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A Parafermion Generalization of Poincaré Supersymmetry

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Abstract

The space–time Poincaré algebra is extended by introducing a four-spinor generator whose components satisfy certain trilinear parafermi commutation relations. The spin content of the irreducible multiplets is analysed in the massive and massless cases, and weight diagrams constructed, for arbitrary order p of the parastatistics. The supersymmetry algebra of Wess and Zumino, and of Salam and Strathdee, is exhibited as the special case of order $p = 1$ in this formulation.

1. Introduction

Despite its apparent lack of success in yielding realistic models and physical predictions, the notion of Fermi–Bose symmetry, or 'supersymmetry', has provided an important stimulus to several areas of particle physics (see review by Fayet and Ferrara 1977). For example, much interest currently attaches to the derived idea of local supersymmetry and the rich possibilities inherent in the 'supergravity' theories (reviews are given by Wess 1977; Freedman and van Nieuwenhuizen 1978).

In parallel with the more applied developments, considerable attention has been given to elucidating the classification and properties of the mathematical structures underlying supersymmetry, namely the graded Lie algebras (see e.g. Corwin *et al.* 1975; Kac 1977; Rittenberg 1977). In particular, in terms of this classification theory (using the notation of Rittenberg 1977), the original Poincaré supersymmetry algebra of Wess and Zumino (1974), and of Salam and Strathdee (1974*a*), can be seen as a contraction of the orthosymplectic graded Lie algebra osp(1, 4); the underlying sp(4) Lie algebra is that of the de Sitter group O(3, 2). Similarly, the full conformal supersymmetry of Wess and Zumino (1974) is just the special linear graded Lie algebra spl(1, 4), the underlying sl(4) Lie algebra being that of the conformal group SU(2, 2) $\sim$ SO(4, 2). Seen from this standpoint, it is very natural to conjecture that the generalizations to the osp(N, 4) and spl(N, 4) graded Lie algebras should represent a nontrivial unification of the space–time de Sitter (or Poincaré when contracted) or conformal symmetries with an internal O(N) or SU(N) symmetry respectively. Such steps were indeed taken in the literature, but without the benefit of hindsight and the classification theorems for graded Lie algebras. Some of the results are described by Fayet and Ferrara (1977); as these authors point out, the spin and internal multiplet structures involved in the proposed generalizations are rather unphysical, even if some of the components are assigned as the unobserved quarks and scalar bosons of the current genre of unified theories.

Perhaps to circumvent the difficulties noted by Fayet and Ferrara (1977), various sorts of generalizations of the original Z_2-graded Lie algebras have been proposed (Omote *et al.* 1976; Mansouri 1977; Lukierski and Rittenberg 1978; Rittenberg and Wyler 1978*a*, 1978*b*). In particular, the 'colour superalgebras' considered by Lukierski and Rittenberg (1978), based on a $Z_2 \oplus Z_2 \oplus \dots \oplus Z_2$ grading, are related to the parastatistics representation of colour (Drühl *et al.* 1970; Greenberg and Nelson 1977).

All the above examples involve generalized Lie algebra brackets which are bilinear in the generators. It is the purpose of this paper to point out another straightforward means of generalizing a graded Lie algebra to a larger algebraic structure. In the spirit of the original formulation of parastatistics (Green 1953), we impose not bilinear but trilinear parafermion commutation relations for the odd generators. The underlying graded Lie algebra and its irreducible representations now realize only a special case of the general trilinear algebra, corresponding to ordinary Fermi statistics. For arbitrary order of the parastatistics, many other representations exist.

In Section 2, we illustrate our construction with the Poincaré supersymmetry algebra and establish the existence of an internal Poincaré-invariant $\mathrm{SU}(2)\times\mathrm{U}(1)$ algebra. In Section 3, we analyse the spin and internal content of the irreducible multiplets in the massive and massless cases ($P_\mu P^\mu > 0$ and $P_\mu P^\mu = 0$ respectively, and $P_0 > 0$), for arbitrary order p of the parafermi statistics, and illustrate with weight diagrams. The original Poincaré supersymmetry corresponds to the $p = 1$ case (Fermi statistics). Some concluding remarks are made in Section 4.

2. Parafermion Generalization of Poincaré Supersymmetry

Supersymmetry can be viewed as the grading of the Poincaré Lie algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(\eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\rho} J_{\nu\sigma}), \tag{1a}$$

$$[J_{\mu\nu}, P_\rho] = \mathrm{i}(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu), \tag{1b}$$

$$[P_\mu, P_\nu] = 0 \tag{1c}$$

(where $J_{\mu\nu}$ ($\mu, \nu = 0,1,2,3$) can be chosen hermitian) by a 4-spinor S_α ($\alpha = 1,2,3,4$) as

$$[J_{\mu\nu}, S_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta, \qquad [P_\mu, S_\alpha] = 0, \tag{2}$$

which is to be made Majorana, that is, $S_\alpha = C_{\alpha\beta}\bar{S}^\beta$. The odd generators S_α satisfy the anticommutation relations

$$\{S_\alpha, S_\beta\} = -(\gamma_\mu C)_{\alpha\beta} P^\mu. \tag{3}$$

Equations (1), (2) and (3) constitute the Poincaré supersymmetry algebra (Wess and Zumino 1974; Salam and Strathdee 1974*a*).

From equations (2) and (3) it is easily verified that

$$[S_\alpha, [S_\beta, S_\gamma]] = -2(\gamma . PC)_{\alpha\beta} S_\gamma + 2(\gamma . PC)_{\alpha\gamma} S_\beta. \tag{4}$$

For the remainder of this paper we shall analyse the algebra $\mathscr{S}$ generated by $J_{\mu\nu}$ and S_α with the commutation relations (1), (2) and (4), to give an example of the way of generalizing graded Lie algebra structures as suggested in the previous section.

To begin with, observe that the operator

$$P^2 = P_\mu P^\mu \tag{5}$$

commutes with the generators and, as in the ordinary Poincaré supersymmetry, serves to label irreducible representations of $\mathscr{S}$. If we imagine decomposing the latter with respect to the Poincaré subalgebra, we should thus find the same mass $m^2 = P^2$ for each irreducible constituent; because of the spinor nature of S_α, however, we might expect to find both integral and half-integral constituent spins. The new algebra thus retains the character of a Bose–Fermi symmetry, as in the Poincaré supersymmetry case.

The spin structure is further elucidated by observing that the bilinear combinations*

$$\Sigma_\pm = \tfrac{1}{4}\bar{S}(1\pm \mathrm{i}\gamma_5), \qquad \Sigma_5 = \tfrac{1}{4}\bar{S}\mathrm{i}\gamma_5\,\gamma.PS, \qquad Q = \tfrac{1}{4}\bar{S}\,\gamma.PS$$

are Poincaré invariant and generate a Lie algebra of $\mathrm{SU}(2)\times\mathrm{U}(1)$:

$$[\Sigma_5,\Sigma_\pm] = \pm P^2\Sigma_\pm, \qquad [\Sigma_+,\Sigma_-] = 2\Sigma_5, \qquad [\Sigma_\pm, Q] = 0 = [\Sigma_5, Q]. \tag{6}$$

Thus, within an irreducible multiplet of $\mathscr{S}$, each constituent spin carries a representation of this internal $\mathrm{SU}(2)\times\mathrm{U}(1)$. Moreover, defining $(S_\pm)_\alpha = \frac{1}{2}(1\pm\mathrm{i}\gamma_5)S_\alpha$, we find

$$[\Sigma_5,(S_\pm)_\alpha] = \pm\tfrac{1}{2}P^2(S_\pm)_\alpha, \tag{7}$$

so that $(S_\pm)_\alpha$ acts as a shifting operator for the SU(2) weights. The connection between S_α and Q is established in Section 3.

Finally, let us justify the assertion that the algebra $\mathscr{S}$ is a parafermion generalization of Poincaré supersymmetry. Defining $[S_\alpha, S_\beta] = S_{\alpha\beta}$, we have from equation (4)

$$[S_{\alpha\beta}, S_{\gamma\delta}] = -2(\gamma.PC)_{\alpha\delta}S_{\beta\gamma} - 2(\gamma.PC)_{\beta\gamma}S_{\alpha\delta} + 2(\gamma.PC)_{\alpha\gamma}S_{\beta\delta} + 2(\gamma.PC)_{\beta\delta}S_{\alpha\gamma}, \tag{8a}$$

$$[S_{\alpha\beta}, S_\gamma] = -2(\gamma.PC)_{\beta\gamma}S_\alpha + 2(\gamma.PC)_{\alpha\gamma}S_\beta, \tag{8b}$$

so that S_α and $S_{\beta\gamma}$ generate an O(5) Lie algebra, with metric $-2(\gamma.PC)$ in the 4×4 sector, -1 in the 1×1 sector and zero elsewhere. However, this is precisely the structure generated by two pairs of parafermion creation and annihilation operators, but with a fixed metric (Kamefuchi and Takahashi 1962; Ryan and Sudarshan 1963).

3. Irreducible Multiplets

In this section we study certain classes of irreducible representations of the extended algebra $\mathscr{S}$, by the Wigner method of induced representations (cf. Salam and Strathdee 1974*b*) but making use of known results from the representation theory of parafermion algebras (Bracken and Green 1972).

* We use the metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ and define $\varepsilon_{0123} = +1$. The Dirac algebra is generated by $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$ and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. Adjoint and conjugate spinors are defined by $\bar{S}^\alpha = (A^{-1})^{\alpha\beta}S_\beta{}^\dagger$ and $(\bar{S}^c)^\alpha = (C^{-1})^{\alpha\beta}S_\beta$ respectively.

(a) Massive Case

Consider the time-like case $P_\mu P^\mu = m^2 > 0$. The second Casimir operator of the algebra (the generalization of the square of the Pauli–Lubanski vector $W_\mu W^\mu$) can be written down precisely as in the Poincaré supersymmetry case (Salam and Strathdee 1974*b*). With the usual definitions

$$\Omega_\mu = \tfrac{1}{4}\bar{S}\mathrm{i}\gamma_5\big(\gamma_\mu - P_\mu/(\gamma\,.\,P)\big)S, \qquad W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma},$$

we have that the vector

$$K_\mu = W_\mu - \Omega_\mu \tag{9}$$

commutes with S_α and P_μ, and its square $K_\mu K^\mu$ is the required Casimir invariant.

The induced representation method proceeds by casting the algebra into the rest frame, where $P_\mu = m(1,0,0,0)$, and finding irreducible multiplets of the 'little algebra' $\hat{\mathscr{S}}$ which leaves P_μ invariant. This is generated by rotations $\boldsymbol{J} = (J_{23}, J_{31}, J_{12})$ and by the spinor generators S_α. Choosing a specific representation of the Dirac algebra such that

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix},$$

where the $\boldsymbol{\sigma}$ are the 2×2 Pauli matrices, we can write the little algebra in terms of the two-component spinor S_a ($a = 1,2$) and its hermitian conjugate $S_a^\dagger$ as

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k, \qquad [J_i, S_a] = \tfrac{1}{2}(\sigma_i)_a{}^b S_b, \tag{10a}$$

$$[S_a, [S_b, S_c]] = 0, \qquad [S_a, [S_b^\dagger, S_c]] = 2m\delta_{ab}S_c, \tag{10b}$$

$$[S_a, [S_b^\dagger, S_c^\dagger]] = 2m\delta_{ab}S_c^\dagger - 2m\delta_{ac}S_b^\dagger. \tag{10c}$$

We can construct multiplets of $\hat{\mathscr{S}}$ as follows (Salam and Strathdee 1974*b*). Introduce a set of states $|j_0\lambda\rangle$ ($\lambda = -j_0, -j_0+1, ..., j_0$), carrying an irreducible representation of SU(2) with 'superspin' j_0. Then, formally, the space spanned by the set $|j_0\lambda\rangle$, $S_a^\dagger|j_0\lambda\rangle$, $S_a|j_0\lambda\rangle$, $S_a^\dagger S_b^\dagger|j_0\lambda\rangle$, ... carries a representation of the rules (10) which must be decomposed into irreducible parts. However, the three equations (10b, c) are now precisely the defining relations of a pair of parafermion creation and annihilation operators, and the combinations S_a, $S_a^\dagger$, $[S_a, S_b]$, $[S_a, S_b^\dagger]$ and $[S_a^\dagger, S_b^\dagger]$ generate the Lie algebra of SO(5) (cf. equations 8). Hence the space spanned by the monomials in S_a and $S_a^\dagger$ decomposes with respect to SO(5).

We do not attempt here a complete treatment of all SO(5) multiplets arising in the parafermion realization (compare e.g. Bracken and Green 1974). Rather, in direct analogy with the Poincaré supersymmetry case, we consider those representations of the parafermion algebra for which there exists a 'vacuum subspace', defined by

$$(2m)^{-\frac{1}{2}}S_a|j_0\lambda\rangle = 0, \qquad (2m)^{-1}S_a S_b^\dagger|j_0\lambda\rangle = p\,\delta_{ab}|j_0\lambda\rangle \tag{11}$$

($\lambda = -j_0, -j_0+1, ..., j_0$), where the positive integer p is the order of parafermi statistics. Bracken and Green (1972) have shown that this class of representations corresponds to irreducible representations of SO(5) labelled $[\frac{1}{2}p, \frac{1}{2}p]$ and they have given an explicit algorithm for constructing the one- and two-particle states in terms of monomials in the $S_a^\dagger$ acting on the vacuum. For our present purposes this construction is to be applied to the vacuum subspace (11), and yields, for each order p, the appropriate irreducible representation of the little algebra $\hat{\mathscr{S}}$.

It should be mentioned that the label p is related to the eigenvalue of the quadratic SO(5) invariant. In terms of $S_{\alpha\beta}$ ($S_\alpha \equiv S_{\alpha 5}$) and the inverse metric $g^{\alpha\beta} = -(2m^2)^{-1}(C^{-1}\gamma.P)^{\alpha\beta}$, this eigenvalue is (cf. equations 8)

$$C_2 = S_{\alpha\beta}\, g^{\beta\gamma} S_{\gamma\delta}\, g^{\delta\alpha} + 2S_{\alpha 5}\, g^{55} S_{5\beta}\, g^{\beta\alpha}, \tag{12}$$

and is, in fact, a Casimir invariant of $\mathscr{S}$ itself, with eigenvalue $2\{2(\frac{1}{2}p)(\frac{1}{2}p+2)\}$ (Bracken and Green 1971).

The problem of constructing irreducible multiplets of $\mathscr{S}$, for each order p, is now solved in principle: the states of arbitrary P_μ are obtained simply by applying the appropriate boosting operator to the rest frame states. We shall not show this in detail, but shall concentrate on the more important task of determining what values of the spin occur in the irreducible multiplets of $\hat{\mathscr{S}}$ and hence of $\mathscr{S}$. To this end, let us consider once again the SO(5) rest frame algebra.

A more perspicuous basis for the six SO(4) generators $S_{\alpha\beta}$ or $[S_a, S_b]$, $[S_a, S_b^\dagger]$ and $[S_a^\dagger, S_b^\dagger]$ is the set $\Sigma_\pm$, Σ_5 and $\mathbf{\Omega}$ defined above (note that, in the rest frame, $\Omega_0 = W_0 = 0$). In fact, taking the linear combinations

$$\Sigma_+ + \Sigma_- = 2m\Sigma_1, \qquad \Sigma_+ - \Sigma_- = 2im\Sigma_2, \qquad \Sigma_5 = m^2\Sigma_3$$

and

$$S_a + \varepsilon_{ab} S_b^\dagger = S_{1a}, \qquad S_a - \varepsilon_{ab} S_b^\dagger = S_{2a},$$

we have

$$[\Sigma_i, \Sigma_j] = \mathrm{i}\varepsilon_{ijk}\Sigma_k, \qquad [\Omega_i, \Omega_j] = m\mathrm{i}\varepsilon_{ijk}\Omega_k, \tag{13a}$$

$$[\Sigma_i, \Omega_j] = 0, \qquad [J_i, \Sigma_j] = 0, \tag{13b}$$

$$[J_i, \Omega_j] = \mathrm{i}\varepsilon_{ijk}\Omega_k, \qquad [\Sigma_i, S_{\kappa a}] = \tfrac{1}{2}(\tau_i)_\kappa{}^\lambda S_{\lambda a}, \tag{13c}$$

$$[\Omega_i, S_{\kappa a}] = \tfrac{1}{2}(\sigma_i)_a{}^b S_{\kappa b}, \qquad [J_i, S_{\kappa a}] = \tfrac{1}{2}(\sigma_i)_a{}^b S_{\kappa b}, \tag{13d}$$

where $\kappa, \lambda = 1, 2$ and the $\boldsymbol{\tau}$ are a set of Pauli matrices. Thus the SO(4) generators are comprised of two commuting SU(2) factors generated by $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, the former being rotation invariant. The spinor $S_{\kappa a}$ is simultaneously doublet under both $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, $\boldsymbol{J}$.

The square of the vector $\boldsymbol{K} = m\boldsymbol{J} - \mathbf{\Omega}$ is the Casimir invariant $K_\mu K^\mu$ in the rest frame. Clearly, the various spin components can be found by recoupling the $\boldsymbol{K}$ and $\mathbf{\Omega}$ spins. It follows from (11) that $\mathbf{\Omega}$ vanishes on the vacuum subspace, whence we have $\boldsymbol{K} = m^2 j_0(j_0+1)$ for the whole irreducible multiplet. Now the irreducible representation $[\frac{1}{2}p, \frac{1}{2}p]$ of SO(5) decomposes into the irreducible representations

$$[\tfrac{1}{2}p, \tfrac{1}{2}p], \qquad [\tfrac{1}{2}p, \tfrac{1}{2}p-1], \qquad \ldots, \qquad [\tfrac{1}{2}p, -\tfrac{1}{2}p]$$

of the SO(4) subgroup, yielding the $\mathbf{\Sigma} \times \mathbf{\Omega}$ spins

$$\sigma \times \omega = \tfrac{1}{2}p \times 0, \qquad (\tfrac{1}{2}p - \tfrac{1}{2}) \times \tfrac{1}{2}, \qquad \ldots, \qquad 0 \times \tfrac{1}{2}p$$

respectively. The spin j_0 constituents (including the vacuum) therefore carry an internal spin $\sigma = \frac{1}{2}p$, with total multiplicity $p+1$. The occurrence of $\omega = \frac{1}{2}$ corresponds to spin constituents $j_0+\frac{1}{2}$ and $j_0-\frac{1}{2}$, carrying internal spin $\sigma = \frac{1}{2}(p-1)$, and so on. Also, the bilinear combination Q, which commutes with $\mathbf{\Sigma}$, $\mathbf{\Omega}$ and $\boldsymbol{J}$ (see equations 6), is given by the difference between the quadratic SO(5) and SO(4) Casimir invariants. Specifically (Bracken and Green 1972) the eigenvalue is

$$Q = m^2(p+p\omega-\omega^2). \tag{14}$$

Finally, we can write down the decomposition of the irreducible multiplet of $\mathscr{S}$ labelled $(m>0, j_0, p)$, with respect to the (Poincaré by internal) subalgebra. Using the notation ${}^{2\sigma+1}j$, we have the spin constituents

$${}^{p+1}j_0\,,\quad {}^{p}\{(j_0+\tfrac{1}{2})+(j_0-\tfrac{1}{2})\}\,,\quad \ldots,\quad {}^{1}\{(j_0+\tfrac{1}{2}p)+(j_0+\tfrac{1}{2}p-1)+\ldots+(j_0-\tfrac{1}{2}p)\}\,,$$

the total number of helicity states being $(2j_0+1)\{\frac{1}{6}(p+1)(p+2)(p+3)\}$.

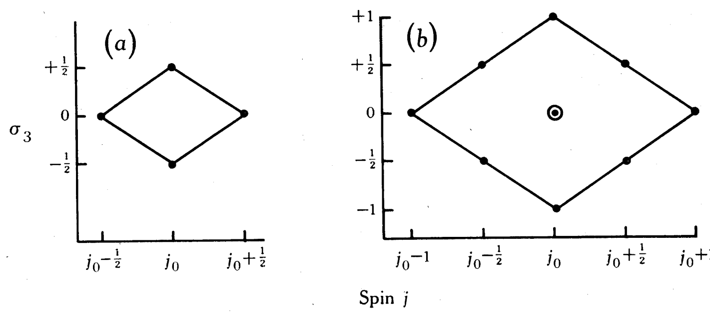

Fig. 1. Weight diagrams showing the spin content of the irreducible multiplet (m, j_0, p) (massive case) for order (*a*) $p = 1$ and (*b*) $p = 2$.

It is convenient to display the spin content of the irreducible multiplet (m, j_0, p) by means of a type of weight diagram in which the spin j is plotted against the third component σ_3 of the internal SU(2). This is done in Figs 1*a* and 1*b* for $p = 1$ and 2 respectively. For order p, the multiplet includes spins between $j_0+\frac{1}{2}p$ and $j_0-\frac{1}{2}p$ or 0, whichever is greater, in steps of $\frac{1}{2}$. The values of σ occurring for fixed j are extracted by noting the various multiplicity changes starting from the lower edge of the pattern and working upwards; conversely the values of ω for fixed σ and σ_3 are extracted by working from left to right.

(*b*) *Massless Case*

We now repeat the analysis for the light-like case $P_\mu P^\mu = m^2 = 0$. Recall that in this case the Pauli–Lubanski vector reduces to $W_\mu = \Lambda P_\mu$ and irreducible representations of the Poincaré group are characterized by an invariant helicity λ. Turning to the algebra $\mathscr{S}$ in the frame $P_\mu = w(1,0,0,1)$, we find that the metric $-2(\gamma . PC)_{\alpha\beta}$ is degenerate. In fact, as noted by Jarvis (1976), in the light-like case for Poincaré supersymmetry the algebra collapses because of the constraint condition $\gamma . PS = 0$. The combinations $\mathbf{\Sigma}$ and Q vanish while $\Omega_\mu = \frac{1}{2}NP_\mu$, where in the standard frame $N = (2w)^{-1}[S_1, S_1^\dagger]$. Also the relation (4) becomes

$$[S_1,[S_1,S_1]] = 0\,, \qquad [S_1,[S_1^\dagger,S_1]] = 2wS_1\,, \tag{15}$$

which is the algebra of a single parafermion operator. In fact N, together with $N_+ = w^{-\frac{1}{2}}S_1$ and $N_- = w^{-\frac{1}{2}}S_1^\dagger$, generates the Lie algebra of SO(3), in agreement with general result (Bracken and Green 1972).

Just as Λ is a Poincaré invariant, so is N. The vector K_μ, which commutes with S_α, takes the form $K_\mu = KP_\mu$, whence

$$K = \Lambda - \tfrac{1}{2}N \tag{16}$$

is a Casimir invariant of the algebra $\mathscr{S}$.

Irreducible multiplets are again obtained by the induced representation method, by constructing irreducible multiplets of the little algebra leaving $P_\mu = w(1,0,0,1)$ invariant. Once again we consider the class of representations for which there exist the vacuum-like conditions

$$w^{-\frac{1}{2}}S_1\,|\,\lambda_0\rangle = 0\,, \qquad w^{-1}S_1S_1^\dagger\,|\,\lambda_0\rangle = p\,|\,\lambda_0\rangle\,, \tag{17}$$

where $|\,\lambda_0\rangle$ denotes a multiplet of the little Lie subalgebra corresponding to 'superhelicity' λ_0. According to the general result (Bracken and Green 1972), the representation of SO(3) obtained for order p is labelled $[\frac{1}{2}p]$. The conditions (17) imply that the vacuum subspace has weight $\frac{1}{2}p$, and that the Casimir invariant K has eigenvalue $\lambda_0 - \frac{1}{2}p$. The remaining states are spanned by

$$S_1^\dagger\,|\,\lambda_0\rangle\,, \quad (S_1^\dagger)^2\,|\,\lambda_0\rangle\,, \quad \ldots, \quad (S_1^\dagger)^p\,|\,\lambda_0\rangle\,.$$

Again the label p is associated with the eigenvalue of the quadratic SO(3) invariant. With the same normalization (12), it is $2(N^2 + N_+N_- + N_-N_+)$ and is in fact a Casimir invariant of $\mathscr{S}$ itself, with eigenvalue $2\{(\frac{1}{2}p)(\frac{1}{2}p+1)\}$.

It is straightforward to determine the constituent helicities within an irreducible multiplet of order p. There is now no longer an internal SU(2), but a U(1) generated by the Poincaré-invariant label N. Because of the invariance of K, the states of N with eigenvalues $\nu = \frac{1}{2}p,\ \frac{1}{2}p-1, \ldots, -\frac{1}{2}p$ must carry helicities $\lambda = \lambda_0,\ \lambda_0-\frac{1}{2}, \ldots, \lambda_0-\frac{1}{2}p$ respectively. In the standard frame, these are precisely the states

$$|\,\lambda_0\rangle\,, \quad S_1^\dagger\,|\,\lambda_0\rangle\,, \quad (S_1^\dagger)^2\,|\,\lambda_0\rangle\,, \quad \ldots, \quad (S_1^\dagger)^p\,|\,\lambda_0\rangle$$

(higher powers not being allowed by the parafermi statistics of order p). States in a general frame may be constructed by applying the appropriate boosts.

We can now write down the decomposition of the irreducible multiplet of $\mathscr{S}$ labelled $(m^2 = 0, \lambda_0, p)$ with respect to the (Poincaré by internal) subalgebra. Using the notation λ_ν we have the constituents

$$(\lambda_0)_{\frac{1}{2}p}\,, \quad (\lambda_0-\tfrac{1}{2})_{\frac{1}{2}p-1}\,, \quad \ldots, \quad (\lambda_0-\tfrac{1}{2}p)_{-\frac{1}{2}p}\,,$$

there being a total of $p+1$ helicity states.

Once again we can display the helicity content of the irreducible multiplets by plotting λ against ν on a weight diagram. Strictly the label ν is redundant but its retention suggests the connection with the massive case. The weight diagrams are given in Figs 2*a* and 2*b* for $p = 1$ and 2 respectively. For order p, the multiplet includes helicities between λ_0 and $\lambda_0 - \frac{1}{2}p$ or 0, whichever is greater, in steps of $\frac{1}{2}$.

Figs 1*a* and 2*a* show that the $p = 1$ multiplets of $\mathscr{S}$ in the massive and massless cases have precisely the same spin and helicity structure as in Poincaré supersymmetry (Jarvis 1976). Green (1953) and Bracken and Green (1972) have shown that in each order p the parafermi generators satisfy an order $p+1$ identity, in addition to the relation (4). For $p = 1$, this is precisely the anticommutation relation (3), thus confirming that the Poincaré supersymmetry is indeed a particular case of the extended algebra corresponding to $p = 1$ (Fermi statistics).

There is one further representation of the algebra $\mathscr{S}$ of possible interest: that in which the spinor generators S_α are trivially represented by 0. This may be described as the 'order $p = 0$' case by analogy. Note that it is not available for Poincaré supersymmetry unless $P_\mu = 0$, the null case.

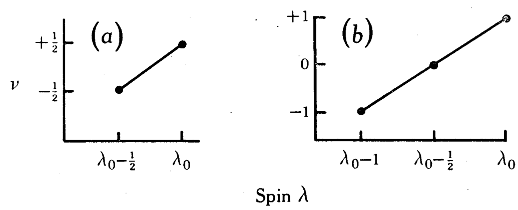

Fig. 2. Weight diagrams showing the spin content of the irreducible multiplet $(m^2=0, \lambda_0, p)$ (massless case) for order (*a*) $p = 1$ and (*b*) $p = 2$.

4. Conclusions

We have considered classes of representations of a parafermion extension of supersymmetry in the massive and massless cases, and have established that the usual supersymmetry occurs as a special case (of order $p = 1$). The spin structure of the irreducible multiplets can be displayed by weight diagrams such as Figs 1 and 2.

Applications such as the construction of explicit matrix representations (Jarvis 1976), the development of the generalized partial wave analysis (Jarvis 1977) and the explicit inclusion of internal symmetry are obviously possible with the order-p irreducible multiplets considered here, and other representations (see e.g. Bracken and Green 1974). A basic problem is to set up the analogue of the superfield representation (Salam and Strathdee 1975); relevant work in this general direction has been done by Omote *et al.* (1976). For example, the usual supergravity theory involves gauge fields of spin 2 and $\frac{3}{2}$ (Deser and van Nieuwenhuizen 1978). Presumably the analogous local version of the extended parafermion supersymmetry for $p = 2$, if it were constructible, would include gauge fields of spin 2, $\frac{3}{2}$ and 1. Work along these lines is in progress.

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