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The Motion of Spinless Particles in General Relativity

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Abstract

A technique is developed, which uses a general energy–momentum tensor, to derive the equations of motion in general relativity. This enables the geodesic equations for spinless particles to be deduced.

Introduction

Nearly 40 years ago, Einstein *et al.* (1938; hereinafter referred to as EIH) showed that the nonlinearity of the general relativistic field equations forces a 'test particle' to move on a geodesic of the space–time. Since then, numerous studies of this problem in general relativity have been published, and the EIH technique has been used successfully to re-establish the credibility of the nonsymmetric unified field theories (Klotz and Russell 1972, 1973; Russell and Klotz 1972).

Nevertheless, the question which remains unresolved is the apparent non-uniqueness of the way in which matter and, in particular, a test particle are to be defined. Should matter be regarded as a singularity of the manifold? This question is especially relevant to unified field theories, in which there are good reasons for excluding singular solutions (see Appendix II of Einstein 1956). In the present paper we apply a method attributed to A. Schild (in unpublished lecture notes delivered at Liverpool in 1963) to find conditions which ensure that a test particle (described merely by a conserved, symmetric energy tensor T) follows a geodesic path.

General Equations of Motion

Consider a time-like (x^4) curve $\mathscr{L}$ in a Riemannian manifold V_4. Using Fermi–Walker (FW) transport, it is possible to choose the coordinates in such a way that $\mathscr{L}$ is given elsewhere by

$$x^i = 0, \qquad x^4 = s,$$

where s is the arc parameter of $\mathscr{L}$ (Synge 1960). Throughout this paper, Latin indices go from 1 to 3, while Greek indices go from 1 to 4. Moreover, if $P(0,0,0,s)$ is a point of $\mathscr{L}$, we can represent any point Q on the space-like geodesics, emanating from P, by the coordinates

$$x^i = n^i \sigma, \qquad x^4 = s.$$

Here n^i is the direction vector of the geodesic at P and σ is the arc distance from P

to Q. Under FW transport, the space-like hypersurface normal to $\mathscr{L}$ at P remains normal to $\mathscr{L}$ along the whole length of $\mathscr{L}$. Effectively, a coordinate system has been constructed which is geodesic in the vicinity of $\mathscr{L}$.

We now consider the motion of a small mass (our test particle) in the field of more massive bodies. The test particle will be characterized by a narrow tube surrounding $\mathscr{L}$, the energy–momentum tensor density $\mathscr{T}^{\mu\nu}$ of the particle being zero outside the tube:

$$\mathscr{T}^{\mu\nu} = (-g)^{\frac{1}{2}} \mathrm{T}^{\mu\nu} = 0, \qquad \text{outside } \mathscr{L}. \tag{1}$$

The metric in the absence of the test particle will be considered to be known and, since the particle moves outside the massive bodies, the energy–momentum tensors of the massive bodies are zero near $\mathscr{L}$.

The affine connection $\Gamma'^{\lambda}_{\mu\nu}$ (Christoffel symbol) consists of a term $\Gamma^{\lambda}_{\mu\nu}$ due to the massive bodies and a correction due to the test particle:

$$\Gamma'^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\nu}. \tag{2}$$

Our approximation of a small test particle will mean that we neglect the term $\delta\Gamma^{\lambda}_{\mu\nu}$. Hence, the affine connection is that of the background space. The components of the affine connection at a point $Q(x^a, s)$, in the geodesic three-space of the point $P(0, s)$ on $\mathscr{L}$, can be given as a Taylor series

$$\Gamma^{\lambda}_{\mu\nu}(Q) = \Gamma^{\lambda}_{\mu\nu}(0) + \Gamma^{\lambda}_{\mu\nu,a}(0)\, x^a + \dots, \tag{3}$$

where the affine connection is evaluated at Q or on $\mathscr{L}$ according as its argument is Q or 0 respectively.

In the interior of the world tube of the test particle we have

$$\mathscr{T}^{\mu\nu}{}_{;\nu} = \mathscr{T}^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\alpha\beta}\, \mathscr{T}^{\alpha\beta} = 0 \tag{4}$$

and hence

$$(x^a \mathscr{T}^{\mu\nu})_{,\nu} = \mathscr{T}^{\mu a} - \Gamma^{\mu}_{\alpha\beta}\, x^a\, \mathscr{T}^{\alpha\beta}, \tag{5a}$$

$$(x^a x^b\, \mathscr{T}^{\mu\nu})_{,\nu} = x^b\, \mathscr{T}^{\mu a} + x^a\, \mathscr{T}^{\mu b} - \Gamma^{\mu}_{\alpha\beta}\, x^a x^b\, \mathscr{T}^{\alpha\beta}. \tag{5b}$$

We can define the space integrals over the three-volume V perpendicular to $\mathscr{L}$ at P and, on using the boundary condition (1) and Gauss's theorem, we obtain

$$\begin{aligned}\int_V (x^a x^b \dots \mathscr{T}^{\mu\nu})_{,\nu}\, \mathrm{d}v &= \int_{\partial V} x^a x^b \mathscr{T}^{\mu c}\, \mathrm{d}S + \int_V (x^a x^b \dots \mathscr{T}^{\mu 4})_{,4}\, \mathrm{d}v, \\ &= \frac{\partial}{\partial x^4} \int_V x^a x^b \dots \mathscr{T}^{\mu\nu}\, \mathrm{d}v.\end{aligned} \tag{6}$$

We may also define the moments $M^{ab\dots c\mu\nu}$ at P by

$$M^{ab\dots c\mu\nu}(s) = \int_V x^a x^b \dots x^c\, \mathscr{T}^{\mu\nu}\, \mathrm{d}v.$$

The moments are defined along $\mathscr{L}$ only, and are functions of s.

Our first assumption about the structure of the test particle is that it is a 'pole–dipole' object, that is, $M^{ab\mu\nu}$ and the higher moments about $\mathscr{L}$ are zero. We will assume that the 'mass' M^{44} of the particle is nonzero:

$$m = M^{44} \neq 0 . \tag{7}$$

Then, using equations (3), (4), (5a) and (5b), we obtain as the equations of motion

$$\partial M^{\mu 4}/\partial x^4 = -\Gamma^{\mu}_{\alpha\beta}(0)\, M^{\alpha\beta} - \Gamma^{\mu}_{\alpha\beta,a}(0)\, M^{a\alpha\beta} , \tag{8}$$

$$\partial M^{a\mu 4}/\partial x^4 = M^{\mu a} - \Gamma^{\mu}_{\alpha\beta}(0)\, M^{a\alpha\beta} . \tag{9}$$

Now, by equation (5b) we have

$$\begin{aligned} M^{ab\mu} &= \int_V x^a\, \mathscr{T}^{b\mu}\, \mathrm{d}v \\ &= \int_V \{x^b\, \mathscr{T}^{\mu a} + (x^a x^b\, \mathscr{T}^{\mu\nu})_{,\nu} + \Gamma^{\mu}_{\alpha\beta}\, x^a x^b\, \mathscr{T}^{\alpha\beta}\}\, \mathrm{d}v . \end{aligned}$$

From equations (3) and (6) we then obtain

$$\begin{aligned} M^{ab\mu} &= -M^{ba\mu} + \partial M^{ab\mu 4}/\partial x^4 + \Gamma^{\mu}_{\alpha\beta}(0)\, M^{ab\alpha\beta} + \Gamma^{\mu}_{\alpha\beta,c}(0)\, M^{abc\alpha\beta} + \dots \\ &= -M^{ba\mu} , \end{aligned}$$

as the higher-order moments are zero. Since $\mathscr{T}^{ab}$ is symmetric we have

$$M^{abc} = M^{acb} ,$$

and thus

$$M^{abc} = -M^{bac} = -M^{bca} = M^{cba} = M^{cab} = -M^{acb} = -M^{abc} .$$

Therefore

$$M^{abc} = 0 . \tag{10}$$

We introduce the following notation for the remaining nonzero moments

$$S^{ab} = 2M^{ab4} = -2M^{ba4} ,$$

$$Q^a = M^{a44} , \qquad M^a = M^{a4} , \qquad m = M^{44} ,$$

where S^{ab} is the 'spin' and m is the 'mass'. In the Fermi coordinate system we have

$$g_{\mu\nu}(0) = \mathrm{diag}(-1, -1, -1, 1) = \eta_{\mu\nu} , \tag{11}$$

$$\Gamma^{i}_{jk}(0) = 0 , \qquad g_{44,4}(0) = 0 , \tag{12}$$

$$\Gamma^{4}_{ij}(0) = \Gamma^{i}_{4j}(0) = \Gamma^{4}_{44}(0) = 0 , \tag{13}$$

$$\Gamma^{4}_{4i}(0) = -\Gamma_{44,i}(0) = \tfrac{1}{2} g_{44,i}(0) , \tag{14}$$

$$\Gamma^{4}_{44,i}(0) = \tfrac{1}{2} g_{44,i4}(0) = \Gamma^{4}_{4i,4}(0) . \tag{15}$$

The four-velocity u^μ and its covariant derivative along $\mathscr{L}$ (four-acceleration) are given by

$$u^\mu = (0,0,0,1) \qquad \text{and} \qquad \mathrm{D}u^\mu/\delta s = \Gamma^\mu_{44}(0) = (a^i, 0) \tag{16}$$

respectively. We now use an overhead dot to denote differentiation with respect to s. Equations (8) and (9) are seen to reduce to 10 ordinary first-order differential equations:

$$\dot{M}^a = -mp^a - p^a{}_{,b}Q^b - q^a_{bc}S^{bc}, \qquad \dot{m} = 2p^a M_a + \dot{p}^a Q_a, \tag{17a, b}$$

$$M^{ab} = p^a Q^b + p^b Q^a, \qquad \dot{S}^{ab} = 2p^b Q^a - 2p^a Q^b, \qquad \dot{Q}^a = M^a + p_b S^{ab}, \tag{18a, b, c}$$

where $\eta_{\mu\nu}$ is used to raise and lower indices, and

$$p^a = \Gamma^a_{44}(0), \qquad q^a_{bc} = \Gamma^a_{c4,b}(0).$$

Geodesic Motion

In his unpublished lecture notes of 1963, Schild observed that if

$$Q^a \equiv 0 \tag{19}$$

then

$$M^a = -p_b S^{ab},$$

which follows here directly from equation (18c). Substituting this result into equation (17b) we have $\dot{m} = -2p^a p^b S_{ab} = 0$, since S^{ab} is skew symmetric. Therefore

$$m = m_0 = \text{const}. \tag{20}$$

For a 'spinless particle' ($S^{ab} = 0$) satisfying Schild's condition (19), equations (17a), (18c) and (20) imply that

$$p^a = 0, \qquad M^a = 0.$$

From the equations (16) we also have in Fermi coordinates

$$\mathrm{D}u^\mu/\delta s = p^\mu = 0,$$

and hence $\mathscr{L}$ corresponds to a geodesic of the space–time.

We now consider the integrability of equations (17) and (18) in terms of functions of $x^4 = s$, regular at the origin $s = 0$. If we assume the initial conditions at $s = 0$ to be

$$S^{ab} = Q^a = M^a = p^a = 0, \qquad m = m_0,$$

and expand these functions as power series in s of the form

$$u = u_0 + u_1 s + u_2 s^2 + \ldots,$$

an inspection of equations (17) and (18) shows that we must have

$$m = m_0 + m_4 s^4 + \ldots, \qquad M^a = M^a_2 s^2 + M^a_3 s^3 + \ldots,$$

$$Q^a = Q^a_3 s^3 + Q^a_4 s^4 + \ldots, \qquad S^{ab} = S^{ab}_6 s^6 + S^{ab}_7 s^7 + \ldots, \qquad p^a = p^a_1 s + p^a_2 s^2 + \ldots.$$

Instead of Schild's condition (19), it therefore seems more appropriate to let

$$S^{ab} \equiv 0$$

and assume $Q^a \not\equiv 0$, that is, we consider an *a priori* spinless distribution of matter. The equations of motion are now

$$\dot{M}^a = -mp^a - p^a{}_{,b} Q^b, \qquad \dot{m} = 2p^a M_a + \dot{p}^a Q_a, \tag{21a, b}$$

$$M^{ab} = p^a Q^b + p^b Q^a, \qquad p^a Q^b - p^b Q^a = 0, \qquad \dot{Q}^a = M^a. \tag{22a, b, c}$$

Equation (22b) implies that either

$$p^a = 0 \qquad \text{or} \qquad Q^a = \lambda p^a, \tag{23a,b}$$

where λ is a scalar function of s. If equation (23b) holds, it follows from equations (21) and (22) that

$$\mathrm{d}^2(\lambda p^a)/\mathrm{d}s^2 = -mp^a - \lambda p^a{}_{,b} p^b. \tag{24}$$

We now assume the initial conditions at $s = 0$ to be

$$Q^a = M^a = 0. \tag{25}$$

We then have

$$Q^a = Q^a_1 s + Q^a_2 s^2 + \dots, \qquad M^a = M^a_1 s + M^a_2 s^2 + \dots,$$

and substitution into equation (22c) demands that $Q^a_1 = 0$. Hence, from equations (21b) and (23b), we see that $\lambda_0 = \lambda_1 = m_1 = 0$.

We conclude that

$$p^a = p^a_0 + p^a_1 s + p^a_2 s^2 + \dots, \qquad m = m_0 + m_2 s^2 + \dots, \tag{26a, b}$$

$$\lambda = \lambda_2 s^2 + \lambda_3 s^3 + \dots, \qquad Q^a = Q_2 s^2 + Q_3 s^3 + \dots. \tag{27a, b}$$

Substituting expansions (26a), (26b) and (27a) into equation (24) and equating constant terms gives

$$2\lambda_2 p^a_0 = -m_0 p^a_0.$$

Hence, if $m_0 \neq 2\lambda_2$, it follows that $p^a_0 = 0$. If we assume that

$$p^a_i = 0, \qquad i \leqslant l-1,$$

the equating of coefficients of s^l in equation (24) produces

$$(l+2)(l+1)\lambda_2 p^a_l = 2m_0 p^a_l.$$

Therefore, by induction, it follows that, if $2m_0 \neq -(l+2)(l+1)\lambda_2$ holds for all integers ($l \geqslant 0$) then

$$p^a \equiv 0.$$

Thus, even under the less stringent conditions (25), a test particle will move on a geodesic path.

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