r} - \tau\,\mathrm{D}^2\boldsymbol{r})^2 + V\} = -\tau\boldsymbol{F}\,.\,\mathrm{D}^2\boldsymbol{r}\,. \tag{3}$$

The only assumption made to this point is that $\boldsymbol{F}$ is conservative in the elementary sense: $\boldsymbol{F} = -\nabla V$. In order to obtain a conservation law from equation (3), we must further assume that $\boldsymbol{F}$ is a constant vector. Then, with $V = -\boldsymbol{F}\,.\,\boldsymbol{r}$,

$$\mathrm{D}\{\tfrac{1}{2}m(\mathrm{D}\boldsymbol{r} - \tau\,\mathrm{D}^2\boldsymbol{r})^2 - \boldsymbol{F}\,.\,(\boldsymbol{r} - \tau\,\mathrm{D}\boldsymbol{r})\} = 0$$

or

$$G \equiv \tfrac{1}{2}m(\mathrm{D}\boldsymbol{r} - \tau\,\mathrm{D}^2\boldsymbol{r})^2 - \boldsymbol{F}\,.\,(\boldsymbol{r} - \tau\,\mathrm{D}\boldsymbol{r}) = \text{const.} \tag{4}$$

The quantity G is obviously a generalization of the usual energy integral, to which it reduces when $\tau = 0$. With the general solution of equation (1)

$$\boldsymbol{r} = \boldsymbol{R}+\boldsymbol{U}t+\boldsymbol{A}\mathrm{e}^{t/\tau}+\boldsymbol{F}t^{2}/2m, \tag{5}$$

where $\boldsymbol{R}$, $\boldsymbol{U}$ and $\boldsymbol{A}$ are constant, we find

$$G = \tfrac{1}{2}mU^{2} -\boldsymbol{F}.\boldsymbol{R} +F^{2}\tau^{2}/2m. \tag{6}$$

The coefficient $\boldsymbol{A}$ of the 'runaway' term in the solution (5) does not appear in equation (6).

The new integral G does not give much additional insight into the nature of the Abraham–Lorentz equation with a constant force, for all information is, of course, contained in the exact solution (5). However, the existence of this integral suggests that there might be a generalization of the energy conservation theorem which is valid for a larger class of forces.

References

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