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Dirac Particles in the Minkowski Space with Torsion

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Abstract

After presenting a simple derivation of the covariant derivative of the Dirac spinor functions, the Dirac equation in the space of constant torsion, Minkowski metric and straight line geodesics is considered. Solutions of the equation are given, showing the particular way in which the energy degeneracy of the states with different spin projections is removed in the presence of torsion.

1. Introduction

While the idea of considering torsion as a physical property of space and time dates back to the work of Cartan (1922), the demonstrated reliability of Einstein's torsion-free theory made any discussion of torsion of rather an academic nature. Recently, however, there has been some renewed interest in torsion (Hehl 1973*a*, 1973*b*) partly caused by the gauge field formulation of the relativity theory, in which the consideration of the tetrad field as gauge potentials leads naturally to connections with torsion.

In the present article we discuss neither the physical nor the mathematical origin of torsion, only the behaviour of spin $\frac{1}{2}$ particles in a torsion field. This can help in understanding interaction between torsion and spin, as well as establishing upper limits on the strength of the torsion field if it exists in nature. We consider only connections with a constant 'pure' torsion that lead to the Minkowski metric and straight line geodesics.

In Section 2 we present a direct derivation of the covariant differentiation of spinors, in Section 3 we give the form of the Dirac equation in the space with pure torsion, and in Section 4 we discuss plane wave solutions of that equation.

2. Covariant Differentiation of Dirac Spinors

While there exist several derivations of covariant differentiation of spinors in the literature (e.g. Majumdar 1962; Brill and Cohen 1966; Hehl 1973*a*, 1973*b*), we feel that it is worth while to describe the following method because of its direct relationship to differentiation of vectors.

The usual covariant derivative of a vector is written as

$$A_{\mu;\nu} = \partial_\nu A_\mu - \Gamma^{\sigma}_{\nu\mu} A_\sigma, \tag{1}$$

where the $\Gamma^{\sigma}_{\nu\mu}$ are components of a connection with respect to a general system of coordinates. If h^i_μ are tetrad fields connecting general coordinates with the unholonomic Minkowski coordinates, we define A_i by

$$A_\mu = h^i_\mu A_i \qquad \text{or} \qquad A_i = h^\mu_i A_\mu,$$

where

$$h^i_\mu h^\sigma_i = \delta^\sigma_\mu \qquad \text{and} \qquad h^i_\mu h^\mu_k = \delta^i_k.$$

The covariant derivative of A_i can be written as

$$A_{i;\nu} = h^\mu_i A_{\mu;\nu} = \partial_\nu A_i - \Gamma^k_{\nu i} A_k, \tag{2}$$

where the relation

$$\Gamma^k_{\nu i} = \Gamma^\sigma_{\nu\mu} h^k_\sigma h^\mu_i - h^\mu_i(\partial_\nu h^k_\mu) \tag{3}$$

follows from comparison of equations (1) and (2). As we assume that the connection is metric, we have

$$\partial_\sigma g_{\mu\nu} - \Gamma^\rho_{\sigma\mu} g_{\rho\nu} - \Gamma^\rho_{\sigma\nu} g_{\mu\rho} = 0$$

or

$$\partial_\sigma g_{ik} - \Gamma^l_{\sigma i} g_{lk} - \Gamma^l_{\sigma k} g_{il} = 0, \tag{4}$$

where g_{ik} is the Minkowski metric: $g_{ik} = 0$ if $i \neq k$ and $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. Equation (4) yields

$$\Gamma^l_{\sigma i} g_{lk} = -\Gamma^l_{\sigma k} g_{il} \qquad \text{or} \qquad \Gamma_{\sigma ik} = -\Gamma_{\sigma ki}.$$

Now consider A_k, $k = 0, 1, 2, 3$, as a column vector $\mathscr{A}$ and write equation (2) in the form

$$\mathscr{A}_{;\nu} = \partial_\nu \mathscr{A} - \tfrac{1}{2}\Gamma_{\nu ik} B^{ik} \mathscr{A}, \tag{5}$$

where the B^{ik} are 4×4 matrices of the form

$$[B^{ik}]^m_n = \delta^i_n g^{km} - \delta^k_n g^{im}.$$

Equations (2) and (5) are equivalent, as

$$\begin{aligned}\tfrac{1}{2}\Gamma_{\nu ik}[B^{ik}]^m_n A_m &= \tfrac{1}{2}\Gamma_{\nu nk} g^{km} A_m - \tfrac{1}{2}\Gamma_{\nu in} g^{im} A_m \\ &= \Gamma_{\nu nk} g^{km} A_m = \Gamma^m_{\nu n} A_m.\end{aligned}$$

The matrices B^{ik} are generators of the Lorentz group in the 4×4 real representation. If a covariant derivative of a function with different transformation properties is required, the generators are simply written in the appropriate representation, while the components of the connection $\Gamma_{\nu ik}$ remain unchanged. Thus the covariant derivative of the Dirac spinor is

$$\psi_{;\nu} = \partial_\nu \psi - \tfrac{1}{2}\Gamma_{\nu ik} C^{ik} \psi,$$

where

$$C^{ik} = \tfrac{1}{4}(\gamma^i\gamma^k - \gamma^k\gamma^i)$$

and the γ^i, $i = 0, 1, 2, 3$, are the Dirac matrices satisfying $\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}$.

3. Minkowski Space with Torsion

We consider a space with a non-flat connection, but with a Minkowski metric. Tetrads can be chosen as Kronecker deltas, and we do not use Greek indices anymore, as the coordinates will be the Minkowski coordinates from now on. If we further require that the geodesics are straight lines, the components of the connection must have zero symmetric parts.

Hence we have

$$\Gamma_{ikl} = -\Gamma_{kil}$$

but, as the connection is still metric, we also have

$$\Gamma_{ikl} = -\Gamma_{ilk}.$$

There are four independent components of such a connection. One can define a four-vector M^i by

$$\Gamma_{jkl} = \varepsilon_{ijkl} M^i, \tag{6}$$

where ε_{ijkl} is the fully antisymmetric tensor (our convention is that $\varepsilon_{0123} = +1$).

Under Lorentz transformations, Γ_{jkl} transforms like a third-order covariant tensor, and M^i like a contravariant vector. Substitution into the Dirac equation gives

$$\mathrm{i}\gamma^i\{\partial_i - \tfrac{1}{2}\Gamma_{ikl}\tfrac{1}{4}(\gamma^k\gamma^l - \gamma^l\gamma^k)\}\psi + m\psi = 0$$

or

$$\mathrm{i}\gamma^i\partial_i\psi + m\psi - \tfrac{1}{4}\mathrm{i}M^i\varepsilon_{ijkl}\gamma^j\gamma^k\gamma^l\psi = 0.$$

Further

$$\varepsilon_{ijkl}\gamma^j\gamma^k\gamma^l = 6g_{ik}\gamma^k\gamma^5,$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, so that the Dirac equation reads

$$\mathrm{i}\gamma^k\partial_k\psi + m\psi - \mathrm{i}T^j g_{jn}\gamma^n\gamma^5\psi = 0, \tag{7}$$

with

$$T^j = \tfrac{3}{2}M^j.$$

It can be readily deduced from the requirement of covariance that under space inversion the time component of T^j changes its sign while the spatial part remains unchanged, and the situation is reversed under time inversion. Of course, this conforms with the definition (6) of the torsion vector, where M^0 is related to the spatial components of the connection, while M^k, $k = 1, 2, 3$, is related to the components with one time index and two spatial indices.

4. Solutions of Dirac Equation with Torsion

We shall now find the plane wave solutions of equation (7), assuming that the T^j are constant. If we have

$$\psi(x) = u(p)\exp(\mathrm{i}x^k p_k),$$

equation (7) yields

$$(\gamma^k p_k + \mathrm{i}\gamma^k\gamma^5 T_k - m)u(p) = 0. \tag{8}$$

This system of four homogeneous linear equations has nontrivial solutions only when the determinant of the system is zero. Let

$$S = \gamma^k p_k + \mathrm{i}\gamma^k\gamma^5 T_k - m.$$

Then $\det S$ is a fourth-order polynomial in the components of p and T (and m) such that it is a Lorentz-invariant expression. We shall find it in a special coordinate system and deduce the general form from the result.

In the rest system of the particle with the third axis in the direction of the spatial part of the torsion vector,

$$S = \gamma^0 p_0 + \mathrm{i}\gamma^0\gamma^5 T_0 + \mathrm{i}\gamma^3\gamma^5 T_3 - m.$$

As

$$\gamma^5 S\gamma^5 = \gamma^0 p_0 + \mathrm{i}\gamma^0\gamma^5 T_0 + \mathrm{i}\gamma^3\gamma^5 T_3 + m$$

and

$$\det(S\gamma^5 S\gamma^5) = (\det S)^2,$$

we have

$$(\det S)^2 = \det(p_0^2 - m^2 - T_0^2 + T_3^2 + 2\mathrm{i}\gamma^0\gamma^3\gamma^5 p_0 T_3).$$

Repeating the procedure with γ^1 replacing γ^5 (γ^1 anticommutes with $\gamma^0\gamma^3\gamma^5$) we obtain

$$(\det S)^4 = \det[(p_0^2 - m^2 - T_0^2 + T_3^2)^2 - 4p_0^2 T_3^2],$$

implying that $\det S$ is the expression in the square brackets. (Ambiguity of the sign is removed by realizing that $\det(p_0\gamma^0) = +p_0^4$.) Changing back to a general system of coordinates we have

$$\det S = (p^2 - m^2 - T^2)^2 + 4\{p^2T^2 - (pT)^2\}, \tag{9}$$

where

$$p^2 = g^{ik}p_i p_k, \qquad T^2 = g^{ik}T_i T_k, \qquad (pT) = g^{ik}p_i T_k.$$

The plane wave solutions must satisfy the equation $\det S = 0$, leading in general to four roots for p_0. We see that the interaction with torsion can remove degeneracy of the energy levels for the two states with different spin directions. We notice that the degeneracy remains when the four-momentum of the particle is aligned with the torsion, i.e. when $T_k = \kappa p_k$. In this case $p^2 - m^2 - T^2 = 0$ determines the energy levels, and the presence of torsion just makes an effective correction to the mass of the particle in the form $p^2 = m^2/(1-\kappa^2)$.

In the particular representation of the Dirac matrices:

$$\gamma^0 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \qquad \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} 0 & -\mathrm{i}\mathbf{I} \\ -\mathrm{i}\mathbf{I} & 0 \end{bmatrix},$$

where $\mathbf{I}$ is the 2×2 unit matrix, $k = 1, 2, 3$ and

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

equation (8) yields

$$(-p_0 + m - \sigma^k T_k)u_1 - (T_0 + \sigma^k p_k)u_2 = 0, \tag{10a}$$

$$(T_0+\sigma^k p_k)u_1+(p_0+m+\sigma^k T_k)u_2 = 0, \tag{10b}$$

where

$$u(p) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Equation (10b) yields the relation between the 'large' and 'small' components of $u(p)$:

$$u_2 = -\frac{(p_0+m-\sigma^k T_k)(T_0+\sigma^k p_k)}{(p_0+m)^2-\mathscr{T}^2}u_1, \tag{11}$$

where

$$\mathscr{T}^2 = \sum_{k=1}^{3} T_k^2.$$

Alternatively, equation (10a) yields

$$u_1 = \frac{(-p_0+m+\sigma^k T_k)(T_0+\sigma^k p_k)}{(m-p_0)^2-\mathscr{T}^2}u_2. \tag{12}$$

In the specific system in which $\det S$ was evaluated, u_1 (or alternatively u_2) may be chosen as the eigenvectors of the third component of the spin:

$$u_1^{(+)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad u_1^{(-)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The energy levels for the two spin states are not in general identical. In fact $u_1^{(+)}$ corresponds to

$$p_0 = T_3+(m^2+T_0^2)^{\frac{1}{2}},$$

while $u_1^{(-)}$ corresponds to

$$p_0 = -T_3+(m^2+T_0^2)^{\frac{1}{2}}.$$

A similar situation holds for the negative energies, where

$$u_2^{(+)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{corresponds to} \quad p_0 = T_3-(m^2+T_0^2)^{\frac{1}{2}},$$

$$u_2^{(-)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad\qquad\qquad p_0 = -T_3-(m^2+T_0^2)^{\frac{1}{2}},$$

and u_1 is given by equation (12).

5. Final Remarks

The expression (9) for $\det S$ has been obtained via the rest system of the particle. Its validity is, however, more general and, in principle, solutions of $\det S = 0$ by space-like energy–momentum vectors are possible. Nevertheless, time-like energy–momentum vectors still play a special role. Once it is assumed that the four-momentum is time-like, four real solutions for p_0 always exist independently of the magnitude of the torsion. Space-like solutions do not exist for weak torsion fields when $m \neq 0$.

The case $m = 0$ would deserve a separate discussion. We shall just comment that the two-component theory is still possible (equation (8) is still γ^5-invariant), but in general the four-momentum of the massless particle does not have to be a null-vector when torsion is present, and there are still four independent solutions of the Dirac equation.

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References

Brill, D. R., and Cohen, J. M. (1966). *J. Math. Phys. (N.Y.)* **7**, 238.
Cartan, E. (1922). *C. R. Acad. Sci.* **174**, 593.
Hehl, F. W. (1973*a*). *Gen. Relativity Gravitation* **4**, 333.
Hehl, F. W. (1973*b*). *Gen. Relativity Gravitation* **5**, 491.
Majumdar, S. D. (1962). *Phys. Rev.* **126**, 2227.

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