

## **Properties of the Inelastic Form Factors of the Nucleus**

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### *Abstract*

Based on an inversion formula for the energy-weighted sum rules, a study is made of the properties of the inelastic form factors of the nucleus. The inversion formula is derived by using a simple representation of the identity operator for a restricted set of non-orthogonal states and it is applicable to both isoscalar and isovector transitions. As a result, it is shown that the longitudinal form factor for a particular multipolarity cannot be explicitly factorized into a product of a function of momentum transfer and a function of excitation energy over the entire range of momentum transfer. Further, the ambiguity arising out of the use of the hydrodynamical model to assign spins of giant resonances is illustrated, taking the isovector electric dipole form factor as an example.

### **1. Introduction**

For many years a simple hydrodynamical model (Tassie 1956) has been extensively used for the analysis of electron scattering data (Überall 1971; Fukuda and Torizuka 1972, 1976; Nagao and Torizuka 1973). The work of Deal and Fallieros (1973) on the energy-weighted sum rules (EWSR) has given an understanding of the success of the hydrodynamical model by establishing that the result from this model is a consequence of the assumption that a single state, called the doorway state, completely dominates the EWSR. However, Deal (1973) has also shown that the single doorway-state dominance of the EWSR is inadequate to describe experimental data at large momentum transfer and the use of two doorway states improves the description of the data considerably. Thus there is a need for corrections to the simple hydrodynamical model. More formally, for isoscalar transitions, Tassie (1975) inverted the EWSR to obtain the transition charge densities and transition form factors as infinite sums of the EWSR, so that the first term of the sum is the result of the hydrodynamical model while the subsequent terms, which become more important as the momentum transfer is increased, correspond to the double doorway correction, triple doorway correction and so on.

In this paper, a rigorous basis for the inversion formula used by Tassie (1975) is established through the use of an identity operator representation for a restricted set of non-orthogonal states, and the formalism is extended to apply also to isovector transitions. Various relations are obtained among the coefficients in the expansion of the form factors, and, as the coefficients can in principle be determined from experiment, these relations can be tested. A proof is then given that the longitudinal form factor for a particular multipolarity cannot be factorized into a product of a function of excitation energy and a function of momentum transfer. Thus the dependence of

the form factor on momentum transfer cannot be described by the hydrodynamical model at all excitation energies. The possibility of such a failure of the hydrodynamical model was previously discussed by Ui and Tsukamoto (1974).

The tacit assumption of the validity of the hydrodynamical model at various ranges of momentum transfers has been used in the past to assign spins of giant resonances (Fukuda and Torizuka 1972; Nagao and Torizuka 1973). In Section 7 here it is shown that this is not a satisfactory method of assigning spins because of an ambiguity arising from the similarity in dependence on momentum transfer between the higher order terms in the form factor for the excitation of a state with spin  $l$  and the first terms of the states of higher spin  $l+2, l+4, \dots$ . A well-known case is the similarity between electric monopole and electric quadrupole transitions.

## 2. Useful Representation of Identity Operator

Suppose a set of states  $|n\rangle$  is a basis for the Hilbert space  $L$  of the many-body system (in general the states  $|n\rangle$  are not orthogonal). Then any arbitrary state  $|a\rangle$  in  $L$  can be written as

$$|a\rangle = \sum_n a_n |n\rangle. \quad (1)$$

We choose basis states satisfying

$$\langle m | W | n \rangle = 0 \quad \text{for } m \neq n, \quad (2)$$

where  $W$  is a positive definite hermitian operator defined over the space  $L$ . Since  $\langle a | W | b \rangle$  for  $|a\rangle, |b\rangle \in L$  has all the properties of an inner product, then from any given arbitrary basis it is always possible to construct a basis satisfying equation (2) by Schmidt orthogonalization (Greub 1975).

Taking the inner product of equation (1) with  $W | m \rangle$ , we have

$$a_m = \langle m | W | a \rangle / \langle m | W | m \rangle. \quad (3)$$

Substituting this result into equation (1) we then obtain

$$|a\rangle = \sum_n \frac{|n\rangle \langle n | W | a \rangle}{\langle n | W | n \rangle} \quad (4)$$

and, since  $|a\rangle$  is an arbitrary state in  $L$ ,

$$\sum_n \frac{|n\rangle \langle n | W | n \rangle}{\langle n | W | n \rangle} = I, \quad (5)$$

where  $I$  is the identity operator. Equation (5) is the completeness relation for a set of non-orthogonal states satisfying the condition (2). Taking the hermitian adjoint of equation (5),

$$\sum_n \frac{W | n \rangle \langle n |}{\langle n | W | n \rangle} = I. \quad (6)$$

We note in passing that for a basis of non-orthogonal states  $|n\rangle$  it is always possible to find (Rowe 1975) another set of states  $|\tilde{n}\rangle$ , called the biorthogonal states, such that

$$\langle n | \tilde{m} \rangle = k \delta_{nm}. \quad (7)$$

The states have the property

$$\sum_n \frac{|n\rangle\langle\tilde{n}|}{\langle n|\tilde{n}\rangle} = I. \quad (8)$$

The set of states  $|\tilde{n}\rangle$  is also a basis. Comparing equations (8) and (7) with (5) and (2) we can identify

$$|\tilde{n}\rangle = W|n\rangle. \quad (9)$$

Using the expansion (5), we can write the matrix element of an operator  $F$  between a state  $|0\rangle$  and a state  $|f\rangle$  as

$$\langle f|F|0\rangle = \sum_n \langle f|n\rangle\langle n|WF|0\rangle/\langle n|W|n\rangle. \quad (10)$$

For a complicated many-body system such as a nucleus, it is impossible to do any practical calculation in the whole Hilbert space  $L$ . Instead it is usual to work with some subspace which we shall call  $L_M$ . We decompose the whole Hilbert space into the subspace  $L_M$  and a complementary subspace  $L_C$ ,

$$L = L_M \oplus L_C, \quad (11)$$

and try to choose  $L_M$  and  $L_C$  so that  $L_M$  is spanned by simple states and  $L_C$  can be ignored.

Any state  $|a\rangle \in L$  can be written as

$$|a\rangle = |\psi_a\rangle + |\phi_a\rangle, \quad (12)$$

where  $|\psi_a\rangle \in L_M$  and  $|\phi_a\rangle \in L_C$ . We note that, given  $L$  and  $L_M$ ,  $L_C$  is not uniquely specified and we complete the specification of  $L_C$  by choosing it such that

$$\langle\psi|W|\phi\rangle = 0 \quad \text{for all} \quad |\psi\rangle \in L_M, |\phi\rangle \in L_C. \quad (13)$$

Since  $\langle a|W|b\rangle$  has all the properties of an inner product, the condition (13) uniquely specifies  $L_C$  which is the orthogonal complement of  $L_M$  with respect to the inner product  $\langle a|W|b\rangle$  (Greub 1975).

We divide the basis states  $|m\rangle$  of  $L$  into two sets,  $|\alpha\rangle$  and  $|\alpha'\rangle$ , and let  $L_M$  be the space spanned by the set  $|\alpha\rangle$ . Then the set  $|\alpha'\rangle$  is a basis for  $L_C$ . Equation (5) thus becomes

$$I = P + Q, \quad (14)$$

with

$$P = \sum_{\alpha} \frac{|\alpha\rangle\langle\alpha|W}{\langle\alpha|W|\alpha\rangle}, \quad Q = \sum_{\alpha'} \frac{|\alpha'\rangle\langle\alpha'|W}{\langle\alpha'|W|\alpha'\rangle}. \quad (15)$$

We note that

$$P^2 = P, \quad Q^2 = Q \quad \text{and} \quad PQ = QP = 0. \quad (16)$$

Equation (10) can be written as

$$\begin{aligned} \langle f|F|0\rangle &= \sum_{\alpha} \langle f|\alpha\rangle\langle\alpha|WF|0\rangle/\langle\alpha|W|\alpha\rangle \\ &\quad + \sum_{\alpha'} \langle f|\alpha'\rangle\langle\alpha'|WF|0\rangle/\langle\alpha'|W|\alpha'\rangle. \end{aligned} \quad (17)$$

If  $\langle \alpha' | WF | 0 \rangle = 0$ , that is, if  $F | 0 \rangle$  lies in  $L_M$ , the second term in equation (17) is zero and the states  $|\alpha'\rangle$  can be effectively ignored.

### 3. Electroexcitation of Nuclei

#### (a) Longitudinal Form Factor

In electron scattering, under the one-photon exchange approximation, the differential cross section is given by (de Forest and Walecka 1966)

$$\frac{d\sigma}{d\Omega} = \frac{4\pi\sigma_M}{1 + (2E_0 \sin^2 \frac{1}{2}\theta)/M_t} \left\{ \frac{q_\mu^4}{q^4} F_L^2(q, \omega) + \left( \frac{q_\mu^2}{2q^2} + \tan^2 \frac{1}{2}\theta \right) F_T^2(q, \omega) \right\}, \quad (18)$$

where  $E_0$  is the incident electron energy,  $\theta$  is the scattered angle,  $M_t$  is the target mass and  $\sigma_M$  is the Mott cross section defined by

$$\sigma_M = \{(\alpha \cos \frac{1}{2}\theta)/(2E_0 \sin^2 \frac{1}{2}\theta)\}^2, \quad (19)$$

with  $\alpha$  the fine structure constant. Also in equation (18),  $q_\mu = (q, i\omega)$  is the four-momentum transfer,  $\omega$  being the energy loss, and  $F_L(q, \omega)$  and  $F_T(q, \omega)$  are respectively the longitudinal and transverse form factors which summarize all the nuclear structure information.

We consider now only the longitudinal form factor  $F_L(q, \omega)$ , which is defined as

$$F_L^2(q, \omega) = \sum_{i=0}^{\infty} \frac{|\langle J_f \| F^i(q) \| J_i \rangle|^2}{(2J_i + 1)}, \quad (20)$$

where

$$F_m^l(q) = \int j_l(qr) Y_{lm}(\Omega) \rho(r) d^3r \quad (21)$$

is the  $2^l$  pole form-factor operator and the reduced matrix element is defined by

$$\langle J_f M_f | F_m^l(q) | J_i M_i \rangle = (-)^{J_f - M_f} \begin{pmatrix} J_f & l & J_i \\ -M_f & m & M_i \end{pmatrix} \langle J_f \| F^l(q) \| J_i \rangle.$$

It is convenient to consider separately the isoscalar form factor with operator

$$F_m^l(q) = \sum_{i=1}^A j_l(qr_i) Y_{lm}(\Omega_i) \quad (22a)$$

and the isovector form factor with operator

$$F_m^l(q) = \sum_{i=1}^A \tau_3(i) j_l(qr_i) Y_{lm}(\Omega_i). \quad (22b)$$

These form factors can also be used in the description of the scattering of hadrons by nuclei, as in the calculations of Alexander and Rinat (1974) and Boridy and Feshbach (1974).

We define the generalized electric multipole operators for the isoscalar case as

$$Q_{lx} = \sum_{i=1}^A Q_{lx}(i) \quad (23a)$$

and for the isovector case as

$$Q_{1x} = \sum_{i=1}^A \tau_3(i) Q_{1x}(i), \quad (23b)$$

where

$$Q_{1x}(i) = r_i^{l+2\alpha} Y_{lm}(\Omega_i). \quad (24)$$

To avoid unnecessary subscripts, we omit the subscript  $m$  from the  $Q_{1x}$  operators and also from  $F_m^l$ . In practice, physical quantities such as cross sections are given using the Wigner-Eckart theorem in terms of reduced matrix elements, as the  $m$  dependence is only geometrical. Since

$$F^l(q) = \sum_{\alpha=0} k_{\alpha}(q) Q_{1x}, \quad (25)$$

where

$$k_{\alpha}(q) = (-)^{\alpha} q^{l+2\alpha} / 2^{\alpha} \alpha! (2l+2\alpha+1)!!, \quad (26)$$

the operators  $Q_{1x}$  operating on the ground state  $|0\rangle$  give all the states excited by electron scattering. These states  $Q_{1x}|0\rangle$  can be used as a set of basis states spanning a subspace  $L_M$  of  $L$ . The treatment is now confined to even nuclei with zero spin in the ground state; extension of this treatment to nuclei with nonzero spin in the ground state is possible, but is more complicated. In order to use the expansion (17) we choose a basis

$$|\alpha\rangle = M_{1x}|0\rangle, \quad (27)$$

where

$$M_{1x} = \sum_{\beta} A_{\alpha\beta} Q_{1\beta} \quad (28)$$

such that

$$\langle 0 | M_{1x}^{\dagger} W M_{1\beta} | 0 \rangle = 0 \quad \text{for } \alpha \neq \beta. \quad (29)$$

We take  $W$  as a scalar under rotations, and then

$$\langle 0 | M_{1x}^{\dagger} W M_{1'\beta} | 0 \rangle = 0 \quad \text{for } l \neq l'$$

and each value of  $l$  can be treated separately, i.e. we now take  $L$  to be the space of states with spin  $l$ . For convenience we usually will omit the subscript  $l$ .

From equations (25) and (28)

$$F^l(q) = \sum_{\alpha} b_{\alpha}(q) M_{\alpha}. \quad (30)$$

The subspace  $L_C$  is spanned by the states  $|\alpha'\rangle$ , which from equations (2) and (27) satisfy

$$\langle \alpha' | W M_{\alpha} | 0 \rangle = 0, \quad (31)$$

and so from equation (30)

$$\langle \alpha' | W F^l(q) | 0 \rangle = 0, \quad (32)$$

which is the condition for only the first terms in equation (17) to be needed. Thus

$$\langle f | F^l(q) | 0 \rangle = \sum_{\alpha} \langle f | M_{\alpha} | 0 \rangle \langle 0 | M_{\alpha}^{\dagger} W F^l(q) | 0 \rangle / \langle 0 | M_{\alpha}^{\dagger} W M_{\alpha} | 0 \rangle. \quad (33)$$

The expansion for the form factor as given previously by Tassie (1975) is a special case of equation (33).

A particular construction of the  $M_\alpha$  satisfying equation (29) is given by

$$M_0 = Q_0, \quad M_\alpha = Q_\alpha + \sum_{\beta=0}^{\alpha-1} A_{\alpha\beta} Q_\beta, \quad (34)$$

where the  $\alpha$  unknowns  $A_{\alpha\beta}$  are the solutions of the  $\alpha$  equations

$$\langle 0 | Q_\beta^\dagger W M_\alpha | 0 \rangle = 0 \quad \text{for } \beta < \alpha. \quad (35)$$

Then

$$\begin{aligned} \langle 0 | M_\alpha^\dagger W F^l(q) | 0 \rangle &= \left\langle 0 \left| M_\alpha^\dagger W \sum_{\beta=0}^{\alpha-1} k_\beta(q) Q_\beta \right| 0 \right\rangle = \sum_{\beta=\alpha} k_\beta(q) \langle 0 | M_\alpha^\dagger W Q_\beta | 0 \rangle \\ &= k_\alpha(q) \langle 0 | M_\alpha^\dagger W Q_\alpha | 0 \rangle + \sum_{\beta=\alpha+1} k_\beta(q) \langle 0 | M_\alpha^\dagger W Q_\beta | 0 \rangle. \end{aligned} \quad (36)$$

From equation (26) it is seen that the first term in (36) predominates at small  $q$ , so that

$$\langle 0 | M_\alpha^\dagger W F^l(q) | 0 \rangle \propto q^{l+2\alpha} \quad \text{as } q \rightarrow 0. \quad (37)$$

#### (b) Choice of $W$

For the isoscalar form factors and multipole operators,  $\langle 0 | Q_{1\alpha} H Q_{1\beta} | 0 \rangle$  and  $\langle 0 | Q_{1\alpha} H F^l(q) | 0 \rangle$  can be evaluated (Ui and Tsukamoto 1974; Tassie 1975) using the standard double commutator technique for EWSR (Ferreira and Sesma 1967). The quantity  $H$  is the nuclear Hamiltonian and the energy of the ground state is taken as zero,

$$H|0\rangle = 0. \quad (38)$$

For isovector transitions, however, the sum rules are complicated by the effects of the exchange currents. These effects have been tackled by Mekjian (1974), de Shalit and Feshbach (1974), Bohr and Mottelson (1975) and Noble (1977) but all their attempts invariably would involve model-dependent assumptions for practical explicit evaluation of the exchange current contributions. Such complications, however, do not hinder the formal development of the inversion technique for both types of transitions; only the final results will be different when the EWSR results are explicitly used.

The results for  $\langle 0 | Q_{1\alpha}^\dagger H Q_{1\beta} | 0 \rangle$  and  $\langle 0 | Q_{1\alpha}^\dagger H F^l(q) | 0 \rangle$  cannot be used directly by taking  $W = H$  because  $W$  must be positive definite, but these results can be used by suitably defining  $W$ , as is shown in the Appendix.

#### 4. Limits of Single-particle Excitation

We now consider some of the consequences that the state  $|f\rangle$  of the system can be written as

$$|f\rangle = |\psi\rangle + |\phi\rangle, \quad (39)$$

where

$$|\psi\rangle \in L_M, \quad |\phi\rangle \in L_C \quad (40)$$

and

$$|\psi\rangle = \sum_{\alpha} |\alpha\rangle \langle \alpha | W | f \rangle / \langle \alpha | W | \alpha \rangle. \quad (41)$$

We define

$$p_\alpha = \langle \alpha | W | f \rangle \langle f | \alpha \rangle / \langle \alpha | W | \alpha \rangle. \quad (42)$$

Then, for any other choice of states  $|\alpha_1\rangle$  spanning  $L_M$  such that

$$\langle \alpha_1 | W | \beta_1 \rangle = 0 \quad \text{for } \alpha_1 \neq \beta_1, \quad (43)$$

we have

$$\sum_\alpha p_\alpha = \sum_{\alpha_1} p_{\alpha_1} = p, \quad (44)$$

since

$$\langle f | \psi \rangle = \sum_\alpha \langle f | \alpha \rangle \langle \alpha | W | f \rangle / \langle \alpha | W | \alpha \rangle = p. \quad (45)$$

Although  $p$  is an invariant, it does depend on the choice of  $W$  and of  $L_M$ . Since

$$\langle f | W | f \rangle = \langle \psi | W | \psi \rangle + \langle \phi | W | \phi \rangle \quad (46)$$

and  $W$  is positive definite, we have

$$\langle \psi | W | \psi \rangle \leq \langle f | W | f \rangle \quad (47)$$

and on substituting equation (41)

$$\sum_\alpha \langle f | W | \alpha \rangle \langle \alpha | W | f \rangle / \langle \alpha | W | \alpha \rangle \leq \langle f | W | f \rangle. \quad (48)$$

In addition to the basis (27) of  $L_M$ , it is convenient to introduce a basis of orthonormal states  $N_\alpha |0\rangle$ , which span  $L_M$  and which can be chosen so that the operator  $W$  is diagonal within  $L_M$ , by constructing operators

$$N_\alpha = \sum_\beta D_{\alpha\beta} Q_\beta \quad (49)$$

such that

$$\langle 0 | N_\alpha^\dagger N_\beta | 0 \rangle = \delta_{\alpha\beta} \quad \text{and} \quad \langle 0 | N_\alpha^\dagger W N_\beta | 0 \rangle = \delta_{\alpha\beta} \omega_\alpha. \quad (50)$$

Then the expansions (39) and (41) for the state  $|f\rangle$  can be written

$$|f\rangle = \sum_\alpha a'_\alpha N_\alpha |0\rangle + |\phi\rangle, \quad (51)$$

where

$$a'_\alpha = \langle 0 | N_\alpha^\dagger W | f \rangle / \omega_\alpha. \quad (52)$$

Then we have

$$p = \sum_\alpha \langle 0 | N_\alpha^\dagger W | f \rangle \langle f | N_\alpha | 0 \rangle / \omega_\alpha. \quad (53)$$

As well as the decomposition (11) of the Hilbert space, we introduce the more usual decomposition

$$L = L_M \oplus L_O, \quad (54)$$

where  $L_O$  is the orthogonal complement of  $L_M$ . Then we can write

$$|f\rangle = |\psi_1\rangle + |\phi_1\rangle, \quad (55)$$

where  $|\psi_1\rangle \in L_M$ ,  $|\phi_1\rangle \in L_O$  and

$$\langle \psi_1 | \phi_1 \rangle = 0. \quad (56)$$

We have

$$|\psi_1\rangle = \sum_{\alpha} a_{\alpha} N_{\alpha} |0\rangle, \quad (57)$$

where

$$a_{\alpha} = \langle 0 | N_{\alpha}^{\dagger} | f \rangle. \quad (58)$$

Then

$$\langle \psi_1 | \psi_1 \rangle = \langle f | \psi_1 \rangle = \sum_{\alpha} |a_{\alpha}|^2 \quad (59)$$

and so

$$\sum_{\alpha} |a_{\alpha}|^2 \leq 1. \quad (60)$$

Using the basis (27) of  $L_M$ , we have

$$|\psi\rangle = \sum_{\alpha} b_{\alpha} M_{\alpha} |0\rangle, \quad (61)$$

with

$$b_{\alpha} = \langle 0 | M_{\alpha}^{\dagger} W | f \rangle / \langle 0 | M_{\alpha}^{\dagger} W M_{\alpha} | 0 \rangle. \quad (62)$$

Then

$$p = \sum_{\alpha} \langle f | M_{\alpha} | 0 \rangle \langle 0 | M_{\alpha}^{\dagger} W | f \rangle / \langle 0 | M_{\alpha}^{\dagger} W M_{\alpha} | 0 \rangle. \quad (63)$$

We can also expand  $|\psi_1\rangle$  using this non-orthogonal basis,

$$|\psi_1\rangle = \sum_{\alpha} b'_{\alpha} M_{\alpha} |0\rangle, \quad (64)$$

and, defining  $T_{\alpha\beta}$  such that

$$\sum_{\beta} T_{\alpha\beta} \langle 0 | M_{\beta}^{\dagger} M_{\gamma} | 0 \rangle = \delta_{\alpha\gamma}, \quad (65)$$

that is,

$$(T^{-1})_{\alpha\beta} = \langle 0 | M_{\alpha}^{\dagger} M_{\beta} | 0 \rangle, \quad (66)$$

we have

$$b'_{\alpha} = \sum_{\beta} T_{\alpha\beta} \langle 0 | M_{\beta}^{\dagger} | f \rangle. \quad (67)$$

Then

$$\langle f | \psi_1 \rangle = \sum_{\gamma\alpha} \langle f | M_{\gamma} | 0 \rangle T_{\gamma\alpha} \langle 0 | M_{\alpha}^{\dagger} | f \rangle. \quad (68)$$

Taking  $W = H$ , it is possible to calculate  $\langle 0 | M_{\alpha}^{\dagger} H M_{\alpha} | 0 \rangle$  and  $\langle 0 | M_{\alpha}^{\dagger} H F^l(q) | 0 \rangle$ , at least for  $\Delta T = 0$  transitions (Tassie 1975), in terms of the ground state density distribution, which can be determined from elastic electron scattering, and then, by fitting equation (33) to the experimental inelastic form factor, to eventually determine  $p$  without making any assumptions about the nuclear structure. However, the quantity of more direct physical significance is  $\langle f | \psi_1 \rangle$ , which is the fraction to which the state  $|f\rangle$  is a single-particle excitation of the ground state, and in order to determine this from an analysis of inelastic scattering  $T_{\gamma\alpha}$  must be known. Unfortunately to calculate  $\langle 0 | M_{\alpha}^{\dagger} M_{\beta} | 0 \rangle$  it is necessary to know the two-body correlation function of the ground state. Thus, although  $p$  can in principle be determined by a model-independent analysis of experimental data, it is not possible to determine  $\langle f | \psi_1 \rangle$  in a model-independent manner.

We now restrict the treatment to eigenstates of  $W$ , namely

$$W|f\rangle = \omega_f|f\rangle, \quad (69)$$

as in practice  $W$  will usually be the Hamiltonian or the operator defined by equation (A3) in the Appendix. Then

$$p_\alpha = \omega_f \langle \alpha | f \rangle^2 / \langle \alpha | W | \alpha \rangle, \quad (70)$$

and thus  $p_\alpha \geq 0$ . From equation (39)

$$\omega_f = \omega_f \langle f | \psi \rangle + \langle \phi | W | \phi \rangle \quad (71)$$

and from equation (45)

$$p = 1 - \langle \phi | W | \phi \rangle / \omega_f. \quad (72)$$

Since

$$\langle \phi | W | \phi \rangle \geq 0, \quad (73)$$

it follows that

$$0 \leq p \leq 1. \quad (74)$$

Comparing equations (58) and (52), we have

$$a'_\alpha = \omega_f a_\alpha / \omega_\alpha. \quad (75)$$

While both  $p$  and  $\langle \psi_1 | \psi_1 \rangle$  lie between 0 and 1, there is no general relation between them. But if  $p = 1$  then  $\langle \psi_1 | \psi_1 \rangle = 1$ , since from equation (72) for  $p = 1$

$$\langle \phi | W | \phi \rangle = 0$$

and so  $|\phi\rangle = 0$  since  $W$  is positive definite. Then

$$|f\rangle = |\psi\rangle = |\psi_1\rangle \quad \text{and} \quad \langle \psi_1 | \psi_1 \rangle = 1.$$

With  $W = H$ ,  $p_\alpha$  is the fraction of the  $\alpha$ th EWSR contributed by the state  $|f\rangle$  and  $p$  is the total fraction the state contributes to all the sum rules. The relation between  $p$  and  $\langle \psi_1 | \psi_1 \rangle$ , the amount of single-particle excitation in  $|f\rangle$ , is complicated. However, from equations (53) and (58)

$$p = \sum_{\alpha} E_f |a_{\alpha}|^2 / E_{\alpha}$$

and if  $|f\rangle$  is the state of lowest energy with a particular set of quantum numbers, such as spin and isospin, then

$$E_f \leq E_{\alpha} \quad \text{for all } \alpha$$

and thus

$$\langle \psi_1 | \psi_1 \rangle \geq p.$$

## 5. Sum Rules and Inequalities

It has been shown in Section 4 that  $p_\alpha$  is the fraction of the  $\alpha$ th EWSR contributed by a state  $|f\rangle$  and  $p = \sum_{\alpha} p_\alpha$  is the total fraction the state  $|f\rangle$  contributes to all the sum rules. In this section a few pertinent relations for these fractions are derived.

From equation (42) we have

$$\sum_f p_\alpha = 1, \quad (76)$$

which can be written more generally as

$$\sum_f \langle \alpha | W | f \rangle \langle f | \beta \rangle = \delta_{\alpha\beta} \langle \alpha | W | \alpha \rangle. \quad (77)$$

For eigenstates of  $W$ , we define

$$c_\alpha(\omega_f) = \omega_f^{\frac{1}{2}} \langle f | \alpha \rangle / \langle \alpha | W | \alpha \rangle^{\frac{1}{2}}. \quad (78)$$

Then

$$|c_\alpha(\omega_f)|^2 = p_\alpha(\omega_f) \quad (79)$$

and the expansion for the form factor, given by equation (33), takes the form

$$\langle f | F^l(q) | 0 \rangle = \sum_\alpha c_\alpha(\omega_f) \omega_f^{-\frac{1}{2}} F_\alpha^l(q), \quad (80)$$

where

$$F_\alpha^l(q) = \langle \alpha | W F^l(q) | 0 \rangle / \langle \alpha | W | \alpha \rangle^{\frac{1}{2}}. \quad (81)$$

From equations (77) and (78),

$$\sum_f c_\alpha^*(\omega_f) c_\beta(\omega_f) = \delta_{\alpha\beta}. \quad (82)$$

This relation has interesting consequences for the analysis of experimental inelastic form factors. The coefficients  $c_\alpha(\omega_f)$  can be determined by fitting equation (80) to experiment, and indications of missing strength of a particular multipole should be given by the extent to which the experimentally determined  $c_\alpha(\omega_f)$  obey equation (82). At present such analysis is confined to determining the extent of exhaustion of the EWSR of the multipole operator, and this corresponds to using equation (82) with  $\alpha = \beta = 0$ .

Another relation (shown in the previous section, e.g. equation (74)) which gives the totality of these fractions over all the order of a particular multipole is

$$\sum_\alpha |c_\alpha(\omega_f)|^2 \leq 1. \quad (83)$$

For the orthogonal states defined by equations (49) and (50), we define

$$c'_\alpha(\omega_f) = \omega_f^{\frac{1}{2}} \langle f | N_\alpha | 0 \rangle / \omega_\alpha^{\frac{1}{2}} = \omega_f^{\frac{1}{2}} a_\alpha^* / \omega_\alpha^{\frac{1}{2}} \quad (84)$$

and note that

$$\sum_\alpha |c'_\alpha(\omega_f)|^2 = \sum_\alpha |c_\alpha(\omega_f)|^2 = p \leq 1 \quad (85)$$

and

$$\sum_f c_\alpha^*(\omega_f) c'_\beta(\omega_f) = \delta_{\alpha\beta}. \quad (86)$$

From the completeness of the eigenstates  $|f\rangle$  of  $W$ , we have

$$\sum_f \langle 0 | N_\alpha^\dagger | f \rangle \langle f | N_\beta | 0 \rangle = \delta_{\alpha\beta} \quad (87)$$

and so using equation (84)

$$\sum_f \omega_f^{-1} c_\alpha^*(\omega_f) c'_\beta(\omega_f) = \omega_\alpha^{-1} \delta_{\alpha\beta}. \quad (88)$$

Similarly from equations (60) and (84)

$$\sum_{\alpha} \omega_{\alpha} |c'_{\alpha}(\omega_f)|^2 \leq \omega_f. \quad (89)$$

From equations (51), (55), (57), (75) and (84), we have

$$\begin{aligned} |\phi_1\rangle &= |\phi\rangle + \sum_{\alpha} (a'_{\alpha} - a_{\alpha}) N_{\alpha} |0\rangle \\ &= |\phi\rangle + \sum_{\alpha} c'^{*}_{\alpha}(\omega_f) (\omega_{\alpha} \omega_f)^{-\frac{1}{2}} (\omega_f - \omega_{\alpha}) N_{\alpha} |0\rangle \end{aligned} \quad (90)$$

and so

$$\langle \phi_1 | W | \phi_1 \rangle = \langle \phi | W | \phi \rangle + \sum_{\alpha} |c'_{\alpha}(\omega_f)|^2 (\omega_f - \omega_{\alpha})^2 / \omega_f. \quad (91)$$

From equations (72) and (85),

$$\langle \phi | W | \phi \rangle = \omega_f \left( 1 - \sum_{\alpha} |c'_{\alpha}(\omega_f)|^2 \right) \quad (92)$$

and so

$$\langle \phi_1 | W | \phi_1 \rangle = \omega_f - \sum_{\alpha} \omega_{\alpha} (2 - \omega_{\alpha} / \omega_f) |c'_{\alpha}(\omega_f)|^2. \quad (93)$$

Since  $W$  is positive definite, we then have

$$\sum_{\alpha} \omega_{\alpha} (2\omega_f - \omega_{\alpha}) |c'_{\alpha}(\omega_f)|^2 \leq \omega_f^2. \quad (94)$$

Another sum of  $|c'_{\alpha}(\omega_f)|^2$  is provided by using equation (90):

$$\langle \phi | \phi \rangle = \langle \phi_1 | \phi_1 \rangle + \sum_{\alpha} |c'_{\alpha}(\omega_f)|^2 (\omega_f - \omega_{\alpha})^2 / \omega_{\alpha} \omega_f. \quad (95)$$

Since

$$\langle \phi_1 | \phi_1 \rangle = 1 - \langle \psi_1 | \psi_1 \rangle = 1 - \sum_{\alpha} |c'_{\alpha}(\omega_f)|^2 \omega_{\alpha} / \omega_f, \quad (96)$$

we have

$$\langle \phi | \phi \rangle = 1 - \sum_{\alpha} (2 - \omega_f / \omega_{\alpha}) |c'_{\alpha}(\omega_f)|^2 \quad (97)$$

and so

$$\sum_{\alpha} (2 - \omega_f / \omega_{\alpha}) |c'_{\alpha}(\omega_f)|^2 \leq 1. \quad (98)$$

For a fixed multipolarity, equation (82) provides limits for the fraction of the EWSR over all the excited states and an illustration of the consequences of it is given in Section 7. The set of relations, i.e. (83), (89), (94) and (98), similarly provides limits to the sum of the fractions of EWSR over all orders of excitations for a particular state  $|f\rangle$ . Equations (94) and (89) are more stringent limits than equation (83). However, the operators  $N_{\alpha}$  are not readily calculable and hence equations (94) and (89) are not so easily tested.

## 6. Non-factorization of Form Factor

In the analyses of giant resonances, for simplicity it is usually assumed that  $F^l(q, E_f)$ , the longitudinal form factor for a particular multipole, factorizes into a product of two independent functions of excitation energy  $E_f$  and momentum transfer

$q$  (Geramb *et al.* 1975; Torizuka *et al.* 1975; Fukuda and Torizuka 1976; Sasao and Torizuka 1977). However, if one assumes

$$F^l(q, E_f) = k(E_f)g(q) \quad (99)$$

then

$$\begin{aligned} \langle 0 | M_\alpha^\dagger W F^l(q) | 0 \rangle &= \sum_f \langle 0 | M_\alpha^\dagger W | f \rangle \langle f | F^l(q) | 0 \rangle \\ &= \left( \sum_f \langle 0 | M_\alpha^\dagger W | f \rangle k(E_f) \right) g(q), \end{aligned} \quad (100)$$

and so there is a  $q$  dependence independent of  $\alpha$ , which contradicts equation (37). Thus equation (99) cannot hold for all  $q$  and all  $E_f$ .

Since as  $q \rightarrow 0$  the form factor is dominated by the  $\alpha = 0$  term in equation (33), the factorization assumption, as indicated by equation (99), may be a good approximation. It follows that it is possible to construct a single state which will exhaust the EWSR for  $Q_{10}$  and, since  $Q_{10}$  is the static limit of the form factor operator as  $q \rightarrow 0$ , this state exhausts the form factor at small  $q$ . This conclusion led to the idea of a single doorway state dominating the form factor (Deal 1973), which gives results agreeing with experiments at moderate momentum transfers. On the other hand, there cannot be a single giant state  $|g\rangle$  dominating the form factor for all  $q$  (Koo and Tassie 1976), i.e. for which

$$\langle n | F^l(q) | 0 \rangle = 0 \quad \text{for} \quad \langle n | g \rangle = 0 \quad \text{for all } q,$$

as the longitudinal form factor for the  $2^l$  pole  $F^l(q, E_f)$  would then factorize.

It is possible that the form factor has the form (99) over a limited range of  $E_f$ , in which case there must be more scattering strength outside that range of  $E_f$  in order to satisfy the sum rule (82).

## 7. Consequences for Analysis of Experiments

In addition to the well-known isovector E1 giant resonances, the existence of other multipole giant resonances appears conceptually reasonable (Bohr and Mottelson 1975), and there are experimental indications that such multipole giant resonances are indeed possible (Hanna 1976). In this section we show, through the use of the results of the previous sections, that one has to be very careful in making multipole assignments on the basis of some model to the giant resonances observed by inelastic scattering. We want to point out that, on the basis of angular distributions of inelastic scattering, it is not possible to distinguish between, on the one hand, a genuine higher multipole resonance corresponding to a resonance in the strength of  $Q_{10}$  and, on the other hand, a higher order excitation of a lower multipole resonance corresponding to a resonance in the strength of some  $M_{l'\alpha}$  for  $l' < l$ .

As an illustration we consider an isovector E1 transition, making sufficient approximations and assumptions to obtain a simple answer. We neglect the contribution of exchange forces to the EWSR, so that the isovector sum rules can be obtained in the same way as the isoscalar sum rules. The centre-of-mass (c.m.) corrections to the EWSR are expected to be small for heavy nuclei because, although the corrections involve two-particle operators, they are weighted by a factor of  $A^{-1}$  (see Deal 1973). For example, the first-order c.m.-corrected isovector sum rule  $S_{EW}(E1)_{c.m.}$  and the

uncorrected sum rule  $S_{EW}(E1)_u$  are in the ratio

$$S_{EW}(E1)_{c.m.}/S_{EW}(E1)_u = 4NZ/A^2. \quad (101)$$

The difference is small for most nuclei, the correction amounting to 5% at most.

For higher order EWSR, however, the correction terms are complicated and cannot be calculated model independently, and to ensure consistency with equation (101) we substitute  $4NZ/A$  in place of  $A$  in the uncorrected sum rules to allow for the c.m. correction. This procedure could cause errors in the calculation of form factors at large  $q$ , but it suffices for our illustration, especially since the accuracy of our description at large  $q$  will be limited by our use of the form factors for a uniform charge distribution.

In the first Born approximation, for a particular multipole, the differential cross section for longitudinal electric excitation to a state  $|f\rangle$  is given by

$$d\sigma/d\Omega = (d\sigma/d\Omega)_{pt} 4\pi(2l+1) |\langle f | F^l(q) | 0 \rangle|^2, \quad (102)$$

where  $(d\sigma/d\Omega)_{pt}$  is the point nuclear cross section and  $\langle f | F^l(q) | 0 \rangle$  is the  $2^l$  pole form factor. Using the inversion technique, we can write the form factor, for a uniform charge distribution, as (Tassie 1975)

$$\begin{aligned} \langle f | F^l(q) | 0 \rangle &= \{3E_f^{-1}(\hbar^2/2m)(4NZ/A)(4\pi)^{-1}\}^{\frac{1}{2}} R^{-1} \\ &\times \left( c_0(E_f) l^{\frac{1}{2}} j_l(qR) + \sum_{\alpha=1}^l (-)^{\alpha} c_{\alpha}(E_f) (2l+4\alpha+1)^{\frac{1}{2}} j_{l+2\alpha}(qR) \right). \end{aligned} \quad (103)$$

This expression differs from that of Tassie (1975) because of a difference in definition of the  $2^l$  pole form factor. Applying equation (103) to E1 isovector transitions, we obtain

$$\begin{aligned} |\langle f | F^1(q) | 0 \rangle|^2 &= \{3E_f^{-1}(\hbar^2/2m)(4NZ/A)(4\pi)^{-1}\} R^{-2} \\ &\times \{ c_0^2(E_f) (j_1(qR))^2 - 2\sqrt{7} c_0(E_f) c_1(E_f) j_1(qR) j_3(qR) + \dots \\ &\quad + c_1^2(E_f) 7(j_3(qR))^2 - 2\sqrt{77} c_1(E_f) c_2(E_f) j_3(qR) j_5(qR) + \dots \\ &\quad + c_2^2(E_f) 11(j_5(qR))^2 - \dots \}. \end{aligned} \quad (104)$$

The first term is the same as the result of the Goldhaber and Teller (1948) model or the hydrodynamical model while the terms with coefficients  $c_0(E_f) c_1(E_f)$ ,  $c_1^2(E_f)$  and  $c_2^2(E_f)$  have similar  $q$  dependences (for  $qR$  not too large) to the first terms of the quadrupole, octupole and E5 transitions respectively.

A method frequently used in the analysis of electron scattering experiments for higher multipole resonances is to subtract the giant isovector dipole resonance contribution according to a model, e.g. the Goldhaber-Teller model or the hydrodynamical model (Hotta *et al.* 1974; Torizuka *et al.* 1975). In our illustration, this would correspond to the subtraction of the first term of equation (104), leaving a contribution having  $q$  dependences similar to E2, E3 or higher multipole contributions, even though here all the transitions are electric dipole. This shows that such a method of analysis is unreliable.

More realistically, one expects that some of the observed strength in inelastic scattering will be due to higher multipole transitions as well as that due to higher order contributions from dipole transitions. The observed strength of a particular multipole is frequently stated in terms of the fraction it contributes to the multipole EWSR, and so we now show how the higher order terms can cause confusion in such an analysis. Substituting equation (103) in (102), we can obtain, using the orthogonality relation (82),

$$\sum_f \frac{E_f(d\sigma/d\Omega)}{(d\sigma/d\Omega)_{pt}} = \frac{3\hbar^2(2l+1)}{2m} \frac{4NZ}{A} \frac{1}{R^2} \sum_\alpha \sum_f |c_\alpha(E_f)|^2 a_\alpha^2 (j_{l+2\alpha}(qR))^2, \quad (105)$$

where  $a_0 = l^{\frac{1}{2}}$  and  $a_{\alpha \neq 0} = (-)^{\alpha}(2l+4\alpha+1)^{\frac{1}{2}}$ . As an example, for E1 transitions

$$\sum_f \frac{E_f(d\sigma/d\Omega)}{(d\sigma/d\Omega)_{pt}} = \frac{3\hbar^2}{2m} \frac{4NZ}{A} \frac{3}{R^2} \{ (j_1(qR))^2 + 7(j_3(qR))^2 + 11(j_5(qR))^2 + \dots \}. \quad (106)$$

We see that subtraction of the dipole resonance according to the Goldhaber-Teller model can lead to considerable error in determining the contribution to the E3, E5, E7, ... EWSR. However, because of the effect of the orthogonality relation (82) there is no spurious contribution to the E2, E4, ... EWSR due to the incorrect treatment of dipole transitions, although from equation (104) we see that there would be errors in the determination of how the E2, E4, ... strength was distributed in excitation energy.

It should be noted that the possible error due to incorrect subtraction of the dipole strength is large. If the second-order isovector dipole transitions are completely excited, it would give a spurious contribution of 100% to the E3 EWSR. Similarly, the  $n$ th term in equation (106) corresponds to  $3/l$  of the E $l$  EWSR with  $l = 2n - 1$ . Previous warnings (Tassie 1976, 1977) about the possibility of such errors gave no estimate of how serious the errors can be.

In the same way as the subtraction of E1 strength according to the Goldhaber-Teller model can lead to error in assigning higher multipole strength, so also can the subtraction of E $l'$  strength according to the hydrodynamical model lead to incorrect identification of E $l$  strength with  $l' < l$ . The extent of the spurious contribution to the first-order E $l$  EWSR will be  $(2l'+1)/l$ , where  $l = l' + 2\gamma$  ( $\gamma = 1, 2, \dots$ ).

The total of all the spurious contributions from all the lower multipolarity transitions  $l' < l$  can be obtained by explicitly summing over  $l'$  in equation (105). For zero-order ( $\alpha = 0$ ) E $l$  transitions,

$$\sum_f \frac{E_f(d\sigma/d\Omega)}{(d\sigma/d\Omega)_{pt}} = \frac{3\hbar^2}{2m} \frac{4NZ}{A} \frac{l(2l+1)}{R^2} (j_l(qR))^2. \quad (107)$$

The lower multiplicities that can contribute spurious components are those where  $l' + 2\alpha = l$ . Then the total of the spurious contributions is

$$\begin{aligned} \sum_{l'}^{l-2} \sum_f \frac{E_f(d\sigma/d\Omega)}{(d\sigma/d\Omega)_{pt}} &= \frac{3\hbar^2}{2m} \frac{4NZ}{AR^2} \sum_{l'}^{l-2} \sum_\alpha \delta_{l,l'+2\alpha} a_\alpha^2 (2l'+1) (j_{l'+2\alpha}(qR))^2 \\ &= \frac{3\hbar^2}{2m} \frac{4NZ}{AR^2} l(2l+1) \frac{1}{2}(l-1) (j_l(qR))^2. \end{aligned} \quad (108)$$

Comparing equations (107) and (108), we see that the total spurious contribution is  $\frac{1}{2}(l-1)$  of the first-order ( $\alpha = 0$ ) E1 EWSR. Therefore, to identify giant resonances of high multipolarity  $l$ , care must be exercised to untangle the large spurious contributions resulting from transitions of lower multiplicities. For instance, for the E3 and E4 transitions the total spurious contributions coming from the higher order excitation of the E1 and E2 transitions are respectively 100% and 150% of the  $\alpha=0$  E3 and E4 sum rules. The spurious contribution becomes alarmingly larger when the multipolarity of interest is large.

We thus conclude that the multipolarity of transitions cannot be determined unambiguously from measurements of angular distributions of inelastic scattering, and the assignment of spins and parities to giant resonances needs confirmation from other reactions such as ( $\alpha, \gamma$ ) or from angular correlation measurements.

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### Appendix

As noted in Section 3*b*, the results for  $\langle 0 | Q_{l\alpha}^\dagger H Q_{l\beta} | 0 \rangle$  and  $\langle 0 | Q_{l\alpha}^\dagger H F^l(q) | 0 \rangle$  cannot be used directly by taking  $W = H$  and some complication is necessary, in particular for the monopole case ( $l = 0$ ), because  $H$  is not positive definite.

Let the eigenstates of  $H$  be designated by  $|k\rangle$ , and introduce projection operators  $P_0$  and  $P_1$  such that

$$I = P_0 + P_1, \quad (\text{A1})$$

where

$$P_0 = |0\rangle\langle 0|, \quad P_1 = \sum_{k \neq 0} |k\rangle\langle k|. \quad (\text{A2})$$

We now take

$$W = P_0 + H P_1 \quad (\text{A3})$$

and then choose the set of states which satisfy the condition (2) of Section 2 to include the ground state  $|0\rangle$ , as is the case for the states chosen in Section 3*a*, since the states for  $l = 0$  include

$$M_{00} |0\rangle = \text{const.} |0\rangle. \quad (\text{A4})$$

Then, for  $m \neq 0$ , we have

$$\langle 0 | W | m \rangle = \langle 0 | m \rangle = 0. \quad (\text{A5})$$

Although the states  $|m\rangle$  are in general not orthogonal to each other, they are orthogonal to the ground state. Then

$$P_0 |m\rangle = |0\rangle\langle 0 | m \rangle = 0 \quad \text{for } m \neq 0. \quad (\text{A6})$$

Since

$$(P_0 + P_1) |m\rangle = |m\rangle, \quad (\text{A7})$$

from equation (A6) we have

$$P_1 |m\rangle = |m\rangle \quad \text{for } m \neq 0. \quad (\text{A8})$$

The expansion (5) can be written as

$$\begin{aligned} I &= \frac{|0\rangle\langle 0| (P_0 + H P_1)}{\langle 0 | (P_0 + H P_1) | 0 \rangle} + \sum_{m \neq 0} \frac{|m\rangle\langle m| (P_0 + H P_1)}{\langle m | (P_0 + H P_1) | m \rangle} \\ &= |0\rangle\langle 0| + \sum_m \frac{|m\rangle\langle m| H}{\langle m | H | m \rangle}, \end{aligned} \quad (\text{A9})$$

using equation (A8) and taking the normalization of the ground state as  $\langle 0 | 0 \rangle = 1$ .

For inelastic scattering with excitation of the nucleus to the final eigenstate  $|f\rangle$ , equation (33) for the form factor with the choice (A3) for  $W$  yields, using  $\langle f | 0 \rangle = 0$ ,

$$\langle f | F^l(q) | 0 \rangle = \sum_{M_{l\alpha} | 0 \rangle \neq | 0 \rangle} \frac{\langle f | M_{l\alpha} | 0 \rangle \langle 0 | M_{l\alpha}^\dagger H F^l(q) | 0 \rangle}{\langle 0 | M_{l\alpha}^\dagger H M_{l\alpha} | 0 \rangle}. \quad (\text{A10})$$

For  $l \neq 0$ ,

$$\langle 0 | M_{l\alpha} | 0 \rangle = 0, \quad (\text{A11})$$

all the states  $M_{l\alpha} | 0 \rangle$  are orthogonal to the ground state and equation (29) becomes

$$\langle 0 | M_{l\alpha}^\dagger H M_{l\beta} | 0 \rangle = 0 \quad \text{for } \alpha \neq \beta \quad (\text{A12})$$

(the  $A_{\alpha\beta}$  are real). The form factor then is

$$\langle f | F^l(q) | 0 \rangle = \sum_{\alpha=0} \frac{\langle f | M_{l\alpha} | 0 \rangle \langle 0 | M_{l\alpha}^\dagger H F^l(q) | 0 \rangle}{\langle 0 | M_{l\alpha}^\dagger H M_{l\alpha} | 0 \rangle}. \quad (\text{A13})$$

For monopoles ( $l = 0$ ) there is a slight complication, since

$$\langle 0 | M_{00} | 0 \rangle \neq 0 \quad (\text{A14})$$

(except for the isovector case in self-conjugate nuclei discussed below), and equation (29) becomes

$$\langle 0 | M_{00}^\dagger M_{0\alpha} | 0 \rangle \equiv \langle 0 | M_{0\alpha} | 0 \rangle = 0 \quad \text{for } \alpha \neq 0, \quad (\text{A15a})$$

$$\langle 0 | M_{0\alpha}^\dagger H M_{0\beta} | 0 \rangle = 0 \quad \text{for } \alpha \neq \beta. \quad (\text{A15b})$$

The form factor then is

$$\langle f | F^0(q) | 0 \rangle = \sum_{\alpha=1} \frac{\langle f | M_{0\alpha} | 0 \rangle \langle 0 | M_{0\alpha}^\dagger H F^0(q) | 0 \rangle}{\langle 0 | M_{0\alpha}^\dagger H M_{0\alpha} | 0 \rangle}. \quad (\text{A16})$$

In the analysis by Tassie (1975), the treatment implied for the monopole case is to take

$$M'_{01} = (4\pi)^{-\frac{1}{2}} \sum_i r_i^2 \quad (\text{A17})$$

and

$$\langle 0 | M_{0\alpha}^\dagger H M'_{0\beta} | 0 \rangle = 0 \quad \text{for } \alpha \neq \beta, \alpha \geq 1, \beta \geq 1. \quad (\text{A18})$$

However, we have

$$M_{0\alpha} = M'_{0\alpha} - \langle 0 | M'_{0\alpha} | 0 \rangle, \quad \alpha \geq 1, \quad (\text{A19})$$

and substituting this relation into equation (A16) and using (38) we obtain

$$\langle f | F^0(q) | 0 \rangle = \sum_{\alpha=1} \frac{\langle f | M'_{0\alpha} | 0 \rangle \langle 0 | M_{0\alpha}^\dagger H F^0(q) | 0 \rangle}{\langle 0 | M_{0\alpha}^\dagger H M'_{0\alpha} | 0 \rangle}, \quad (\text{A20})$$

which agrees with the result given by Tassie (1975).

Giant resonances are frequently taken to be the appropriate multipole operator acting on the ground state, but we note that the giant isoscalar monopole state is not  $M'_{01} | 0 \rangle$  but

$$M_{01} | 0 \rangle = (M'_{01} - \langle 0 | M'_{01} | 0 \rangle) | 0 \rangle, \quad (\text{A21})$$

as given by Kirson (1976).

Equation (A16) holds also for the isovector case in self-conjugate nuclei ( $T_3 = 0$ ) because, although the inequality (A14) does not apply, we have

$$M_{00} | 0 \rangle = 2T_3 | 0 \rangle = 0. \quad (\text{A22})$$

