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Flux Conservation, Magnetohydrodynamic and Force-free Approximations for Relativistically Streaming Plasmas

R. R. Burman

Department of Physics, University of Western Australia, Nedlands, W.A. 6009.

Abstract

The purpose of this paper is to clarify the flux conservation, magnetohydrodynamic and force-free approximations, for plasmas whose component species may be relativistically streaming, by paying particular attention to the conditions for their validity and to their interrelationships. All three approximations involve consideration of inertial effects, either in the generalized Ohm law or in the equation of motion of the plasma as a whole. All three imply that the electric field component parallel to the magnetic field is small. The significance of the approximations for pulsar magnetospheric theory is commented on.

Introduction

Certain electrical properties of plasmas are often expressed by generalized Ohm laws which relate the conduction current density $\boldsymbol{i}$ to quantities of the form $\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B}$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electrical field and magnetic induction, c is the vacuum speed of light and $\boldsymbol{u}_B$ is some velocity field. Such laws are complicated when given their full generality, and so approximations are required. For example, if various terms, including the inertial contribution, are negligible in the generalized Ohm law, there results the much-used magnetohydrodynamic approximation

$$\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B}\approx\boldsymbol{0}, \tag{1}$$

where $\boldsymbol{v}$ is the plasma's bulk velocity. This implies that $\boldsymbol{E}.\boldsymbol{B}\approx 0$, an approximation that is needed in pulsar magnetospheric theory in order to satisfy conditions near the star. Although the magnetohydrodynamic condition is often used in that subject, it is a stronger approximation than is required to get $\boldsymbol{E}.\boldsymbol{B}\approx 0$: it is sufficient to neglect inertial and other terms in the generalized Ohm law to the extent of effecting a reduction to

$$\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B}\approx\boldsymbol{0}. \tag{2}$$

Another approximation often used in pulsar magnetospheric theory is the force-free condition, which is valid when the electromagnetic force density acting on the bulk plasma is much greater than the plasma's inertial force density. This condition also implies that $\boldsymbol{E}.\boldsymbol{B}\approx 0$.

All three of the approximations mentioned above involve consideration of inertial effects, either in the generalized Ohm law or in the equation of motion of the plasma as a whole. Of course, estimation of the importance of inertial effects, relative to other effects, in the latter equation is quite a different matter from in the former:

the two equations represent different linear combinations of the equations of motion of the individual species. All three approximations are important in pulsar magnetospheric theory, but their ranges of validity and their interrelationships are often not made clear in the literature. Authors sometimes begin by simply neglecting the inertial forces in the equations of motion of individual species, thus obscuring the significance of inertial effects in those linear combinations of the equations of motion of the individual species that form the generalized Ohm law and the equation of motion of the plasma as a whole.

The approximation procedures required can be expected to be quite different according to whether the plasmas are highly charge-separated or not. In a highly charge-separated plasma, one is dealing with essentially a single species in any particular zone and so is concerned with a single equation of motion rather than with linear combinations.

An attempt to improve the situation was made by Ardavan (1976). He derived a generalized Ohm law for relativistically streaming binary plasmas in which the electrons and ions are nonrelativistic in the local proper frame of the bulk plasma, and he investigated its reduction to the magnetohydrodynamic approximation. But his conditions for the reduction appear to be inadequate: in particular, he claimed that the magnetohydrodynamic approximation follows irrespective of the degree of charge separation—a claim that is difficult to accept for the reason described in the previous paragraph, and one that was borne out by later work (Burman 1977*a*). Also, Ardavan failed to distinguish between conditions for the approximation (1) to hold and those for $\boldsymbol{E}.\boldsymbol{B} \approx 0$ to hold, and so was led to make an unjustified deduction concerning $\boldsymbol{E}.\boldsymbol{B}$ near the star and the generation of an electrically driven stellar wind (see his Section III).

The purpose of this paper is to clarify the three approximations mentioned above by paying particular attention to the conditions for their validity and to their interrelationships.

Generalized Ohm Law

Burman *et al.* (1976) introduced a technique for treating energy dissipation arising from 'frictional' interactions in plasmas. They made no restriction on the number or kinds of species present, and left the nature of the 'frictional' interactions unspecified. The foundation of the technique lies in the expansions of the relative momentum densities and the frictional forces as linear combinations of three *non-orthogonal* vector fields. The basic physical idea is to choose these vector fields, so far as possible, to be generalized currents relating naturally to different types of diffusion processes that occur in plasmas, such as electric current flow and ambipolar diffusion. One is likely to be more interested in such quantities than in the momentum densities of individual species. The technique has been extended to plasmas with relativistic streaming velocities (Burman 1977*a*) and, further, to general systems of vector fluxes and forces in the thermodynamics of irreversible processes (Burman 1977*b*).

Let ρ_r and $\boldsymbol{v}_r$ denote the mass density and fluid velocity of the rth species in an N-component plasma. Write $\boldsymbol{J}_r \equiv \rho_r(\boldsymbol{v}_r - \boldsymbol{v})$, where $\boldsymbol{v}$ is the local barycentric velocity of the medium as a whole. The force per unit mass, acting on the rth species, that arises from 'frictional' interaction with other species will be denoted by $\boldsymbol{F}_r$. Since in ordinary space no more than three vectors can be linearly independent, the N relative momentum densities and the N frictional forces can be expressed as linear combinations of three basic vector fields:

$$\boldsymbol{J}_r = \sum_{i=1}^{3} S_{ri}\,\boldsymbol{m}_i \qquad \text{and} \qquad \boldsymbol{F}_r = \sum_{i=1}^{3} A_{ri}\,\boldsymbol{m}_i. \tag{3}$$

Summations will be over r from 1 to N, unless otherwise specified.

Let $-Q$ denote the rate of frictional work done per unit volume, namely $\sum \boldsymbol{J}_r \cdot \boldsymbol{F}_r$. This now becomes

$$\sum_{i=1}^{3}\sum_{j=1}^{3} K_{(ij)}\,\boldsymbol{m}_i \cdot \boldsymbol{m}_j, \qquad \text{where} \qquad K_{ij} \equiv -\sum S_{ri} A_{rj} \tag{4}$$

and the symmetric and skew parts are denoted by $K_{(ij)}$ and $K_{[ij]}$ respectively. Thus far, so long as they are linearly independent, the three $\boldsymbol{m}_i$ are arbitrary. The next step is to restrict their choice in such a way that Q takes the form of a sum of squares; that is, the $\boldsymbol{m}_i$ are chosen so that $K_{(ij)}$ is diagonal: $K_{(ij)} = \theta_i^{-1}\,\delta_{ij}$. The $\boldsymbol{m}_i$ represent nine quantities, of which three are mere normalization functions and three are required in order to satisfy the diagonalization condition. Thus, in the absence of further conditions, three quantities remain free, and one can choose arbitrarily $\boldsymbol{m}_1$, say, together with the angle between $\boldsymbol{m}_2$ and $\boldsymbol{m}_1$. Because $K_{(ij)}$ is diagonal, Q is reduced to a sum of three squared terms, each corresponding to one of the generalized currents $\boldsymbol{m}_i$; thus

$$Q = \sum_{i=1}^{3} \boldsymbol{m}_i^2/\theta_i. \tag{5}$$

Let $\boldsymbol{m}_1$ be equivalent to $\boldsymbol{i}$, the electric conduction current density in the medium, and write $\theta_1 \equiv \sigma$. The first term in Q is $\boldsymbol{i}^2/\sigma$, corresponding to Joule heating, and σ defines the electrical conductivity of the medium. The remaining terms in Q are subject to different physical interpretations in different types of plasmas.

Write $p_i \equiv \sum (\kappa_r/\gamma_r\,\rho_r) S_{r1}\,S_{ri}$, where κ_r is the ratio of charge to rest mass of the rth species and γ_r is the Lorentz factor corresponding to $\boldsymbol{v}_r$. Consider the following linear combination of the equations of motion of the plasma species (Burman 1977*a*):

$$-\sum S_{r1}\,\boldsymbol{F}_r = \boldsymbol{E} + c^{-1}\boldsymbol{u}_B \times \boldsymbol{B} + \boldsymbol{E}', \tag{6}$$

where

$$\boldsymbol{u}_B \equiv \boldsymbol{v} + \eta \boldsymbol{i} + p_2\,\boldsymbol{m}_2 + p_3\,\boldsymbol{m}_3, \qquad \boldsymbol{E}' \equiv -\sum S_{r1}\,\boldsymbol{A}_r, \tag{7a, b}$$

in which $\eta \equiv p_1$ and $\rho_r \boldsymbol{A}_r \equiv \partial_t(\rho_r\,\boldsymbol{v}_r) + \nabla \cdot (\rho_r\,\boldsymbol{v}_r\,\boldsymbol{v}_r)$, with ∂_t denoting partial differentiation with respect to time. Equation (6) can be thought of as a generalized Ohm law with $\boldsymbol{E}'$ an equivalent electric field arising from the inertial terms in the equations of motion of the individual species: the right-hand side of (6) has the usual form for a generalized Ohm law, while the left-hand side is a linear combination of frictional forces generalizing the usual expression $\boldsymbol{i}/\sigma$.

It was pointed out by Burman *et al.* (1976) that a case in which $K_{[ij]}$ vanishes is the simple collisional relaxation model, in which the $\boldsymbol{F}_r$ are linear sums of the velocity differences between species, with the reciprocity relationships (Delcroix 1965) being valid. When $K_{[ij]}$ vanishes, it follows from the expansion of $\boldsymbol{F}_r$, the definition of K_{ij} and $K_{(ij)} = \theta_i^{-1}\,\delta_{ij}$ that $-\sum S_{ri}\,\boldsymbol{F}_r = \boldsymbol{m}_i/\theta_i$: taking appropriate linear combinations of expressions for the $\boldsymbol{F}_r$ gives expressions for the $\boldsymbol{m}_i$; in particular, for $i = 1$, the generalized Ohm law with $\boldsymbol{i}/\sigma$ on the left-hand side results.

The above discussion covers some of the general technique introduced by Burman *et al.* (1976) and later extended to the relativistic case by Burman (1977*a*). The generalized Ohm law (6) will now be used in a discussion of the flux conservation, magnetohydrodynamic and force-free approximations and their interrelationships.

Approximations

Consider the special case in which the 'frictional' term on the left-hand side of the generalized Ohm law (6) is negligible: this is so if the condition

$$|\sum S_{r1}\boldsymbol{F}_r| \ll |c^{-1}\boldsymbol{v}\times\boldsymbol{B}| \tag{8}$$

is satisfied. If the relationships $K_{[12]} = 0 = K_{[13]}$ are satisfied then the left side of equation (6) becomes $\boldsymbol{i}/\sigma$; in these circumstances, the special case under discussion corresponds to an effectively infinite conductivity, and the condition (8) is replaced by

$$|\boldsymbol{i}/\sigma| \ll |c^{-1}\boldsymbol{v}\times\boldsymbol{B}|. \tag{9}$$

With (8), the generalized Ohm law (6) is reduced to

$$\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B}+\boldsymbol{E}' \approx \boldsymbol{0}. \tag{10}$$

In general, some frictional effects remain in this equation, through the terms contained in the definition (7a) for $\boldsymbol{u}_B$; further simplifications might be available when those terms are known.

Suppose that the term $\boldsymbol{E}'$, representing inertial effects in the generalized Ohm law, is negligible: this is so if the condition

$$|\boldsymbol{E}'| \ll |c^{-1}\boldsymbol{v}\times\boldsymbol{B}| \tag{11}$$

is satisfied; of course, inertial effects may not be negligible in other places, such as the equation of motion of the plasma as a whole. With the conditions (8) and (11), the generalized Ohm law is reduced to

$$\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B} \approx \boldsymbol{0}. \tag{12}$$

This can be regarded as a flux conservation theorem: it implies that magnetic flux through any surface that moves at each of its points with velocity $\boldsymbol{u}_B$ is conserved. Of course, $\boldsymbol{u}_B$ can be quite different from $\boldsymbol{v}$, and so equation (12) does *not* mean that flux is 'frozen-in' to the motion of plasma.

The approximation (12) implies the important property $\boldsymbol{E}.\boldsymbol{B} \approx 0$: the plasma is acting in such a way as to nullify approximately the component $E_{\|}$ of $\boldsymbol{E}$ along $\boldsymbol{B}$. When (12) is valid, the plasma particles are subject to relatively little electromagnetic acceleration along $\boldsymbol{B}$ lines. If the inertial term $\boldsymbol{E}'$ is significant in the generalized Ohm law, so that the flux conservation approximation (12) fails, then charged plasma particles may be powerfully accelerated along $\boldsymbol{B}$ lines. Even the condition $E_{\|} \approx 0$ does not preclude a relatively small residual value of $E_{\|}$ from providing substantial acceleration of charged particles along $\boldsymbol{B}$ lines (Mestel 1973).

In general, some frictional and inertial effects remain in the flux conservation theorem (12), through the terms contained in $\boldsymbol{u}_B$. Equation (12) can be written

$$\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B}+\eta c^{-1}\boldsymbol{i}\times\boldsymbol{B}+c^{-1}(p_2\boldsymbol{m}_2+p_3\boldsymbol{m}_3)\times\boldsymbol{B}\approx\boldsymbol{0}. \tag{13}$$

Suppose that all except the first two terms in the flux conservation approximation, in the form (13), are negligible. This is so if

$$|\boldsymbol{u}_B-\boldsymbol{v}|\ll|\boldsymbol{v}|. \tag{14}$$

With the conditions (8), (11) and (14), the generalized Ohm law is reduced to

$$\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B}\approx\boldsymbol{0}. \tag{15}$$

This is the 'magnetohydrodynamic' approximation, which implies that magnetic flux is frozen-in to the medium, meaning that flux through any surface fixed in the plasma, and so moving at each of its points with the local barycentric velocity of the plasma as a whole, is conserved. Of course, the magnetohydrodynamic condition (15) implies $E_{\parallel}\approx 0$, but is a stronger condition than is necessary for this result: the flux conservation theorem (12) is sufficient.

Relativistic plasmas may well be non-neutral, so the total electric current density $\boldsymbol{j}$ can be split into the sum of conduction and convection current densities. This separation is not uniquely defined (Chapman and Cowling 1970), but for fully ionized plasmas the choice

$$\boldsymbol{j}=\boldsymbol{i}+\nu\boldsymbol{v}, \tag{16}$$

where ν is the net charge density of the medium, is permissible and will be made here. Attention will now be restricted to fully ionized plasmas.

The total electromagnetic force density on a fully ionized plasma is given by, using equation (16),

$$\nu\boldsymbol{E}+c^{-1}\boldsymbol{j}\times\boldsymbol{B}=\nu(\boldsymbol{E}+c^{-1}\boldsymbol{v}\times\boldsymbol{B})+c^{-1}\boldsymbol{i}\times\boldsymbol{B}. \tag{17}$$

If the flux conservation approximation (13) is applicable then equation (17) becomes

$$\nu\boldsymbol{E}+c^{-1}\boldsymbol{j}\times\boldsymbol{B}\approx c^{-1}(1-\nu\eta)\boldsymbol{i}\times\boldsymbol{B}-c^{-1}\nu(p_2\boldsymbol{m}_2+p_3\boldsymbol{m}_3)\times\boldsymbol{B}. \tag{18a}$$

If the magnetohydrodynamic approximation (15) is applicable then (17) becomes

$$\nu\boldsymbol{E}+c^{-1}\boldsymbol{j}\times\boldsymbol{B}\approx c^{-1}\boldsymbol{i}\times\boldsymbol{B}; \tag{18b}$$

for fully ionized plasmas, the magnetohydrodynamic approximation implies that there is a balance between the electric force on the plasma and the magnetic force on the *convection* current, leaving the magnetic force on the *conduction* current as the only dynamically effective electromagnetic force acting on the plasma as a whole.

Consider plasmas in which the frictional forces satisfy $\sum\rho_r\boldsymbol{F}_r\approx\boldsymbol{0}$ in the equation of motion of the plasma as a whole:

$$\partial_t(\rho\boldsymbol{v})+\nabla.(\rho\boldsymbol{v}\boldsymbol{v})+\nabla.\boldsymbol{p}\approx\nu\boldsymbol{E}+c^{-1}\boldsymbol{j}\times\boldsymbol{B}, \tag{19}$$

where $\hat{\boldsymbol{p}} \equiv \sum \rho_r(\boldsymbol{v}_r - \boldsymbol{v})(\boldsymbol{v}_r - \boldsymbol{v})$, and partial pressures and their gradients have been omitted. Now consider situations in which the displacement current is not predominant in the Ampère–Maxwell law. The Gauss and Ampère laws give

$$\nu \boldsymbol{E} + c^{-1}\boldsymbol{j}\times\boldsymbol{B} \sim \{(\nabla.\boldsymbol{E})\boldsymbol{E} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}\}/4\pi. \tag{20}$$

In nonrelativistic plasmas $\boldsymbol{E}^2$ is usually very much smaller than $\boldsymbol{B}^2$; in relativistic plasmas $\boldsymbol{E}^2$ and $\boldsymbol{B}^2$ will often be of similar magnitudes. Suppose that the electromagnetic energy density is much less than the kinetic energy density of the plasma's bulk flow; that is,

$$\boldsymbol{B}^2/4\pi\rho\boldsymbol{v}^2 \ll 1. \tag{21}$$

In this case, the right-hand side of equation (19) will typically be very small compared with the second term on the left, so that (19) reduces to

$$\partial_t(\rho\boldsymbol{v}) + \nabla.(\rho\boldsymbol{v}\boldsymbol{v}) + \nabla.\hat{\boldsymbol{p}} \approx \boldsymbol{0}. \tag{22}$$

This expresses an approximate balance of the plasma's inertial force density by the divergence of its non-electromagnetic stress tensor; electromagnetic effects are negligible here in the equation of bulk motion. Suppose, on the other hand, that the electromagnetic energy density is much greater than the kinetic energy densities of the bulk and relative flows; that is,

$$\boldsymbol{B}^2/4\pi \gg \max\{\rho\boldsymbol{v}^2, \rho_r(\boldsymbol{v}_r - \boldsymbol{v})^2, \rho|\boldsymbol{v}|L/T\}, \tag{23}$$

where L and T denote length and time scales for variation of macroscopic plasma properties. In this case, the left-hand side of equation (19) will typically be very small compared with the right-hand side, so that (19) reduces to

$$\nu\boldsymbol{E} + c^{-1}\boldsymbol{j}\times\boldsymbol{B} \approx \boldsymbol{0}; \tag{24}$$

this is the electromagnetic 'force-free' approximation. Like the flux conservation approximation, the force-free approximation implies $E_{\|} \approx 0$.

Consider fully ionized plasmas when the magnetohydrodynamic approximation is valid: the magnetic force on the conduction current is the only dynamically effective electromagnetic force acting on the bulk medium, and the equation of motion (19) becomes (from equation 18b)

$$\partial_t(\rho\boldsymbol{v}) + \nabla.(\rho\boldsymbol{v}\boldsymbol{v}) + \nabla.\hat{\boldsymbol{p}} \approx c^{-1}\boldsymbol{i}\times\boldsymbol{B}. \tag{25}$$

This equation shows that, for a fully ionized plasma in which the magnetohydrodynamic approximation is valid, the force-free approximation will also be valid when, typically,

$$|c^{-1}\boldsymbol{i}\times\boldsymbol{B}|L \gg \max\{\rho\boldsymbol{v}^2, \rho_r(\boldsymbol{v}_r - \boldsymbol{v})^2, \rho|\boldsymbol{v}|L/T\}. \tag{26}$$

For binary plasmas, only the first of the $\boldsymbol{m}_i$, namely $\boldsymbol{i}$, is needed in the expression $\sum_{i=1}^{3} S_{ri}\boldsymbol{m}_i$ for $\boldsymbol{J}_r$. Now restrict attention to cases in which $K_{[12]} = 0$ and $K_{[13]} = 0$, so that $-\sum S_{r1}\boldsymbol{F}_r = \boldsymbol{i}/\sigma$. Here the generalized Ohm law takes the familiar form with $\boldsymbol{i}/\sigma$ on the left-hand side; also, $\boldsymbol{u}_B = \boldsymbol{v} + \eta\boldsymbol{i}$. The flux conservation approximation (13) is replaced by

$$E + c^{-1}v \times B + \eta c^{-1} i \times B \approx 0 . \tag{27}$$

This implies

$$vE + c^{-1} j \times B \approx (1 - v\eta) c^{-1} i \times B ; \tag{28}$$

that is, the net electromagnetic force density on the plasma is approximately equal to the force density on the conduction current multiplied by $(1 - v\eta)$. The flux conservation approximation (27) is valid if the conditions (8) and (11) on the frictional and inertial effects are satisfied. If, in addition, $|\eta i| \ll |v|$ then the magnetohydrodynamic approximation is valid.

Restrict attention further to electron–proton plasmas in situations in which time-derivative terms, both the displacement current and parts of inertial contributions, do not predominate. Let subscripts e and i denote quantities referring to electrons and protons; in particular m_e and m_i are the electron and proton rest masses and n_{0e} and n_{0i} are their proper number densities. Let ω_{0e} denote the electron proper angular plasma frequency. Impose the restrictions

$$\gamma_e m_e / \gamma_i m_i \ll \gamma_i n_{0i} / \gamma_e n_{0e} \lesssim 1 , \tag{29}$$

meaning that the protons are not too greatly less numerous than the electrons (and so highly charge-separated plasmas are excluded); that the electrons, if relativistic, are not too much more relativistic than the protons ($\gamma_e/\gamma_i \ll 2000$ must hold); and that the protons do not provide the predominant contribution to the charge density. It follows (Burman 1977*a*) that, typically,

$$|E'| / |c^{-1} v \times B| \sim (c/\omega_{0e} L)^2 . \tag{30}$$

Thus, the inertial term E' in the generalized Ohm law will be important when plasma gradients are so sharp that $(c/\omega_{0e} L)^2 \sim 1$, but, under certain conditions, E' can be neglected when $(c/\omega_{0e} L)^2 \ll 1$. This result is relevant to nonrelativistic and relativistically streaming electron–proton plasmas that are not too highly charge-separated.

Ardavan's (1976) restriction to a nonrelativistic relative streaming velocity for the two species has not been imposed. If, in addition to the conditions stated above, the frictional term is neglected in the generalized Ohm law then the flux conservation approximation results. If, further to these restrictions, the Lorentz factor γ of the bulk plasma is much greater than unity and, if the electron streaming velocity in the local proper frame of the bulk plasma is not both near to c in magnitude and antiparallel to v, then the magnetohydrodynamic approximation results (Burman 1977*a*).

For sufficiently highly charge-separated plasmas, the condition $E_{\parallel} \approx 0$ means neglect of the components parallel to B of the inertial and frictional terms in the equation of motion of the predominant species. The condition $E + c^{-1} v \times B \approx 0$ requires neglect of those complete terms. These approximations are quite a different matter from neglecting, relative to other terms in the generalized Ohm law, those linear combinations of inertial and frictional forces of the separate species which enter that law. In charge-separated plasmas, the convection current density vv is the total current density, so the magnetohydrodynamic approximation implies that $vE + c^{-1} j \times B \approx 0$; this equation has the same mathematical form as the force-free approximation for non-charge-separated plasmas, but for charge-separated plasmas it merely re-expresses the magnetohydrodynamic approximation.

Remarks on Pulsar Magnetospheres

The canonical pulsar model consists of a rotating magnetized neutron star with the magnetic axis inclined to the rotation axis, but a self-consistent model of the magnetosphere is still lacking. Both the vacuum model, in which the particles are regarded as test charges in the vacuum field, and the zero-inertia model, in which inertial terms are completely neglected, are unacceptable.

For the *axisymmetric* vacuum model, Goldreich and Julian (1969) pointed out that the component $E_{\|}$ of $\boldsymbol{E}$ along $\boldsymbol{B}$ is sufficiently powerful to pull charges out of the star, so creating a dense magnetosphere. The solution for the non-axisymmetric vaccum model with a dipolar magnetic field on the stellar surface was obtained by Deutsch (1955), in the context of the theory of normal magnetic stars. This enabled Mestel (1971) to extend the argument of Goldreich and Julian to the *oblique rotator* model, in which the magnetic and rotation axes are not aligned or antiparallel. (This was also done by Cohen and Toton (1971), but their paper appears to contain errors, as Mestel (1971) pointed out.) Subsequent investigations of the physics of neutron star surfaces have suggested that most pulsars might not emit positive ions (Ruderman 1971).

The argument of Goldreich and Julian (1969) poses the problem of studying magnetospheres that are sufficiently dense near the star to make $E_{\|} \approx 0$ there. Various authors have investigated charge-separated plasmas with inertial terms, and other non-electromagnetic terms, neglected, so that the equation of motion is just $\boldsymbol{E} + c^{-1}\boldsymbol{v} \times \boldsymbol{B} \approx \boldsymbol{0}$. But in the last few years it has become clear that inertial effects are crucial in pulsar magnetospheres. The need for the fields to be nonsingular at the light cylinder (the surface where the corotation speed equals c) leads to difficulties when inertial effects are neglected.

Mestel (1973) derived an equation for the magnetic field in a steadily rotating magnetosphere when the magnetohydrodynamic and force-free approximations are satisfied. Endean (1974) showed that the equation follows from just the steady-rotation and force-free conditions, together with the boundary condition of infinite conductivity on the stellar surface; he commented that the magnetohydrodynamic condition is not required, which is true for non-charge-separated plasmas, but for charge-separated plasmas the magnetohydrodynamic and force-free conditions are equivalent. Mestel (1973) applied his equation for the magnetic field to the 'cylindrical pulsar' model, in which quantities do not vary in the direction of the rotation axis. He showed that if the equation is valid everywhere outside the star then there is no flow of energy across the light cylinder; furthermore, the solutions between the light cylinder and infinity are standing waves and so require a reflector at infinity (Mestel *et al.* 1976).

Analysis of generalized Ohm laws (Ardavan 1976; Burman 1977*a*) has shown that, for electron–proton plasmas that are not too highly charge-separated (and that may or may not be relativistically streaming), inertial effects will prevent reduction to the flux conservation approximation when plasma gradients are sufficiently sharp. This might occur in a small neighbourhood of the light cylinder, where the length scale for variation of macroscopic plasma properties could become very small.

From the stellar surface to just inside the light cylinder, the length scale for variation of macroscopic plasma properties can be estimated to be of the same order of magnitude as the radial distance: so long as frictional and pressure gradient effects are negligible in the generalized Ohm law, the *flux conservation* approximation could be valid there, thus meeting the Goldreich–Julian (1969) requirement of $E_{\|} \approx 0$ near the star; it is, of course, not necessary for the magnetohydrodynamic approximation to be valid in order to have $E_{\|} \approx 0$. In the outer part of that region, the plasma's Lorentz factor may well be large compared with unity and the *magnetohydrodynamic* approximation could be valid.

Note that the magnetohydrodynamic approximation is more likely to be valid if $\gamma \gg 1$. That the approximation may fail when $\gamma \sim 1$ led Ardavan (1976) to assert that $E_{\|}$ is nonzero in a neighbourhood of a pulsar's surface, where conditions for an electrically driven stellar wind will exist. But reduction to the flux conservation theorem, giving $E_{\|} \approx 0$, does *not* require the plasma to be relativistic; also it is easily seen that Ardavan's work, prior to his taking $\gamma \gg 1$, already implies $E_{\|} \approx 0$ (see particularly his expression 19). Thus, Ardavan's assertion that there exists a neighbourhood of the stellar surface in which $E_{\|}$ is nonzero was not justified. Of course, the condition $E_{\|} \approx 0$ does not preclude a relatively small residual value of $E_{\|}$ from providing substantial acceleration of charged particles along $\boldsymbol{B}$ lines (Mestel 1973).

Concluding Remarks

Certain electrical properties of plasmas are often expressed by generalized Ohm laws which relate the conduction current density $\boldsymbol{i}$ to quantities of the form $\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B}$. When the term in $\boldsymbol{i}$ and the remaining (inertial and pressure gradient) terms in the generalized Ohm law are negligible, that law reduces to $\boldsymbol{E}+c^{-1}\boldsymbol{u}_B\times\boldsymbol{B}\approx\boldsymbol{0}$. Since this condition implies that magnetic flux through any surface moving with velocity $\boldsymbol{u}_B$ at each of its points is conserved, it may be regarded as a flux conservation approximation. Of course, $\boldsymbol{u}_B$ can be quite different from the plasma's bulk velocity $\boldsymbol{v}$, and so the condition does *not* mean that magnetic flux is frozen-in to the medium.

The flux conservation approximation implies that $\boldsymbol{E}.\boldsymbol{B}\approx 0$: the plasma is acting in such a way as to nullify approximately the component $E_{\|}$ of $\boldsymbol{E}$ along $\boldsymbol{B}$. In these circumstances, plasma particles are subjected to relatively little electromagnetic acceleration along $\boldsymbol{B}$ lines. If the inertial term, for example, in the generalized Ohm law is significant then the flux conservation approximation fails and hence $E_{\|}\neq 0$; plasma particles then may be powerfully accelerated along $\boldsymbol{B}$ lines. Even the condition $E_{\|}\approx 0$ does not preclude a relatively small residual value of $E_{\|}$ from providing substantial acceleration of charged particles along $\boldsymbol{B}$ lines.

When the net electromagnetic force density acting on the bulk plasma is large compared with inertial terms in the equation of bulk motion, then the force-free approximation is applicable: $\nu\boldsymbol{E}+c^{-1}\boldsymbol{j}\times\boldsymbol{B}\approx\boldsymbol{0}$.

The above three approximations all involve consideration of inertial effects. They are all important in pulsar magnetospheric theory, but the conditions for their validity and their interrelationships are often obscured in the literature. I hope that this paper has helped to clarify these matters.

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