Aust. J. Phys., 1978, **31**, 353–65

A Lagrangian with Higher Derivatives for Arbitrary Spin Fields

E. A. Jeffery

Division of Applied Organic Chemistry, CSIRO,
P.O. Box 4331, Melbourne, Vic. 3001.

Abstract

A Lagrangian that generalizes the Dirac spin $\frac{1}{2}$ Lagrangian is given. This contains up to $2j$th order derivatives for spin j, but can be readily quantized for free fields. Noether's theorem is generalized and the results are shown to conform with other known quantization procedures for arbitrary spin particles. Use of the higher derivative Lagrangian overcomes some of the difficulties encountered by others in rigorously deriving a covariant Feynman propagator.

1. Introduction

In a recent paper (Jeffery 1978) component minimization of the Bargmann–Wigner wavefunction was shown to yield the higher derivative equations

$$(-E+\mu)^{2j}\,\psi_{1,2}^{[2j]} = (-\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[2j]}\,\psi_{3,4}^{[2j]}, \tag{1a}$$

$$(E+\mu)^{2j}\,\psi_{3,4}^{[2j]} = (\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[2j]}\,\psi_{1,2}^{[2j]}, \tag{1b}$$

where $\boldsymbol{P}$ and E are $-\mathrm{i}\nabla$ and $\mathrm{i}\,\partial/\partial t$, μ is the rest mass, $\boldsymbol{\sigma}$ is the Pauli spin matrix vector and j is the spin quantum number. The symbol $[2j]$ indicates the $(2j+1)^2$ dimensional $2j$th induced matrix or the corresponding $2j$th rank totally symmetric spinor. A definition of induced matrices, together with the explicit values of $(\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[2]}$, $(\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[3]}$ and $\psi_{1,2}^{[2j]}$, are set out in the Appendix (for a more detailed treatment of induced and other invariant matrices see Littlewood 1958).

Equations (1) are equivalent to those given by Joos (1962) and Weinberg (1964). They are also directly derivable from the Dirac spin $\frac{1}{2}$ equations and invariant matrix theory upon noting that $\psi_{\alpha\beta\ldots\tau}$ transforms in the same way as a symmetrized product of single spinors $\phi_\alpha, \chi_\beta, \ldots, \varepsilon_\tau$.

To remove nonphysical solutions, equations (1) were transformed to

$$(-ES+R)\psi_{1,2}^{[2j]} = (-\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[2j]}\,\psi_{3,4}^{[2j]}, \tag{2a}$$

$$(ES+R)\psi_{3,4}^{[2j]} = (\boldsymbol{P}\,.\,\boldsymbol{\sigma})^{[2j]}\,\psi_{1,2}^{[2j]}, \tag{2b}$$

where

$$R = \sum_{l=0}^{\infty} \frac{(2j)!\,\mu^{2j}}{(2j-2l)!\,(2l)!}\left(1+\frac{\boldsymbol{P}^2}{\mu^2}\right)^{l}, \qquad S = \sum_{l=0}^{\infty} \frac{(2j)!\,\mu^{2j-1}}{(2j-2l-1)!\,(2l+1)!}\left(1+\frac{\boldsymbol{P}^2}{\mu^2}\right)^{l}. \tag{3}$$

The Hamiltonian form of equations (2) is shown in Section 2 below to be equivalent, within a similarity transformation, to that of Weaver *et al.* (1964) and that of Mathews (1966).

There are several thorough discussions of second quantization of $2(2j+1)$ component fields in the literature (e.g. Weinberg 1964, 1969; Mathews and Ramakrishnan 1967; Nelson and Good 1968; Weaver 1968; Mathews 1971). However, these have not developed a Lagrangian formalism in a rigorous manner. Weinberg, for example, takes an intermediate path between a Lagrangian formalism and a pure S-matrix approach. Where Lagrangians have been developed in detail for arbitrary spin equations, the fields have often contained more than $2(2j+1)$ components and the differential operators have been linear; examples are Hurley's (1971, 1974) equations which contain $6j+1$ components and Frank's (1973) treatment which is based on the Gel'fand–Naimark formalism for linear representation of the Lorentz group. By contrast Belinicher (1975) has developed a Lagrangian formalism and Feynman rules from six axioms; although not linear the equations contain only up to second powers in the momentum operators. Singh and Hagen (1974) have extended the original Fierz–Pauli program for constructing high spin Lagrangians, by introducing auxiliary conditions which disappear in the free-field case. Upon taking the Galilean limit (i.e. low velocities) their equations give those of Hagen and Hurley (1970). Doria (1977) has developed a free-field Lagrangian by using the properties of Sauter spinors. His equations contain linear operators and in general have wave-functions with more than $2(2j+1)$ components.

There are not many treatments of higher derivative Lagrangians in the literature but a notable exception is the work of Coelho de Souza and Rodrigues (1969). These authors have considered canonical formalism, including Hamilton's equation, transformation theory and Poisson brackets for classical field theory with higher time and/or spatial derivatives. Their work will serve as a spring-board for the Lagrangian formalism described here.

The object of this paper is to show that equations (2) can be second-quantized within a rigorous Lagrangian formalism by a procedure paralleling the spin $\frac{1}{2}$ theory. Emphasis is placed on those aspects of field theory which are a consequence of the higher derivative Lagrangian, and areas adequately covered by other theories are avoided. Interacting fields are not considered.

A Hamiltonian form of the single-particle equations (2) is described first in Section 2 because this suggests a suitable generalization of the Dirac Lagrangian (Section 3). The Euler–Lagrange equations of motion, commutation relationships, Noether's theorem, the field-theoretic Hamiltonian and the Feynman propagator are then discussed in the subsequent sections.

2. Single-particle Hamiltonian

Definition of an inverse operator S^{-1} by

$$SS^{-1}\psi(x) = \psi(x) \tag{4}$$

allows rearrangement of equations (2) to the Hamiltonian form

$$E\begin{bmatrix} \psi_{1,2}^{[2j]} \\ (-)^{2j+1}\psi_{3,4}^{[2j]} \end{bmatrix} = \begin{bmatrix} R & (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & -R \end{bmatrix} S^{-1}\begin{bmatrix} \psi_{1,2}^{[2j]} \\ (-)^{2j+1}\psi_{3,4}^{[2j]} \end{bmatrix}. \tag{5}$$

Since the free fields may be expanded in terms of plane waves there is no difficulty with a precise meaning for S^{-1}.

Table 1 shows values of R and S up to spin 3. When $-\mathrm{i}\nabla$ is substituted for $\boldsymbol{P}$ then R and S are formally identical with Chebyshev polynomials in μ/∇ of the first and second kind, multiplied by ∇^{2j} and ∇^{2j-1} respectively. With $\sinh\phi = \mu/|\boldsymbol{P}|$ the results of Table 1 are summarized by $ES = |\boldsymbol{P}|^{2j}\cosh 2j\phi$ and $R = |\boldsymbol{P}|^{2j}\sinh 2j\phi$ for half-integer j, and by $ES = |\boldsymbol{P}|^{2j}\sinh 2j\phi$ and $R = |\boldsymbol{P}|^{2j}\cosh 2j\phi$ for integer j.

Table 1. Values of *R* and *S* up to spin 3

The operators R and S are defined by equations (3)

Spin j	R	S
1/2	μ	1
1	$2\mu^2+\boldsymbol{P}^2$	2μ
3/2	$4\mu^3+3\boldsymbol{P}^2\mu$	$4\mu^2+\boldsymbol{P}^2$
2	$8\mu^4+8\boldsymbol{P}^2\mu^2+\boldsymbol{P}^4$	$8\mu^3+4\boldsymbol{P}^2\mu$
5/2	$16\mu^5+20\mu^3\boldsymbol{P}^2+5\mu\boldsymbol{P}^4$	$16\mu^4+12\mu^2\boldsymbol{P}^2+\boldsymbol{P}^4$
3	$32\mu^6+48\mu^4\boldsymbol{P}^2+18\mu^2\boldsymbol{P}^4+\boldsymbol{P}^6$	$32\mu^5+32\mu^3\boldsymbol{P}^2+6\mu\boldsymbol{P}^4$

The Hamiltonian H of equation (5) is diagonalizable via the transformation

$$THT^{-1} = \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}, \tag{6}$$

where

$$T = \begin{bmatrix} (ES+R) & (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ -(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & (-)^{2j+1}(ES+R) \end{bmatrix} C^{-1},$$

$$T^{-1} = \begin{bmatrix} (ES+R) & (-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & (-)^{2j+1}(ES+R) \end{bmatrix} C^{-1},$$

with

$$C = \{2ES(ES+R)\}^{\frac{1}{2}},$$

by using the relation $R^2-E^2S^2 = (-\boldsymbol{P}^2)^{2j}$. The columns of T^{-1} are solutions of equation (5), provided the first $2j+1$ columns operate on positive-energy plane waves and the second $2j+1$ columns on negative-energy plane waves.

The Weaver *et al.* (1964) Hamiltonian H_{W} and the similar form of Mathews (1966) are seen to be equivalent to that of equation (5) by diagonalizing to $M^{-1}S_{\mathrm{W}}^{-1}H_{\mathrm{W}}S_{\mathrm{W}}M$ and then transforming to H:

$$T^{-1}M^{-1}S_{\mathrm{W}}^{-1}H_{\mathrm{W}}S_{\mathrm{W}}MT = H, \qquad (7$$

with

$$M^{-1} = 2^{-\frac{1}{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad M = 2^{-\frac{1}{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

The operators S_{W} and S_{W}^{-1} are listed for various j by Weaver *et al.* (1964) in their Table 1. This is a different procedure from that of Krajcik and Nieto (1977), who have also diagonalized H_{W}. Since all the equations describe the same physics, their equivalence is necessary. Indeed, one hopes that all arbitrary spin equations for free fields, based on the Poincaré group, are equivalent whether they evolve from $2(2j+1)$ component theories, linear operator theories or otherwise. The formal minimal coupling substitution makes the equations non-equivalent. Does this suggest that some of the problems in describing interacting fields are centred around too great a reliance on the minimal coupling substitution itself?

3. General Lagrangian

The form of the single-particle Hamiltonian suggests that a reasonable generalization of the Dirac spin $\frac{1}{2}$ Lagrangian density may be

$$\mathscr{L} = -\bar{\psi}\begin{bmatrix} (R-ES) & (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ (-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & (R+ES) \end{bmatrix} S^{-1}\psi. \tag{8}$$

Here ψ is the $2j$th rank column spinor given in equation (5), while $\bar{\psi}$ is the conjugate transpose of ψ multiplied by the diagonal matrix having $2j+1$ entries of 1 and $2j+1$ entries of -1 down its diagonal for half-integer spins, and $2(2j+1)$ entries of 1 down its diagonal for integer spin. This is a generalization of the definition $\bar{\psi} = \psi^{\dagger}\gamma_4$ given by Lurié (1968) for spin $\frac{1}{2}$, with $\psi^{\dagger}$ being multiplied by the $2j$th induced equivalent of γ_4 for spin j.

The inclusion of S^{-1} in the Lagrangian is partly optional. However, the results then parallel those for spin $\frac{1}{2}$ more closely than otherwise. Also $\mathscr{L}$ is a scalar with the dimensions of energy density and leads to a positive definite field-theoretic Hamiltonian. Furthermore, if S^{-1} is excluded, the occupation numbers contain S, which must somehow be absorbed into their definition and makes them momentum dependent.

4. Euler–Lagrange Equations of Motion

In the Lagrangian density given by equation (8) every field component of ψ and its derivatives occur with S^{-1}, so the primitive fields are taken to be $S^{-1}\psi$ and its derivatives and, of course, $\bar{\psi}$. Then, assuming a summation over α, we have

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial(S^{-1}\psi_\alpha)}\delta(S^{-1}\psi_\alpha) + \frac{\partial\mathscr{L}}{\partial\partial_\mu(S^{-1}\psi_\alpha)}\delta\partial_\mu(S^{-1}\psi_\alpha) + \ldots + \frac{\partial\mathscr{L}}{\partial\partial_\mu^{2j}(S^{-1}\psi_\alpha)}\delta\partial_\mu^{2j}(S^{-1}\psi_\alpha) + \frac{\partial\mathscr{L}}{\partial\bar{\psi}_\alpha}\delta\bar{\psi}_\alpha, \tag{9}$$

where the abbreviation

$$N^{-1}\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}\left(\frac{\partial\mathscr{L}}{\partial\,\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_n}(S^{-1}\psi_\alpha)}\right) = \partial_\mu^n\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu^n(S^{-1}\psi_\alpha)}\right) \tag{10}$$

has been made. By summing over repeated indices $\mu_1, \mu_2, \ldots, \mu_{2j}$, a field variable with mixed indices would occur more than once. To counteract this, the left-hand side of equation (10) is multiplied by an inverse permutation operator N^{-1}, where N is the number of ways of arranging the indices.

Repeated integration by parts of equation (9) over 4-space, together with the assumption of periodicity of ψ and all its derivatives at the boundaries, leads to the generalized Euler–Lagrange equations of motion

$$\frac{\partial \mathscr{L}}{\partial(S^{-1}\psi_\alpha)} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial\partial_\mu(S^{-1}\psi_\alpha)}\right) + \partial_\mu^2\left(\frac{\partial \mathscr{L}}{\partial\partial_\mu^2(S^{-1}\psi_\alpha)}\right) - \ldots + (-)^{2j}\partial_\mu^{2j}\left(\frac{\partial \mathscr{L}}{\partial\partial_\mu^{2j}(S^{-1}\psi_\alpha)}\right) = 0, \tag{11a}$$

$$\partial\mathscr{L}/\partial\bar{\psi}_\alpha = 0. \tag{11b}$$

These equations used with the Lagrangian density (8) will reproduce the field equations (2) or their conjugate transposes.

Coelho de Souza and Rodrigues (1969) have defined momenta conjugate to the field variables $\partial_t^m \phi_\alpha$ $(m = 0, 1, \ldots, 2j)$ by equations which are equivalent to

$$\pi_{\alpha/m} = \sum_{r=0}^{2j-m} (-)^r \partial_\mu^r(\partial\mathscr{L}/\partial\,\partial_\mu^r\,\partial_t^m\,\phi_\alpha). \tag{12}$$

The value of $\pi_{\alpha/0}$ is zero as a consequence of the Euler–Lagrange equations of motion.

For the Lagrangian density (8) there is only one conjugate momentum field, namely $\pi_{\alpha/1}$ which is given by

$$\pi_{\alpha/1} = \mathrm{i}S(\bar{\psi}\lambda)_\alpha, \tag{13}$$

with

$$\lambda = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},$$

I being the $(2j+1)^2$ identity matrix. Hence the equal-time commutation relations are

$$[S^{-1}\psi_\alpha(\boldsymbol{x},t),\, S\bar{\psi}_\beta(\boldsymbol{x}',t)\lambda_{\beta\beta}]_\pm = \delta_{\alpha\beta}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{14}$$

where the plus or minus subscript refers to anticommutation (for j half-integer) or commutation (for j integer) respectively. All other field components are assumed to commute.

The field-theoretic Hamiltonian could be obtained from the Lagrangian and the conjugate momentum (12) by following Coelho de Souza and Rodrigues (1969) and *defining*

$$H = \int \mathrm{d}^3x \sum_{\alpha,m} (\pi_{\alpha/m}\,\partial_t^m\,\phi_\alpha - \mathscr{L}). \tag{15}$$

However, there is no need to define H in this manner. By generalizing Noether's theorem, the right-hand side of equation (15) is proved to be conserved in time and is the fourth component of a vector identifiable with the 4-momentum. The proof follows.

5. Noether's Theorem for Higher Derivative Lagrangians

For a general Lagrangian density $\mathscr{L} = \mathscr{L}(\phi_\alpha, \partial_\mu\phi_\alpha, \ldots, \partial_\mu^{2j}\phi_\alpha, x)$ we have

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_\alpha}\delta\phi_\alpha + \frac{\partial\mathscr{L}}{\partial\,\partial_\mu\phi_\alpha}\delta\,\partial_\mu\phi_\alpha + \ldots + \frac{\partial\mathscr{L}}{\partial\,\partial_\mu^{2j}\phi_\alpha}\delta\,\partial_\mu^{2j}\phi_\alpha, \tag{16}$$

where the sum over α is assumed and the notation (10) has been adopted. Elimination of $\partial\mathscr{L}/\partial\phi_\alpha$ using the Euler–Lagrange equation

$$\frac{\partial\mathscr{L}}{\partial\phi_\alpha} = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\phi_\alpha}\right) - \partial_\mu^2\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^2\phi_\alpha}\right) + \ldots + (-)^{2j}\,\partial_\mu^{2j}\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^{2j}\phi_\alpha}\right) \tag{17}$$

gives

$$\delta\mathscr{L} = \sum_{n=1}^{2j} I_n, \tag{18a}$$

with

$$I_n = \left\{\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^n\phi_\alpha}\,\delta\,\partial_\mu^n\phi_\alpha + (-)^{n-1}\,\partial_\mu^n\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^n\phi_\alpha}\right)\delta\phi_\alpha\right\}. \tag{18b}$$

The differential operator ∂_μ can be broken down into spatial parts ∂_i ($i = 1,2,3$) and a temporal part ∂_t, that is,

$$I_n = \sum_{r=0}^{n}\left\{\frac{\partial\mathscr{L}}{\partial\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha}\,\delta\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha + (-)^{n-1}\,\partial_i^{n-r}\,\partial_t^r\left(\frac{\partial\mathscr{L}}{\partial\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha}\right)\delta\phi_\alpha\right\}. \tag{19}$$

Upon repeated integration by parts over 3-space and use of the boundary conditions, the completely spatial factors ($r = 0$) cancel and equation (19) integrates to

$$\int I_n\,\mathrm{d}^3x = \int \mathrm{d}^3x \sum_{r=1}^{n}\left\{(-)^{n-r}\,\partial_i^{n-r}\left(\frac{\partial\mathscr{L}}{\partial\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha}\right)\delta\,\partial_t^r\,\phi_\alpha\right.$$
$$\left. + (-)^{n-1}\,\partial_i^{n-r}\,\partial_t^r\left(\frac{\partial\mathscr{L}}{\partial\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha}\right)\delta\phi_\alpha\right\}. \tag{20}$$

The easily derivable formula

$$A(\partial_t^r B) + (-)^{r-1}(\partial_t^r A)B = \sum_{l=0}^{r-1}(-)^l\,\partial_t\{(\partial_t^l A)(\partial_t^{r-l-1}B)\} \tag{21}$$

allows equation (20) to be rewritten as

$$\int I_n\,\mathrm{d}^3x = \int \mathrm{d}^3x \sum_{r=1}^{n}(-)^{n-r}\left(\sum_{l=0}^{r-1}(-)^l\,\partial_t\left\{\partial_i^{n-r}\,\partial_t^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_i^{n-r}\,\partial_t^r\,\phi_\alpha}\right)\delta\,\partial_t^{r-l-1}\,\phi_\alpha\right\}\right). \tag{22}$$

Summation over n from 1 to $2j$ and substitution into equation (18a) followed by regrouping of the terms then gives

$$\int\delta\mathscr{L}\,\mathrm{d}^3x = \int \mathrm{d}^3x\,\partial_t(\pi_{\alpha/m}\,\delta\,\partial_t^{m-1}\,\phi_\alpha), \tag{23}$$

where sums over α and m are implied.

The action W_{21} between 4-points x_1 and x_2 is given by

$$W_{21} = \int_{x_1}^{x_2}\mathscr{L}\,\mathrm{d}^4x, \tag{24}$$

so that

$$\delta W_{21} = \delta\left(\int_{x_1}^{x_2}\mathscr{L}\,\mathrm{d}^4x\right) = \int(\delta\mathscr{L})\,\mathrm{d}^4x + \int\mathscr{L}\,\delta(\mathrm{d}^4x). \tag{25}$$

Now we have that $\delta(\mathrm{d}^4x) = \partial_\mu(\mathrm{d}^4x)\,\delta x^\mu = (\mathrm{d}^3x)_\mu\,\delta x^\mu$, where $(\mathrm{d}^3x)_\mu$ is the surface vector $(\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}t,\ \mathrm{d}x_1\,\mathrm{d}x_3\,\mathrm{d}t,\ \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t,\ \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)$. Also the spatial integral for $\delta\mathscr{L}$ is given by equation (23). Thus equation (25) becomes

$$\delta W_{21} = \int \mathrm{d}^4x\,\partial_t(\pi_{\alpha/m}\,\delta\,\partial_t^{m-1}\,\phi_\alpha) + \int \mathscr{L}\,\delta x^\mu\,(\mathrm{d}^3x)_\mu. \tag{26}$$

The second integral on the right-hand side of this equation is a hypersurface integral and will vanish unless $\mu = 4$ when periodic boundary conditions are imposed over 3-space. Hence

$$\delta W_{21} = \left[\int_{t_2} - \int_{t_1}\right]\mathrm{d}^3x\,(\pi_{\alpha/m}\,\delta\,\partial_t^{m-1}\,\phi_\alpha + \mathscr{L}\,\mathrm{d}t). \tag{27}$$

For space–time displacements we have $\delta\,\partial_t^{m-1}\,\phi_\alpha = -\delta x^\mu\,\partial_\mu\,\partial_t^{m-1}\,\phi_\alpha$, so that

$$\delta W_{21} = \delta x^\mu\left[\int_{t_2} - \int_{t_1}\right]\mathrm{d}^3x\,(-\pi_{\alpha/m}\,\partial_\mu\,\partial_t^{m-1}\,\phi_\alpha - \mathrm{i}\,\delta_{\mu 4}\,\mathscr{L}). \tag{28}$$

Since $\delta W_{21} = 0$ and the displacements δx^μ are arbitrary, the 4-vector in equation (28) must be conserved over time. This vector is identified as the 4-momentum and its spatial and temporal parts are

$$-\int \mathrm{d}^3x\,\pi_{\alpha/m}\,\nabla\partial_t^{m-1}\,\phi_\alpha, \qquad \int \mathrm{d}^3x\,(\pi_{\alpha/m}\,\partial_t^m\,\phi_\alpha - \mathscr{L}). \tag{29}$$

These are identified as the field-theoretic momentum and the energy respectively.

The conserved current and charge of a field with a higher derivative Lagrangian are also easily derived. Supposing ϕ_α has an internal symmetry such that the Lagrangian is invariant under the infinitesimal *global* gauge transformation ($\theta \neq \theta(x)$)

$$\phi_\alpha \to \phi_\alpha + \mathrm{i}\theta\phi_\alpha, \qquad \phi_\alpha^* \to \phi_\alpha^* - \mathrm{i}\theta\phi_\alpha^*, \tag{30}$$

so that

$$\partial_\mu^n\,\phi_\alpha \to \partial_\mu^n\,\phi_\alpha + \mathrm{i}\theta\,\partial_\mu^n\,\phi_\alpha, \qquad \partial_\mu^n\,\phi_\alpha^* \to \partial_\mu^n\,\phi_\alpha^* - \mathrm{i}\theta\,\partial_\mu^n\,\phi_\alpha^*, \tag{31}$$

then

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\theta} = {}& \frac{\partial\mathscr{L}}{\partial\phi_\alpha}\mathrm{i}\phi_\alpha + \frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\phi_\alpha}\mathrm{i}\,\partial_\mu\,\phi_\alpha + \frac{\partial\mathscr{L}}{\partial\,\partial_\mu^2\,\phi_\alpha}\mathrm{i}\,\partial_\mu^2\,\phi_\alpha \ldots \\ & - \frac{\partial\mathscr{L}}{\partial\phi_\alpha^*}\mathrm{i}\phi_\alpha^* - \frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\phi_\alpha^*}\mathrm{i}\,\partial_\mu\,\phi_\alpha^* - \frac{\partial\mathscr{L}}{\partial\,\partial_\mu^2\,\phi_\alpha^*}\mathrm{i}\,\partial_\mu^2\,\phi_\alpha^* - \ldots.\end{aligned} \tag{32}$$

Equation (32) together with the relation (see equation 21)

$$\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^n\,\phi_\alpha}\partial_\mu^n\,\phi_\alpha + (-)^{n-1}\,\partial_\mu^n\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^n\,\phi_\alpha}\right)\phi_\alpha = \sum_{l=0}^{n-1}(-)^l\,\partial_\mu\left\{\partial_\mu^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu^n\,\phi_\alpha}\right)\partial_\mu^{n-l-1}\,\phi_\alpha\right\} \tag{33}$$

and the Euler–Lagrange equations of motion (17) result in the expression

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial\theta} = \partial_\mu\Bigg(\sum_{n=1}^{2j}\sum_{l=0}^{n-1}(-)^l\Bigg\{&\partial_\nu^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\partial_\nu^{n-1}\,\phi_\alpha}\right)\partial_\nu^{n-l-1}\,\phi_\alpha \\ & - \partial_\nu^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\partial_\nu^{n-1}\,\phi_\alpha^*}\right)\partial_\nu^{n-l-1}\,\phi_\alpha^*\Bigg\}\Bigg).\end{aligned} \tag{34}$$

Then $\partial\mathscr{L}/\partial\theta = 0$ means that the current j_μ, given by

$$j_\mu = -\mathrm{i}\sum_{n=1}^{2j}\sum_{l=0}^{n-1}(-)^l\left\{\partial_\nu^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\partial_\nu^{n-1}\,\phi_\alpha}\right)\partial_\nu^{n-l-1}\,\phi_\alpha - \partial_\nu^l\left(\frac{\partial\mathscr{L}}{\partial\,\partial_\mu\,\partial_\nu^{n-1}\,\phi_\alpha^*}\right)\partial_\nu^{n-l-1}\,\phi_\alpha^*\right\}, \tag{35}$$

is conserved. The charge Q can be obtained by repeated integration by parts of j_4 over 3-space. With great care in the use of the permutation factor N (see equation 10) it comes to

$$Q = -\mathrm{i}\int j_4\,\mathrm{d}^3x = -\mathrm{i}\int \mathrm{d}^3x\,(\pi_{\alpha/m}\,\partial_t^{m-1}\,\phi_\alpha - \pi_{\alpha/m}^*\,\partial_t^{m-1}\,\phi_\alpha^*)\,. \tag{36}$$

6. Field-theoretic Hamiltonian

Now that Noether's theorem for a Lagrangian with higher derivatives is established, the Hamiltonian for the field can be obtained.

From equations (8), (13) and (15) the Hamiltonian density $\mathscr{H}$ is found to be

$$\mathscr{H} = \bar{\psi}\begin{bmatrix} R & (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ (-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & R \end{bmatrix} S^{-1}\psi + (S\bar{\psi}\lambda E S^{-1}\psi - \bar{\psi}\lambda E S S^{-1}\psi)\,. \tag{37}$$

Upon repeated integration by parts over 3-space, the last term disappears, because of the boundary conditions, so that the Hamiltonian becomes

$$H = \int \mathrm{d}^3x\,\bar{\psi}\begin{bmatrix} R & (\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ (-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & R \end{bmatrix} S^{-1}\psi \tag{38}$$

or, using the field equation (5),

$$H = \int \mathrm{d}^3x\,\bar{\psi}\lambda E\psi. \tag{39}$$

For half-integer spins, $\bar{\psi}\lambda$ equals $\psi^\dagger$ and, for integer spins, it equals $\psi^\dagger\lambda$.

To describe the Hamiltonian in term of creation and annihilation operators, ψ and $\bar{\psi}$ are first expanded as a Fourier sum of plane waves. A suitable set of $2j+1$ orthogonal positive-energy plane wave solutions, for a particular momentum $\boldsymbol{k}$, is the columns of the matrix

$$\left(\frac{R_k + E_k S_k}{2R_k}\right)^{\frac{1}{2}}\begin{bmatrix} \mathscr{I} \\ \mathscr{S} \end{bmatrix}\exp\{\mathrm{i}(\boldsymbol{k}.\boldsymbol{x} - E_k t)\} = u_{k\sigma}\exp\{\mathrm{i}(\boldsymbol{k}.\boldsymbol{x} - E_k t)\}\,, \tag{40a}$$

with

$$\mathscr{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{[2j]}, \qquad \mathscr{S} = \frac{(-)^{2j+1}}{R_k + E_k S_k}\begin{bmatrix} k_z & k_- \\ k_+ & -k_z \end{bmatrix}^{[2j]}, \tag{40b}$$

where R_k and S_k are R and S as given by equations (3) and Table 1, but with the operator $\boldsymbol{P}^2$ replaced by $\boldsymbol{k}^2$, and E_k is the energy associated with the momentum $\boldsymbol{k}$. The subscript σ refers to the magnetic spin quantum number which ranges from $+j$ for the first column to $-j$ for the last column of the matrix in equation (40a). The $2j+1$ orthogonal negative-energy solutions are the columns of

$$\left(\frac{R_k + E_k S_k}{2R_k}\right)^{\frac{1}{2}}\begin{bmatrix} \mathscr{S} \\ \mathscr{I} \end{bmatrix}\exp\{-\mathrm{i}(\boldsymbol{k}.\boldsymbol{x} - E_k t)\} = v_{k\sigma}\exp\{-\mathrm{i}(\boldsymbol{k}.\boldsymbol{x} - E_k t)\}\,. \tag{41}$$

These wavefunctions are normalized in the sense

$$\bar{u}_{k\sigma} u_{k\sigma} = 1\,, \qquad \bar{v}_{k\sigma} v_{k\sigma} = (-)^{2j}. \tag{42}$$

The Fourier expansions of ψ and $\bar{\psi}$ as plane waves are then

$$\psi(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} \left(u_{k\sigma} a_{k\sigma} \exp(\mathrm{i}k.x) + v_{k\sigma} b_{k\sigma}^\dagger \exp(-\mathrm{i}k.x)\right), \tag{43a}$$

$$\bar{\psi}(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} \left(\bar{u}_{k\sigma} a_{k\sigma}^\dagger \exp(-\mathrm{i}k.x) + \bar{v}_{k\sigma} b_{k\sigma} \exp(\mathrm{i}k.x)\right), \tag{43b}$$

where $a_{k\sigma}$, $b_{k\sigma}$, $a_{k\sigma}^\dagger$ and $b_{k\sigma}^\dagger$ are operators. When these expansions are substituted into equation (39) the Hamiltonian becomes

$$H = \sum_{k,\sigma} E_k \{a_{k\sigma}^\dagger a_{k\sigma} + (-)^{2j} b_{k\sigma} b_{k\sigma}^\dagger\}\,, \tag{44}$$

where we have used the results

$$\bar{u}_{k\sigma} \lambda u_{k\sigma} = E_k S_k / R_k\,, \qquad \bar{v}_{k\sigma} \lambda v_{k\sigma} = (-)^{2j+1} E_k S_k / R_k. \tag{45}$$

From equations (14) and (43) the commutation relations obeyed by the creation and annihilation operators are

$$[a_{k\sigma}, a_{k'\sigma'}^\dagger]_\pm = \delta_{kk'} \delta_{\sigma\sigma'}\,, \qquad [b_{k\sigma}, b_{k'\sigma'}^\dagger] = \delta_{kk'} \delta_{\sigma\sigma'}\,, \tag{46}$$

with $[a_{k\sigma}, b_{k'\sigma'}]_\pm$, $[a_{k\sigma}, a_{k'\sigma'}]_\pm$ and $[b_{k\sigma}, b_{k'\sigma'}]$ all zero. Hence

$$H = \sum_{k,\sigma} E_k (a_{k\sigma}^\dagger a_{k\sigma} + b_{k\sigma}^\dagger b_{k\sigma})\,, \tag{47}$$

where the zero-point energy has been omitted. Similarly the momentum is given by

$$\sum_{k,\sigma} \boldsymbol{k} (a_{k\sigma}^\dagger a_{k\sigma} + b_{k\sigma}^\dagger b_{k\sigma}). \tag{48}$$

The charge Q of the field can be derived from equation (36), and for our Lagrangian is

$$Q = \int \mathrm{d}^3 x\, \bar{\psi} \lambda \psi = \sum_{k,\sigma} (a_{k\sigma}^\dagger a_{k\sigma} - b_{k\sigma}^\dagger b_{k\sigma}). \tag{49}$$

Space reflection, time reflection and charge conjugation have been considered by others (e.g. Weinberg 1964) and their effects on H, Q and the momentum can be deduced from equations (47)–(49) by standard methods. Therefore, these operations are not discussed.

7. Feynman Propagator

In order to obtain the Feynman propagator, the general commutation rules between two 4-points x_1 and x_2 are derived first. Accordingly, ψ and $\bar{\psi}$ are separated into their positive frequency (annihilation) and negative frequency (creation) parts, denoted by $(+)$ and $(-)$ superscripts respectively:

$$\psi^{(+)}(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} u_{k\sigma} a_{k\sigma} \exp(ik.x), \tag{50a}$$

$$\bar{\psi}^{(-)}(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} \bar{u}_{k\sigma} a_{k\sigma}^{\dagger} \exp(-ik.x), \tag{50b}$$

$$\psi^{(-)}(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} v_{k\sigma} b_{k\sigma}^{\dagger} \exp(-ik.x), \tag{50c}$$

$$\bar{\psi}^{(+)}(x) = V^{-\frac{1}{2}} \sum_{k,\sigma} \left(\frac{R_k}{E_k S_k}\right)^{\frac{1}{2}} \bar{v}_{k\sigma} b_{k\sigma} \exp(ik.x). \tag{50d}$$

The commutators $[S^{-1}\psi_\alpha^{(+)}(x_1),\ S\bar{\psi}_\beta^{(-)}(x_2)]_\pm$, $[S^{-1}\psi_\alpha^{(-)}(x_1),\ S\bar{\psi}_\beta^{(+)}(x_2)]_\pm$ and $[S^{-1}\psi_\alpha(x_1),\ S\bar{\psi}_\beta(x_2)]_\pm$ are now derived using the expressions (50) and equations (46). First consider

$$[S^{-1}\psi_\alpha^{(+)}(x_1), S\bar{\psi}_\beta^{(-)}(x_2)]_\pm = V^{-1} \sum_{k,\sigma} \left(u_{k\sigma}\bar{u}_{k\sigma}\right)_{\alpha\beta} \left(\frac{R_k}{E_k S_k}\right) \exp\{ik.(x_1 - x_2)\}. \tag{51}$$

Now the spin sums are

$$\sum_\sigma u_{k\sigma}\bar{u}_{k\sigma} = \begin{bmatrix} (R_k + E_k S_k) & -(\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} & (R_k - E_k S_k) \end{bmatrix} (2R_k)^{-1}, \tag{52a}$$

$$\sum_\sigma v_{k\sigma}\bar{v}_{k\sigma} = \begin{bmatrix} (-)^{2j}(R_k - E_k S_k) & -(\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} & (-)^{2j}(R_k + E_k S_k) \end{bmatrix} (2R_k)^{-1}. \tag{52b}$$

Substitution of the result (52a) into equation (51) leads to

$$[S^{-1}\psi_\alpha^{(+)}(x_1), S\bar{\psi}_\beta^{(-)}(x_2)]_\pm = \mathscr{T}_{\alpha\beta}(SV)^{-1} \sum_k (2E_k)^{-1} \exp\{ik.(x_1 - x_2)\}, \tag{53a}$$

where the matrix

$$\mathscr{T}_{\alpha\beta} = \begin{bmatrix} (R+ES) & -(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} & (R-ES) \end{bmatrix}_{\alpha\beta} \tag{53b}$$

has been converted to its operator form and taken outside the summation over $\boldsymbol{k}$. Similarly, the other two commutators are

$$[S^{-1}\psi_\alpha^{(-)}(x_1), S\bar{\psi}_\beta^{(+)}(x_2)]_\pm = -\mathscr{T}_{\alpha\beta}(SV)^{-1} \sum_k (2E_k)^{-1} \exp\{-ik.(x_1 - x_2)\}, \tag{54}$$

$$[S^{-1}\psi_\alpha(x_1), S\bar{\psi}_\beta(x_2)]_\pm$$
$$= \mathscr{T}_{\alpha\beta}(SV)^{-1} \sum_k (2E_k)^{-1} \left(\exp\{ik.(x_1 - x_2)\} - \exp\{-ik.(x_1 - x_2)\}\right). \tag{55}$$

The Feynman propagator $F_{\alpha\beta}(x_1 - x_2)$ is defined by

$$F_{\alpha\beta}(x_1 - x_2) = \langle 0 | TS^{-1}\psi_\alpha(x_1) S\bar{\psi}_\beta(x_2) | 0 \rangle, \tag{56}$$

where T is the time-ordering operator such that

$$TS^{-1}\psi_\alpha(x_1)S\bar{\psi}_\beta(x_2) = S^{-1}\psi_\alpha(x_1)S\bar{\psi}_\beta(x_2), \qquad t_1 > t_2, \tag{57a}$$

$$= (-)^{2j}S\bar{\psi}_\beta(x_2)S^{-1}\psi_\alpha(x_1), \qquad t_2 > t_1. \tag{57b}$$

Now

$$\langle 0|TS^{-1}\psi_\alpha(x_1)S\bar{\psi}_\beta(x_2)|0\rangle = \langle 0|[S^{-1}\psi_\alpha^{(+)}(x_1), S\bar{\psi}_\beta^{(-)}(x_2)]_\pm|0\rangle, \quad t_1 > t_2, \tag{58a}$$

$$= -\langle 0|[S^{-1}\psi_\alpha^{(-)}(x_1), S\bar{\psi}_\beta^{(+)}(x_2)]_\pm|0\rangle, \quad t_2 > t_1, \tag{58b}$$

so that

$$F_{\alpha\beta}(x) = \theta(x_0)\mathscr{T}_{\alpha\beta}(SV)^{-1}\sum_k (2E_k)^{-1}\exp(\mathrm{i}k.x)$$
$$+\theta(-x_0)\mathscr{T}_{\alpha\beta}(SV)^{-1}\sum_k (2E_k)^{-1}\exp(-\mathrm{i}k.x). \tag{59}$$

The step functions $\theta(x_0)$ and $\theta(-x_0)$ are equal to unity for $x_0 > 0$ and $x_0 < 0$ respectively, and are zero otherwise. They can be taken through the matrix operator which contains only first-order derivatives in time. The reason is that when the matrix operates on $\theta(x_0)$ and $\theta(-x_0)$ it produces delta functions, since $\partial_t\theta(t) = \delta(t)$ and $\partial_t\theta(-t) = -\delta(t)$. The factors that result involve $\sum\sin(\boldsymbol{k}.\boldsymbol{x})$ which vanishes (cf. p. 138 of the text by Lurié 1968). We have not experienced here the problems encountered by Weinberg (1964) when deriving a covariant Feynman propagator because our theory has been based on equations like (2) and (8) which contain only first-order derivatives in time, rather than based on equations (1) which contain higher-order derivatives in time for spins greater than $\frac{1}{2}$.

As a consequence of the previous discussion the Feynman propagator can now be written as

$$F(x) = \mathscr{T}(SV)^{-1}\sum_k (2E_k)^{-1}\{\theta(x_0)\exp(\mathrm{i}k.x) + \theta(-x_0)\exp(-\mathrm{i}k.x)\}, \tag{60}$$

where the subscripts $\alpha\beta$ have been omitted. The summation over $\boldsymbol{k}$ is well known and is determined from the calculus of residues by converting the temporal part into an integral over the complex k_0 plane. The result is

$$F(x) = (2\pi)^{-4}\int \mathrm{d}^4k\,\frac{\mathscr{T}_k}{S_k}\,\frac{\exp(\mathrm{i}k.x)}{k^2+\mu^2}, \tag{61a}$$

where

$$\mathscr{T}_k = \begin{bmatrix} (R_k + k_0 S_k) & -(\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} \\ -(-\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} & (R_k - k_0 S_k) \end{bmatrix} \tag{61b}$$

and the summation over $\boldsymbol{k}$ space has been converted to an integral. This is the form of the Feynman propagator that would be most useful for computation purposes but it can be converted to the form

$$F(x) = (2\pi)^{-4}\int \mathrm{d}^4k\,\{S_k\exp(\mathrm{i}k.x)/\mathscr{T}_k^*\}, \tag{62a}$$

where

$$\mathscr{T}_k^* = \begin{bmatrix} (R_k - k_0 S_k) & (\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} \\ (-\boldsymbol{k}.\boldsymbol{\sigma})^{[2j]} & (R_k + k_0 S_k) \end{bmatrix}. \tag{62b}$$

The propagator in momentum space is then

$$F(k) = S_k/(\mathscr{T}_k^* - \mathrm{i}\varepsilon). \tag{63}$$

8. Conclusions

The extension of the Dirac spin $\frac{1}{2}$ equation by induced matrix formulation leads simply and directly to arbitrary spin field equations and a general free-field Lagrangian in which the results for spin $\frac{1}{2}$ are paralleled very closely. Lagrangians with higher derivatives can be quite easily handled along lines similar to those for ones with linear derivatives only. Whether the types of equations presented here will still be physically realistic and not give rise to acausality upon the introduction of electromagnetic coupling remains to be seen.

Note added in proof

An important paper by Musicki (1978) on canonical formalism in field theory with derivatives of higher order has recently appeared.

Acknowledgments

The author thanks Dr R. A. Lee of this Division and Mr C. H. J. Johnson of the Division of Chemical Physics, CSIRO, for discussions of early drafts of the paper.

References

Belinicher, B. I. (1975). *Theor. Math. Phys.* (*USSR*) **20**, 849.
Coelho de Souza, L. M. C., and Rodrigues, P. R. (1969). *J. Phys.* A (Ser. 2) **2**, 304.
Doria, F. A. (1977). *J. Math. Phys.* (*New York*) **18**, 564.
Frank, V. (1973). *Nucl. Phys.* B **59**, 429.
Hagen, C. R., and Hurley, W. J. (1970). *Phys. Rev. Lett.* **24**, 1381.
Hurley, W. J. (1971). *Phys. Rev.* D **4**, 3605.
Hurley, W. J. (1974). *Phys. Rev.* D **10**, 1185.
Jeffery, E. A. (1978). *Aust. J. Phys.* **31**, 137.
Joos, H. (1962). *Fortschr. Phys.* **10**, 65.
Krajcik, R. A., and Nieto, M. M. (1977). *Phys. Rev.* D **15**, 426.
Littlewood, D. E. (1958). 'The Theory of Group Characters and Matrix Representations of Groups', Ch. 10 (Oxford: London).
Lurié, D. (1968). 'Particles and Fields' (Interscience: New York).
Mathews, P. M. (1966). *Phys. Rev.* **143**, 978.
Mathews, P. M. (1971). *Lect. Theor. Phys.* C **12**, 139.
Mathews, P. M., and Ramakrishnan, S. (1967). *Nuovo Cimento* **50**, A339.
Musicki, D. (1978). *J. Phys.* A **1**, 39.
Nelson, T. J., and Good, R. H., Jr (1968). *Rev. Mod. Phys.* **40**, 508.
Singh, L. P. S., and Hagen, C. R. (1974). *Phys. Rev.* D **9**, 898.
Weaver, D. L. (1968). *Nuovo Cimento* A **53**, 667.
Weaver, D. L., Hammer, C. L., and Good, R. H., Jr (1964). *Phys. Rev.* **135**, B241.
Weinberg, S. (1964). *Phys. Rev.* **133**, B1318.
Weinberg, S. (1969). *Phys. Rev.* **181**, 1893.

Appendix

Consider a transformation of a spinor (x, y) by a 2×2 matrix,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{A1}$$

This induces a transformation in the multispinor $(x, y)^{[n]}$ given by

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n]} \begin{pmatrix} x \\ y \end{pmatrix}^{[n]} = \begin{pmatrix} x' \\ y' \end{pmatrix}^{[n]}, \tag{A2}$$

where

$$(x, y)^{[n]} = (x^n, n^{\frac{1}{2}} x^{n-1} y, \{n(n-1)/1.2\}^{\frac{1}{2}} x^{n-2} y^2, ..., y^n).$$

Equation (A2) serves to define the nth induced matrix of the 2×2 matrix and is symbolized by a superscript $[n]$. For example, with the help of equation (A2), the matrices $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2]}$ and $(\boldsymbol{P}.\boldsymbol{\sigma})^{[3]}$ are respectively

$$\begin{bmatrix} P_z^2 & \surd 2 P_z P_- & P_-^2 \\ \surd 2 P_z P_+ & (-P_z^2 + P_+ P_-) & -\surd 2 P_z P_- \\ P_+^2 & -\surd 2 P_z P_+ & P_z^2 \end{bmatrix}, \tag{A3a}$$

$$\begin{bmatrix} P_z^3 & \surd 3 P_z^2 P_- & \surd 3 P_z P_-^2 & P_-^3 \\ \surd 3 P_z^2 P_+ & (-P_z^3 + 2P_z P_+ P_-) & (-2P_z^2 P_- + P_+ P_-^2) & -\surd 3 P_z P_-^2 \\ \surd 3 P_z P_+^2 & (-2P_z^2 P_+ + P_+^2 P_-) & (P_z^3 - 2P_z P_+ P_-) & \surd 3 P_z^2 P_- \\ P_+^3 & -\surd 3 P_+^2 P_z & \surd 3 P_+ P_z^2 & -P_z^3 \end{bmatrix}. \tag{A3b}$$

The $2j$th rank totally symmetric spinor $\psi_{1,2}^{[2j]}$ in the text has similar transformation properties to the multispinor $(x, y)^{[2j]}$ and is defined by

$$\psi_{1,2}^{[2j]} = \begin{bmatrix} \psi_{11...11} \\ (2j)^{\frac{1}{2}} \psi_{11...12} \\ \{2j(2j-1)/1.2\}^{\frac{1}{2}} \psi_{11...22} \\ \vdots \\ \psi_{22...22} \end{bmatrix}. \tag{A4}$$

Manuscript received 16 March 1978