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Commutation Rules for Generalized Pauli Spin Matrices

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Abstract

The algebra is developed for matrices involved in $2(2j+1)$-component arbitrary spin equations. These matrices can act as generators for the unitary group, and are shown to deserve the name 'generalized Pauli spin matrices'. Their commutation and anticommutation rules are derived from those for the ordinary Pauli spin matrices by a method termed mixed induced multiplication.

Introduction

Field equations for arbitrary spin fields have been discussed earlier (Jeffery 1978*a*), and a Lagrangian for arbitrary spin was considered in the preceding paper, hereinafter referred to as Paper I (Jeffery 1978*b*, present issue pp. 353–65). These formulations contained operators constructed from the induced matrices of those for the spin $\frac{1}{2}$ theory. In particular, the matrix $\boldsymbol{P}.\boldsymbol{\sigma}$ occurring in the spin $\frac{1}{2}$ theory was replaced by the $2j$th induced matrix of $\boldsymbol{P}.\boldsymbol{\sigma}$ in the spin j theory ($\boldsymbol{\sigma}$ is the Pauli spin matrix vector and $\boldsymbol{P}$ is the differential vector operator $-\mathrm{i}\nabla$).

As described in the Appendix of Paper I, induced matrices result when the transformation of a spinor (x, y) by a 2×2 matrix according to

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{1}$$

induces a transformation in a multispinor $(x, y)^{[n]}$ given by

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n]}\begin{pmatrix} x \\ y \end{pmatrix}^{[n]} = \begin{pmatrix} x' \\ y' \end{pmatrix}^{[n]}, \tag{2}$$

where

$$(x,y)^{[n]} = (x^n,\, n^{\frac{1}{2}}x^{n-1}y,\, \{n(n-1)/1\,.\,2\}^{\frac{1}{2}}x^{n-2}y^2,\, \dots,\, y^n)\,.$$

Equation (2) serves to define the nth induced matrix, symbolized by a superscript $[n]$ on the 2×2 matrix. The explicit forms of $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2]}$ and $(\boldsymbol{P}.\boldsymbol{\sigma})^{[3]}$ are given by equations (A3) of Paper I, and in general $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}$ is a $(2j+1)\times(2j+1)$ matrix.

An alternative way of constructing $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]}$ is to define a set of basic matrices $\sigma_{ij\dots n}$ with $2j$ indices such that $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2j]} = \sigma_{ij\dots n}P_iP_j\dots P_n$, where $i, j, \dots, n$ can be chosen from 1, 2 or 3 (referring to x, y and z respectively) and repeated indices are summed. Thus the basic matrices for $\sigma_{ij} P_i P_j$ to be equivalent to $(\boldsymbol{P}.\boldsymbol{\sigma})^{[2]}$ are

$$\sigma_{11} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \sigma_{12} = \begin{bmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix}, \tag{3a}$$

$$\sigma_{13} = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{1}{2}} \\ 0 & -\sqrt{\frac{1}{2}} & 0 \end{bmatrix}, \qquad \sigma_{22} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{3b}$$

$$\sigma_{23} = \begin{bmatrix} 0 & -\sqrt{\frac{1}{2}}\mathrm{i} & 0 \\ \sqrt{\frac{1}{2}}\mathrm{i} & 0 & \sqrt{\frac{1}{2}}\mathrm{i} \\ 0 & -\sqrt{\frac{1}{2}}\mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3c}$$

These together with the usual spin 1 matrices

$$\sigma_{10} = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{bmatrix}, \qquad \sigma_{20} = \begin{bmatrix} 0 & -\sqrt{\frac{1}{2}}\mathrm{i} & 0 \\ \sqrt{\frac{1}{2}}\mathrm{i} & 0 & -\sqrt{\frac{1}{2}}\mathrm{i} \\ 0 & \sqrt{\frac{1}{2}}\mathrm{i} & 0 \end{bmatrix}, \tag{4a}$$

$$\sigma_{30} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{4b}$$

constitute a particular set of generators for the unitary group in three dimensions, written $U(3)$. Notice that the identity matrix is $\sigma_{11}+\sigma_{22}+\sigma_{33}$.

The unitary group, particularly $U(3)$, has been intensively studied owing to its significance in the classification of elementary particles. However, the generators of $U(3)$ defined above are not the usual ones (cf. Gell-Mann and Ne'eman 1964). Furthermore it is tedious to derive commutation rules via the more familiar generators but these rules for the $\sigma_{ij\ldots n}$ are needed to develop an arbitrary spin theory.

The present paper shows how to derive the algebra for the required matrices via a procedure called mixed induced multiplication. This procedure allows the $\sigma_{ij\ldots n}$ and their commutation rules to be obtained directly from the Pauli spin matrices and their commutation rules.

Mixed Induced Multiplication

Consider the second induced matrix of a general 2×2 matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[2]} = \begin{bmatrix} a^2 & \sqrt{2}\,ac & c^2 \\ \sqrt{2}\,ab & (ad+bc) & \sqrt{2}\,cd \\ b^2 & \sqrt{2}\,bd & d^2 \end{bmatrix}. \tag{5}$$

For two different 2×2 matrices, the *mixed induced multiplication*, symbolized by $[\times]$, is defined by

$$\begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix}[\times]\begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 a_2 & \sqrt{\tfrac{1}{2}}(a_1 c_2 + a_2 c_1) & c_1 c_2 \\ \sqrt{\tfrac{1}{2}}(a_1 b_2 + a_2 b_1) & \tfrac{1}{2}(a_1 d_2 + b_1 c_2 + a_2 d_1 + b_2 c_1) & \sqrt{\tfrac{1}{2}}(c_1 d_2 + c_2 d_1) \\ b_1 b_2 & \sqrt{\tfrac{1}{2}}(b_1 d_2 + b_2 d_1) & d_1 d_2 \end{bmatrix}. \quad (6)$$

More generally the mixed induced matrix of n different 2×2 matrices $A_1, A_2, \ldots, A_n$, where A_r has the elements a_r, b_r, c_r and d_r, is constructed by permutating the indices $1, 2, \ldots, n$ between the elements of the ordinary nth induced matrix. A more precise definition is obtained by considering the n transformations of the spinor (x, y),

$$A_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad A_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad \ldots, \quad A_n\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \quad (7)$$

Then the mixed induced transformation of the multispinor $(x, y)^{[n]}$ is given by

$$A_1[\times]A_2[\times]\ldots[\times]A_n\begin{bmatrix} x \\ y \end{bmatrix}^{[n]} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}[\times]\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}[\times]\ldots[\times]\begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad (8a)$$

where

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}[\times]\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}[\times]\ldots[\times]\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 x_2 \ldots x_n \\ n^{-\frac{1}{2}} \mathrm{P}^{\alpha\beta\ldots\tau}_{12\ldots n} y_\alpha x_\beta \ldots x_\tau \\ \{n(n-1)/1.2\}^{-\frac{1}{2}} \mathrm{P}^{\alpha\beta\gamma\ldots\tau}_{123\ldots n} y_\alpha y_\beta x_\gamma \ldots x_\tau \\ \vdots \\ y_1 y_2 \ldots y_n \end{bmatrix}. \quad (8b)$$

$\mathrm{P}^{\alpha\beta\ldots\tau}_{12\ldots n}$ is the permutation $\alpha\beta\ldots\tau$ of the indices $12\ldots n$ and repeated indices are summed. For example, $\mathrm{P}^{\alpha\beta}_{12} y_\alpha x_\beta$ is equal to $y_1 x_2 + y_2 x_1$. The matrix $A_1[\times]A_2[\times]\ldots[\times]A_n$ is the mixed induced matrix of $A_1, A_2, \ldots, A_n$.

The advantage of defining such a form of multiplication is that it leads to a remarkably simple matrix algebra. Thus, where $A, B, C, \ldots$ and $P, Q, R, \ldots$ are 2×2 matrices, I is the 2×2 identity matrix and l and m are constants, the following relations can be easily proved:

$$A[\times]B = B[\times]A, \quad (9a)$$

$$(A+B)[\times]C = A[\times]C + B[\times]C, \quad (9b)$$

$$(A+B)^{[n]} = A^{[n]} + nA^{[n-1]}[\times]B + \{n(n-1)/1.2\}A^{[n-2]}[\times]B^{[2]} + \ldots + B^{[n]}, \quad (9c)$$

$$(A-lI)(A-mI) = A^{[2]} - (l+m)A[\times]I + lmI^{[2]}, \quad (9d)$$

$$PA[\times]QB+QA[\times]PB = 2(P[\times]Q)(A[\times]B), \tag{9e}$$

$$PA[\times]QB[\times]RC+QA[\times]PB[\times]RC+RA[\times]QB[\times]PC+PA[\times]RB[\times]QC$$
$$+QA[\times]RB[\times]PC+RA[\times]PB[\times]QC = 6(P[\times]Q[\times]R)(A[\times]B[\times]C). \tag{9f}$$

The 'binomial' rule (9c) gives simplicity to the 'De Moivre' formula

$$(\cos\tfrac{1}{2}\theta - \mathrm{i}\boldsymbol{\sigma}.\boldsymbol{n}\sin\tfrac{1}{2}\theta)^{[2j]} = \exp(-\mathrm{i}\theta\boldsymbol{n}.\boldsymbol{J}), \tag{10a}$$

which was described in the earlier paper (Jeffery 1978*a*). For example, for spin $j = 1$ the left-hand side of (10a) is equal to

$$(\cos^2\tfrac{1}{2}\theta)I^{[2]} - 2\mathrm{i}(\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta)(\boldsymbol{\sigma}.\boldsymbol{n})[\times]I - (\sin^2\tfrac{1}{2}\theta)(\boldsymbol{\sigma}.\boldsymbol{n})^{[2]}. \tag{10b}$$

Equations (9e) and (9f) are particularly important for obtaining the commutation rules for σ_{ij} and σ_{ijk}.

Generalized Pauli Spin Matrices

The Pauli spin matrices σ_1, σ_2 and σ_3 along with the identity matrix σ_0 are

$$\sigma_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \sigma_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \sigma_2 = \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}, \quad \sigma_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{11}$$

The σ_{ij} of equations (3) and (4) are then given by

$$\sigma_{ij} = \sigma_i[\times]\sigma_j \tag{12}$$

and more generally

$$\sigma_{ij\ldots n} = \sigma_i[\times]\sigma_j[\times]\ldots[\times]\sigma_n. \tag{13}$$

The nth induced matrix of the general 2×2 matrix $(a_0+\boldsymbol{a}.\boldsymbol{\sigma})^{[n]}$ is given by

$$(a_0+\boldsymbol{a}.\boldsymbol{\sigma})^{[n]} = a_i a_j \ldots a_n \sigma_{ij\ldots n}, \tag{14}$$

where repeated indices are summed and can take on the values 0, 1, 2 or 3.

Commutation Rules

Consider first the commutator $[\sigma_{ij}, \sigma_{kl}]$ where i, j, k, and l are all chosen from 1, 2 and 3. From equation (12),

$$[\sigma_{ij}, \sigma_{kl}] = [\sigma_i[\times]\sigma_j, \sigma_k[\times]\sigma_l] \tag{15}$$

and therefore, with the help of the rule (9e),

$$[\sigma_{ij}, \sigma_{kl}] = \tfrac{1}{2}\big(\sigma_i\sigma_k[\times]\sigma_j\sigma_l + \sigma_j\sigma_k[\times]\sigma_i\sigma_l - \sigma_k\sigma_i[\times]\sigma_l\sigma_j - \sigma_l\sigma_i[\times]\sigma_k\sigma_j\big). \tag{16}$$

By the well-known property of the Pauli spin matrices

$$\sigma_\alpha \sigma_\beta = \mathrm{i}\varepsilon_{\alpha\beta s}\sigma_s + \delta_{\alpha\beta}, \tag{17}$$

equation (16) then becomes

$$[\sigma_{ij}, \sigma_{kl}] = \mathrm{i}\varepsilon_{iks}\delta_{jl}\sigma_{s0} + \mathrm{i}\varepsilon_{jls}\delta_{ik}\sigma_{s0} + \mathrm{i}\varepsilon_{jks}\delta_{il}\sigma_{s0} + \mathrm{i}\varepsilon_{ils}\delta_{jk}\sigma_{s0} \tag{18}$$

or

$$[\sigma_{ij}, \sigma_{kl}] = \mathrm{i}\,\mathrm{P}_{ij}^{\alpha\beta}\,\mathrm{P}_{kl}^{\gamma\delta}\,\varepsilon_{\alpha\gamma s}\delta_{\beta\delta}\sigma_{s0}, \tag{19a}$$

where the permutation operator $\mathrm{P}_{ij}^{\alpha\beta}$ requires α, β to be summed over as many permutations of i,j as possible and the right-hand side is summed over the dummy index s for $s = 1, 2, 3$. Similarly, the anticommutator is found to be

$$[\sigma_{ij}, \sigma_{kl}]_+ = \mathrm{P}_{ij}^{\alpha\beta}\,\mathrm{P}_{kl}^{\gamma\delta}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \varepsilon_{\alpha\gamma s}\varepsilon_{\beta\delta t}\sigma_{st}), \tag{19b}$$

where $s, t = 1, 2, 3$. When one of the indices is zero, these results become

$$[\sigma_{ij}, \sigma_{k0}] = \mathrm{i}\varepsilon_{iks}\sigma_{js} + \mathrm{i}\varepsilon_{jks}\sigma_{is}, \qquad [\sigma_{ij}, \sigma_{k0}]_+ = \delta_{ik}\sigma_{j0} + \delta_{jk}\sigma_{i0}. \tag{20}$$

In general the commutators contain only odd multiples of the Levi–Civita tensor whereas the anticommutators contain only even multiples or none. In principle the general formulae can be obtained by substituting from the relation

$$\sigma_\mu \sigma_\nu = \delta_{\mu 0}\sigma_\nu + \delta_{\nu 0}\sigma_\mu - \delta_{\mu 0}\delta_{\nu 0} + \mathrm{i}\delta_{\mu\nu s}^{123}\sigma_s, \tag{21}$$

(where μ and ν can be 0, 1, 2 or 3) into the formula

$$\begin{aligned}
&[\sigma_{\mu\nu\ldots\eta}, \sigma_{\mu'\nu'\ldots\eta'}]_\pm \\
&\quad = (n!)^{-1}\,\mathrm{P}_{\mu\nu\ldots\eta}^{\alpha\beta\ldots\gamma}\,\mathrm{P}_{\mu'\nu'\ldots\eta'}^{\alpha'\beta'\ldots\gamma'}\big(\sigma_\alpha\sigma_{\alpha'}[\times]\sigma_\beta\sigma_{\beta'}[\times]\ldots[\times]\sigma_\gamma\sigma_{\gamma'} \\
&\qquad\qquad \pm\sigma_{\alpha'}\sigma_\alpha[\times]\sigma_{\beta'}\sigma_\beta[\times]\ldots[\times]\sigma_{\gamma'}\sigma_\gamma\big).
\end{aligned} \tag{22}$$

However, the principles have been established and there is nothing to be gained by giving the involved general formulae. For interest the complete results for the $U(4)$ generators are listed in Appendix 1, together with the σ_{ijk} matrices.

There are a total of $(n+3)!/n!\,3!$ matrices when $\sigma_{ij\ldots n}$ has n indices each of which can have values 0–4 (see Appendix 2). Some of these matrices are linear combinations of the others, but $(n+1)^2$ independent matrices can always be obtained. Thus for σ_{ij} there are 10 matrices whereas $U(3)$ needs only 9 generators. A possible independent set can be obtained by excluding the identity matrix σ_{00}, because this is equal to $\sigma_{11}+\sigma_{22}+\sigma_{33}$. Similarly, an independent set of matrices for generating $U(n)$ can be obtained by excluding all matrices with more than one zero in their indices. To prove this consider the four matrices

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \tag{23}$$

which are the linear combinations $\frac{1}{2}(\sigma_0+\sigma_3)$, $\frac{1}{2}(\sigma_1+\mathrm{i}\sigma_2)$, $\frac{1}{2}(\sigma_0-\sigma_3)$ and $\frac{1}{2}(\sigma_1-\mathrm{i}\sigma_2)$ respectively. Mixed induced multiplication with n matrices, each of which is selected from the four matrices (23), will produce an $(n+1)^2$ matrix of which only one element is nonzero; for example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{[n-1]} [\times] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n^{-\frac{1}{2}} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{24}$$

By a suitable selection of the n matrices, any particular element of the $(n+1)^2$ matrix can be made the nonzero element. Hence an arbitrary $(n+1)^2$ matrix is obtained by a linear combination of these various mixed induced matrices. From the definition of mixed induced matrix multiplication it follows that similar linear combinations of mixed induced matrices from σ_0, σ_1, σ_2 and σ_3 will also produce an arbitrary $(n+1)^2$ matrix. Now the total number of basic matrices $\sigma_{ij\dots n}$ having n indices which can take on values 0, 1, 2 or 3 is $(n+3)!/n!\,3!$, whereas only $(n+1)^2$ are needed as generators for $U(n+1)$. Hence $(n+1)!/(n-2)!\,3!$ matrices are superfluous. The superfluous matrices can be expressed as linear combinations of the others. From the identity $\sigma_{00} = \sigma_{11}+\sigma_{22}+\sigma_{33}$ and the definition (8) there is little difficulty in proving

$$\sigma_{00ij\dots k} = \sigma_{11ij\dots k} + \sigma_{22ij\dots k} + \sigma_{33ij\dots k}.$$

Hence all matrices with more than one zero in their indices can be expressed as linear combinations of the other matrices; for example,

$$\begin{aligned} \sigma_{00000} &= \sigma_{11000}+\sigma_{22000}+\sigma_{33000} \\ &= (\sigma_{11110}+\sigma_{11220}+\sigma_{11330})+(\sigma_{22110}+\sigma_{22220}+\sigma_{22330}) \\ &\quad +(\sigma_{33110}+\sigma_{33220}+\sigma_{33330}). \end{aligned} \tag{25}$$

There are exactly $(n+1)^2$ matrices having one or no zero in their indices (Appendix 2) and, since a linear combination of these has been proved capable of generating any $(n+1)^2$ dimensional matrix, they must all be independent and can be used as generators of $U(n+1)$.

Miscellaneous Relations

Many other relations between the generalized matrices can be obtained from the properties of the Pauli spin matrices. For example, since

$$\sigma_1 \sigma_2 \sigma_3 = \mathrm{i} \tag{26}$$

then

$$\sigma_1^{[n]} \sigma_2^{[n]} \sigma_3^{[n]} = \mathrm{i}^n \tag{27a}$$

or

$$\sigma_{11\dots 1} \sigma_{22\dots 2} \sigma_{33\dots 3} = \mathrm{i}^n. \tag{27b}$$

Similarly, for $i,j = 1,2,3$ and $i \neq j$,

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \tag{28}$$

so that

$$\sigma_i^{[n]}\sigma_j^{[n]} = (-)^n \sigma_j^{[n]}\sigma_i^{[n]} \tag{29a}$$

or

$$\sigma_{ii\ldots i}\sigma_{jj\ldots j} + (-)^{n-1}\sigma_{jj\ldots j}\sigma_{ii\ldots i} = 0. \tag{29b}$$

However, these types of relations will not be considered further.

The operators $\pi(-i\partial)$ and $\bar{\pi}(-i\partial)$ given by Weinberg (1964) have already been noted to be equivalent to $[E-\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]}$ and $[E+\boldsymbol{P}.\boldsymbol{\sigma}]^{[2j]}$ (Jeffery 1978*a*). Furthermore, Weinberg's generalized Dirac matrices $\gamma^{\mu_1\mu_2\cdots\mu_{2j}}$ are easily found to be given by

$$\gamma^{\mu_1\mu_2\ldots\mu_{2j}} = \begin{bmatrix} 0 & (-)^{2j+Z}\sigma_{\mu_1\mu_2\ldots\mu_{2j}} \\ \sigma_{\mu_1\mu_2\ldots\mu_{2j}} & 0 \end{bmatrix}, \tag{30}$$

where Z is the number of zeros in the indices $\mu_1\,\mu_2\ldots\mu_{2j}$. Thus the commutation and anticommutation relations for the $\gamma^{\mu_1\mu_2\cdots\mu_{2j}}$ matrices are completely determined by those for the $\sigma_{\mu_1\mu_2\ldots\mu_{2j}}$ matrices.

Lastly, differentiation rules for the matrices are easily established; for example,

$$\frac{\partial}{\partial a}\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n]} = n\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n-1]}[\times]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{31a}$$

and generally

$$\frac{\partial^r}{\partial a^r}\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n]} = \frac{n!}{(n-r)!}\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{[n-r]}[\times]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{[r]}. \tag{31b}$$

When the nth induced matrix is written in the form (see equation 14)

$$(a_0 + \boldsymbol{a}.\boldsymbol{\sigma})^{[n]} \equiv a_i a_j \ldots a_n \sigma_{ij\ldots n},$$

it is obvious that

$$\sigma_{ij\ldots n} = \frac{1}{\mathrm{P}_{ij\ldots n}^{\alpha\beta\ldots\tau}}\frac{\partial^n}{\partial a_\alpha \partial a_\beta \ldots \partial a_\tau}\Big(a_0 + \boldsymbol{a}.\boldsymbol{\sigma}\Big)^{[n]}.$$

When a, b, c and d are functions of x and differentiation is carried out with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} a(x) & c(x) \\ b(x) & d(x) \end{bmatrix}^{[n]} = n\begin{bmatrix} a(x) & c(x) \\ b(x) & d(x) \end{bmatrix}^{[n-1]}[\times]\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix} a(x) & c(x) \\ b(x) & d(x) \end{bmatrix}, \tag{32}$$

a result which can be easily generalized to higher derivatives and more than one variable.

References

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Appendix 1

The complete set of commutation rules for the $U(4)$ generators is presented here. In the following commutators and anticommutators i, j, k, l, m and n can take the values 1, 2 or 3, while zero indices are shown explicitly.

$$[\sigma_{ijk}, \sigma_{lmn}] = \tfrac{1}{6}\mathrm{i}\, \mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \mathrm{P}^{\delta\tau\sigma}_{lmn}(\varepsilon_{\alpha\delta s}\,\delta_{\beta\tau}\,\delta_{\gamma\sigma}\,\sigma_{s00} - \tfrac{1}{3}\varepsilon_{\alpha\delta s}\,\varepsilon_{\beta\tau t}\,\varepsilon_{\gamma\delta v}\,\sigma_{stv}), \tag{A1a}$$

$$[\sigma_{ijk}, \sigma_{lmn}]_+ = \tfrac{1}{6}\mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \mathrm{P}^{\delta\tau\sigma}_{lmn}(\tfrac{1}{3}\delta_{\alpha\delta}\,\delta_{\beta\tau}\,\delta_{\gamma\sigma} - \varepsilon_{\alpha\delta s}\,\varepsilon_{\beta\tau t}\,\delta_{\gamma\sigma}\,\sigma_{st0}), \tag{A1b}$$

$$[\sigma_{ijk}, \sigma_{lm0}] = \tfrac{1}{3}\mathrm{i}\, \mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \mathrm{P}^{\delta\tau}_{lm}\, \varepsilon_{\alpha\delta s}\,\delta_{\beta\tau}\,\sigma_{s\gamma 0}, \tag{A2a}$$

$$[\sigma_{ijk}, \sigma_{lm0}]_+ = \tfrac{1}{6}\mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \mathrm{P}^{\delta\tau}_{lm}(\delta_{\alpha\delta}\,\delta_{\beta\tau}\,\sigma_{\gamma 00} - \varepsilon_{\alpha\delta s}\,\varepsilon_{\beta\tau t}\,\sigma_{st\gamma}), \tag{A2b}$$

$$[\sigma_{ij0}, \sigma_{lm0}] = \tfrac{1}{3}\mathrm{i}\, \mathrm{P}^{\alpha\beta}_{ij}\, \mathrm{P}^{\gamma\tau}_{lm}(\varepsilon_{\alpha\gamma s}\,\sigma_{s\beta\tau} + \varepsilon_{\alpha\gamma s}\,\delta_{\beta\tau}\,\sigma_{s00}), \tag{A3a}$$

$$[\sigma_{ij0}, \sigma_{lm0}]_+ = \tfrac{1}{6}\mathrm{P}^{\alpha\beta}_{ij}\, \mathrm{P}^{\gamma\tau}_{lm}(\delta_{\alpha\gamma}\,\delta_{\beta\tau} + 2\delta_{\alpha\gamma}\,\sigma_{\beta\tau 0} - \varepsilon_{\alpha\gamma s}\,\varepsilon_{\beta\tau t}\,\sigma_{st0}), \tag{A3b}$$

$$[\sigma_{ijk}, \sigma_{l00}] = \tfrac{1}{3}\mathrm{i}\, \mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \varepsilon_{\alpha l s}\,\sigma_{s\beta\gamma}, \qquad [\sigma_{ijk}, \sigma_{l00}]_+ = \tfrac{1}{3}\mathrm{P}^{\alpha\beta\gamma}_{ijk}\, \delta_{\alpha l}\,\sigma_{\beta\gamma 0}, \tag{A4}$$

$$[\sigma_{ij0}, \sigma_{k00}] = \tfrac{2}{3}\mathrm{i}\, \mathrm{P}^{\alpha\beta}_{ij}\, \varepsilon_{\alpha k s}\,\sigma_{s\beta 0}, \qquad [\sigma_{ij0}, \sigma_{k00}]_+ = \tfrac{2}{3}\mathrm{P}^{\alpha\beta}_{ij}(\delta_{\alpha k}\,\sigma_{\beta 00} + \sigma_{ijk}), \tag{A5}$$

$$[\sigma_{i00}, \sigma_{j00}] = \tfrac{2}{3}\mathrm{i}\, \varepsilon_{ijs}\,\sigma_{s00}, \qquad [\sigma_{i00}, \sigma_{j00}]_+ = \tfrac{2}{3}(\delta_{ij} + 2\sigma_{ij0}). \tag{A6}$$

The σ_{ijk} matrices in equations (A1)–(A6) are as follows.

$$\sigma_{111} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \sigma_{112} = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\tfrac{1}{3}\mathrm{i} & 0 \\ 0 & \tfrac{1}{3}\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix},$$

$$\sigma_{113} = \begin{bmatrix} 0 & 0 & \sqrt{\tfrac{1}{3}} & 0 \\ 0 & \tfrac{2}{3} & 0 & -\sqrt{\tfrac{1}{3}} \\ \sqrt{\tfrac{1}{3}} & 0 & -\tfrac{2}{3} & 0 \\ 0 & -\sqrt{\tfrac{1}{3}} & 0 & 0 \end{bmatrix}, \qquad \sigma_{122} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \tfrac{1}{3} & 0 \\ 0 & \tfrac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\sigma_{123} = \begin{bmatrix} 0 & 0 & -\sqrt{\tfrac{1}{3}}\mathrm{i} & 0 \\ 0 & 0 & 0 & \sqrt{\tfrac{1}{3}}\mathrm{i} \\ \sqrt{\tfrac{1}{3}}\mathrm{i} & 0 & 0 & 0 \\ 0 & -\sqrt{\tfrac{1}{3}}\mathrm{i} & 0 & 0 \end{bmatrix}, \qquad \sigma_{133} = \begin{bmatrix} 0 & \sqrt{\tfrac{1}{3}} & 0 & 0 \\ \sqrt{\tfrac{1}{3}} & 0 & -\tfrac{2}{3} & 0 \\ 0 & -\tfrac{2}{3} & 0 & \sqrt{\tfrac{1}{3}} \\ 0 & 0 & \sqrt{\tfrac{1}{3}} & 0 \end{bmatrix},$$

$$\sigma_{222}=\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix},\quad \sigma_{223}=\begin{bmatrix} 0 & 0 & -\sqrt{\frac{1}{3}} & 0 \\ 0 & \frac{2}{3} & 0 & \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & 0 & -\frac{2}{3} & 0 \\ 0 & \sqrt{\frac{1}{3}} & 0 & 0 \end{bmatrix},$$

$$\sigma_{233}=\begin{bmatrix} 0 & -\sqrt{\frac{1}{3}}\,i & 0 & 0 \\ \sqrt{\frac{1}{3}}\,i & 0 & \frac{2}{3}i & 0 \\ 0 & -\frac{2}{3}i & 0 & -\sqrt{\frac{1}{3}}\,i \\ 0 & 0 & \sqrt{\frac{1}{3}}\,i & 0 \end{bmatrix},\quad \sigma_{333}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\sigma_{110}=\begin{bmatrix} 0 & 0 & \sqrt{\frac{1}{3}} & 0 \\ 0 & \frac{2}{3} & 0 & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & 0 & \frac{2}{3} & 0 \\ 0 & \sqrt{\frac{1}{3}} & 0 & 0 \end{bmatrix},\quad \sigma_{120}=\begin{bmatrix} 0 & 0 & -\sqrt{\frac{1}{3}}\,i & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{1}{3}}\,i \\ \sqrt{\frac{1}{3}}\,i & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}}\,i & 0 & 0 \end{bmatrix},$$

$$\sigma_{130}=\begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{1}{3}} \\ 0 & 0 & -\sqrt{\frac{1}{3}} & 0 \end{bmatrix},\quad \sigma_{220}=\begin{bmatrix} 0 & 0 & -\sqrt{\frac{1}{3}} & 0 \\ 0 & \frac{2}{3} & 0 & -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{3}} & 0 & \frac{2}{3} & 0 \\ 0 & -\sqrt{\frac{1}{3}} & 0 & 0 \end{bmatrix},$$

$$\sigma_{230}=\begin{bmatrix} 0 & -\sqrt{\frac{1}{3}}\,i & 0 & 0 \\ \sqrt{\frac{1}{3}}\,i & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}}\,i \\ 0 & 0 & -\sqrt{\frac{1}{3}}\,i & 0 \end{bmatrix},\quad \sigma_{330}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{100}=\begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 & 0 \\ \sqrt{\frac{1}{3}} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \sqrt{\frac{1}{3}} \\ 0 & 0 & \sqrt{\frac{1}{3}} & 0 \end{bmatrix},\quad \sigma_{200}=\begin{bmatrix} 0 & -\sqrt{\frac{1}{3}}\,i & 0 & 0 \\ \sqrt{\frac{1}{3}}\,i & 0 & -\frac{2}{3}i & 0 \\ 0 & \frac{2}{3}i & 0 & -\sqrt{\frac{1}{3}}\,i \\ 0 & 0 & \sqrt{\frac{1}{3}}\,i & 0 \end{bmatrix},$$

$$\sigma_{300}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},\quad \sigma_{000}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Appendix 2

The matrices σ_{ij} for $i,j = 1,2,3$ are

$$(\sigma_{11},\sigma_{12},\sigma_{13})\ (\sigma_{22},\sigma_{23})\ \sigma_{33},$$

that is, a total of $3+2+1 = 6$ matrices; similarly the matrices σ_{ijk} for $i,j = 1,2,3$ are

$$(\sigma_{111},\sigma_{112},\sigma_{113};\sigma_{122},\sigma_{123};\sigma_{133})\ (\sigma_{222},\sigma_{223};\sigma_{233})\ \sigma_{333},$$

that is, a total of $(3+2+1)+(2+1)+1 = 10$ matrices. Let us therefore define the series

$$^{(1)}\textstyle\sum n = 1+2+3+\ldots+n, \tag{A7a}$$

$$^{(2)}\textstyle\sum n = 1+(1+2)+\ldots+(1+2+\ldots+n) \tag{A7b}$$

and, in general,

$$^{(r)}\textstyle\sum n = {}^{(r-1)}\sum 1 + {}^{(r-1)}\sum 2 + \ldots + {}^{(r-1)}\sum n, \tag{A7c}$$

with

$$^{(0)}\textstyle\sum n = 1+1+\ldots+1 = n. \tag{A7d}$$

There is no difficulty in proving that

$$^{(r)}\textstyle\sum n = \{(n+r)!/(r+1)!\,(n-1)!\}. \tag{A8}$$

The $\sigma_{ij\ldots m}$ matrices for spin j have $2j$ indices. The number of these with no zero indices is $^{(2j-1)}\sum 3$, and the number with one and only one zero index is $^{(2j-2)}\sum 3$, giving a total of

$$^{(2j-1)}\textstyle\sum 3 + {}^{(2j-2)}\sum 3 = (2j+1)^2,$$

with the help of equation (A8). This is exactly the number required for a non-degenerate representation of $U(2j+1)$. The total number of matrices in $\sigma_{ij\ldots m}$ is $^{(2j-1)}\sum 4$, of which $^{(2j-3)}\sum 4$ are superfluous.

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