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Simplified DWBA for Heavy Ion Induced Transfer at High Energies*

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Abstract

The treatment of the distorted wave Born approximation (DWBA) for transfer reactions at high energies between complex nuclei due to Braun-Munzinger and Harney (1974) is simplified by introducing eikonal-like representations for the elastic scattering states. Simple expressions for the differential cross sections are derived. The model includes recoil approximately and takes into account the strong absorption of the nuclear cores. Reasonable values for the parameters of the model wavefunctions are determined by comparison with the usual optical model wavefunctions. Angular distributions calculated for the model are compared with the results of exact finite-range DWBA calculations and experimental data for transitions to the ground state and excited states of ^{13}C in the reaction $^{12}C(^{14}N,{}^{13}N)^{13}C$ at several energies. The model reproduces the general features of the exact calculations, giving reasonable fits for the transitions to the ground state and the $1d_{5/2}$ ($3\cdot85$ MeV) state. The transition to the $2s_{1/2}$ ($3\cdot09$ MeV) state appears to be anomalous as in the case of the full DWBA theory.

1. Introduction

Although heavy ion induced transfer reactions at energies well above the Coulomb barrier have proved very useful in forming nuclear states which are not easily accessible by other processes, reliable spectroscopic information has not been readily extracted from the generally rather featureless angular distributions. A number of authors have taken up the early suggestion of Dodd and Greider (1965, 1969) that the absence of structure in the angular distributions can be explained by a proper treatment of kinematics in the distorted wave Born approximation (DWBA), and several approximate methods of including such 'recoil effects' have been devised (Buttle and Goldfarb 1971; Nagarajan 1972; Baltz and Kahana 1974; Hauge 1974; Braun-Munzinger and Harney 1974; Braun-Munzinger *et al.* 1974).

More recently full finite-range DWBA calculations (De Vries and Kubo 1973; Low and Tamura 1975; Nair *et al.* 1975) have confirmed that recoil effects play an important role in determining the character of angular distributions. Although the smooth distributions are not useful signatures of angular momentum transfer, there is evidence (Ford *et al.* 1974; Nair *et al.* 1975) that spectroscopic factors found with the full DWBA theory are reliable. Nevertheless, there are still anomalies (De Vries *et al.* 1974; Nair *et al.* 1974) and the question of whether other reaction mechanisms make significant contributions to the observed distributions is open. The length of

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full DWBA calculations has discouraged their use in wide surveys of reactions. In the author's opinion there remains a need for simple models where distributions can be systematically studied for a wide range of reactions and input parameters.

In this paper a simplified DWBA model, which is intended to reproduce the essential features of the complete theory, is presented. The model is an extension of the approximate treatment of recoil due to Braun-Munzinger and Harney (1974). The main innovation is the adoption of an eikonal-like representation for the scattering states (McCarthy and Pursey 1961), which enables the radial integrals in the DWBA amplitude to be evaluated analytically. A brief outline of the contents of the paper follows.

In Section 2 the DWBA amplitude is defined and the original ansatz of Dodd and Greider (1965, 1969) permitting the separation of coordinate integrations is recalled.

In Section 3 a simplified form of the transfer function is taken from the work of Braun-Munzinger and Harney (1974). The essential approximation, introduced first by Buttle and Goldfarb (1966), is that the final bound state wavefunction can be represented by a Hankel function in the transfer region of configuration space. As much as possible the notation and conventions of Braun-Munzinger and Harney are used throughout this paper.

The eikonal model for the scattering states is introduced in Section 4. The scattering wavefunctions of the model are compared with typical optical model wavefunctions in the energy range where strong absorption dominates, in order to estimate reasonable values for the parameters of the model. All angular integrations and sums over magnetic quantum numbers are then performed, yielding relatively simple expressions for the cross sections.

As a first test of the model, the theory is applied in Section 5 to the reaction $^{12}C(^{14}N,^{13}N)^{13}C$ at incident energies of 78, 100 and 155 MeV. Transitions to the ground and excited $2s_{1/2}$ and $1d_{5/2}$ states of the final ^{13}C nucleus are considered. The predictions of the model are compared both with experimental data (von Oertzen *et al.* 1970; De Vries *et al.* 1974; Nair *et al.* 1975) and also with the results of full DWBA calculations (De Vries and Kubo 1973; De Vries *et al.* 1974; Nair *et al.* 1975). The conclusions are presented in Section 6.

2. DWBA Amplitude with Recoil

We consider the DWBA amplitude T_{fi} for the transfer process

$$\mathsf{A}+(\mathsf{b}+\mathsf{c}) \rightarrow (\mathsf{A}+\mathsf{c})+\mathsf{b}, \tag{1}$$

where the cluster c is transferred from the nucleus a = (b+c) to form the final nuclear state of B = (A+c). If $\boldsymbol{r}$ is the vector displacement of b relative to A, and $\boldsymbol{r}'$ the vector displacement of c relative to b, the amplitude T_{fi} takes the form

$$T_{fi} = \int d^3r \int d^3r' \, \chi_f^{(-)*}(\boldsymbol{k}_f, \boldsymbol{r}A/B - \boldsymbol{r}'c/B) \times \phi_f^*(\boldsymbol{r}+\boldsymbol{r}') \, V_{bc}(\boldsymbol{r}') \, \phi_i(\boldsymbol{r}') \, \chi_i^{(+)}(\boldsymbol{k}_i, \boldsymbol{r}+\boldsymbol{r}'c/a) \,. \tag{2}$$

Here ϕ_i and ϕ_f are the bound state wavefunctions for c in the nuclei a and B respectively; the distorted wave $\chi_i^{(+)}$ describes the elastic scattering of a and A in the initial state while $\chi_f^{(-)*}$ describes the scattering of b and B in the final state; V_{bc}

is the potential between the transferred particle c and the nucleus b; and $A, a, \ldots$ denote the masses of the particles A, a, ...

In deriving equation (2) from the exact expression of formal scattering theory, it is assumed that the scattering is essentially a three-body process in which the internal degrees of freedom of the cores b and A are ignored. While the structure of b and A may be easily included in the bound states ϕ_{i} and ϕ_{f} by introducing appropriate spectroscopic factors, the representation of the scattering states $\chi_{\mathrm{i}}^{(+)}$ and $\chi_{\mathrm{f}}^{(-)}$ in the region where the transfer takes place as functions of single position vectors is more problematical. It is not obvious that optical model potentials give sufficiently accurate representations of the collision between two complex nuclei. Choices other than V_{bc} for the interaction responsible for the transition are also possible.

The six-dimensional integration of equation (2) is difficult to handle numerically. Early theories (Buttle and Goldfarb 1966; Schmittroth *et al.* 1970) simplified the evaluation of (2) by omitting from the distorted waves the terms $\boldsymbol{r}'c/B$ and $\boldsymbol{r}'c/a$ which are proportional to the small mass ratios c/B and c/a. In this *no-recoil approximation*, the amplitude (2) becomes

$$T_{\mathrm{fi}}^{0} = \int \mathrm{d}^3 r\, \chi_{\mathrm{f}}^{(-)*}(\boldsymbol{k}_{\mathrm{f}}, \boldsymbol{r}A/B)\, \chi_{\mathrm{i}}^{(+)}(\boldsymbol{k}_{\mathrm{i}}, \boldsymbol{r})\, G_{\mathrm{fi}}^{0}(\boldsymbol{r})\,. \tag{3}$$

The nuclear structure information is contained in the transfer function

$$G_{\mathrm{fi}}^{0} = \int \mathrm{d}^3 r'\, \phi_{\mathrm{f}}^{*}(\boldsymbol{r}+\boldsymbol{r}')\, V_{\mathrm{bc}}(\boldsymbol{r}')\, \phi_{\mathrm{i}}(\boldsymbol{r}')\,. \tag{4}$$

The amplitude (3) has the same form as the zero-range DWBA for light projectiles.

Some time ago it was pointed out by Dodd and Greider (1965, 1969) that the effects of recoil may be included in an approximate way, while preserving the computational advantages of equations (3) and (4). The essential idea is to retain the recoil terms only in the rapidly varying phases of the distorted waves, where their neglect would cause the most significant error. From the condition that transfer takes place in a restricted region of configuration space, corresponding to grazing collisions of the incident nuclei, one can show that at high energies the phases of the distorted waves in this region may be represented by the phases of equivalent plane waves, and so we may write

$$\chi_{\mathrm{f}}^{(-)}(\boldsymbol{k}_{\mathrm{f}}, \boldsymbol{r}A/B - \boldsymbol{r}'c/B) \approx \chi_{\mathrm{f}}^{(-)}(\boldsymbol{k}_{\mathrm{f}}, \boldsymbol{r}A/B)\exp(-\mathrm{i}\boldsymbol{q}_{\mathrm{f}}\,.\,\boldsymbol{r}'c/B)\,, \tag{5a}$$

$$\chi_{\mathrm{i}}^{(+)}(\boldsymbol{k}_{\mathrm{i}}, \boldsymbol{r}+\boldsymbol{r}'c/a) \approx \chi_{\mathrm{i}}^{(+)}(\boldsymbol{k}_{\mathrm{i}}, \boldsymbol{r})\exp(\mathrm{i}\boldsymbol{q}_{\mathrm{i}}\,.\,\boldsymbol{r}'c/a)\,. \tag{5b}$$

The 'local momenta' $\boldsymbol{q}_{\mathrm{i}}$ and $\boldsymbol{q}_{\mathrm{f}}$ in the transfer region need not equal the asymptotic momenta $\boldsymbol{k}_{\mathrm{i}}$ and $\boldsymbol{k}_{\mathrm{f}}$. With the expressions (5) the transfer function becomes

$$G_{\mathrm{fi}} = \int \mathrm{d}^3 r'\, \exp(\mathrm{i}\boldsymbol{Q}\,.\,\boldsymbol{r}')\, \phi_{\mathrm{f}}^{*}(\boldsymbol{r}+\boldsymbol{r}')\, V_{\mathrm{bc}}(\boldsymbol{r}')\, \phi_{\mathrm{i}}(\boldsymbol{r}')\,, \tag{6}$$

which differs from the G_{fi}^{0} of equation (4) by an additional recoil phase factor $\exp(\mathrm{i}\boldsymbol{Q}\,.\,\boldsymbol{r}')$ with recoil momentum

$$\boldsymbol{Q} = \boldsymbol{q}_{\mathrm{i}}\, c/a + \boldsymbol{q}_{\mathrm{f}}\, c/B\,. \tag{7}$$

Taking into account the spin s and j_{i} and j_{f}, the sum of the spin and orbital angular momentum for the particle c in the initial and final nuclear states, we may define a transfer function G_{lm} for the transfer of definite orbital angular momentum l and third component m by

$$G_{lm}(\boldsymbol{r}) = \sum_{m_{\mathrm{i}}m_{\mathrm{f}}} \hat{l}\hat{J}_{\mathrm{a}}(-1)^{s+j_{\mathrm{f}}+m_{\mathrm{f}}} \begin{pmatrix} l_{\mathrm{f}} & l_{\mathrm{i}} & l \\ m_{\mathrm{f}} & -m_{\mathrm{i}} & -m \end{pmatrix} \begin{pmatrix} l_{\mathrm{f}} & l_{\mathrm{i}} & l \\ j_{\mathrm{i}} & j_{\mathrm{f}} & s \end{pmatrix} G_{\mathrm{fi}}(\boldsymbol{r}). \tag{8}$$

A partial transition amplitude β_{lm} for the transfer of definite orbital angular momentum is then defined by

$$\beta_{lm} = \int \mathrm{d}^3 r\, \chi_{\mathrm{f}}^{(-)*}(\boldsymbol{k}_{\mathrm{f}}, \boldsymbol{r}A/B)\, \chi_{\mathrm{i}}^{(+)}(\boldsymbol{k}_{\mathrm{i}}, \boldsymbol{r})\, G_{lm}(\boldsymbol{r}). \tag{9}$$

If the scattering states are spin independent then the differential cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(2\pi\hbar^2)^2} \frac{aA}{a+A} \frac{bB}{b+B} \frac{k_{\mathrm{f}}(2J_{\mathrm{B}}+1)}{k_{\mathrm{i}}(2J_{\mathrm{a}}+1)(2J_{\mathrm{A}}+1)} \sum_{lm} |\beta_{lm}|^2. \tag{10}$$

3. Simplification of Transfer Function

Several methods have been developed to simplify the transfer function of equation (6) (Braun-Munzinger and Harney 1974; Nagarajan 1972; Sawaguri and Tobocman 1967). Here we shall adopt the method of Braun-Munzinger and Harney (1974), based on the work of Buttle and Goldfarb (1966).

In order to separate the dependence on the variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ in the integral of equation (6), the radial wavefunction of the bound state $\phi_{\mathrm{f}} = u_{l_{\mathrm{f}}} \mathrm{Y}_{l_{\mathrm{f}}}^{m_{\mathrm{f}}}$ is replaced by its asymptotic form

$$\mathrm{i}^{l_{\mathrm{f}}} u_{l_{\mathrm{f}}}(\boldsymbol{r}+\boldsymbol{r}') \approx N_{\mathrm{f}}\, \mathrm{h}_{l_{\mathrm{f}}}^{(1)}(\mathrm{i}\alpha_{\mathrm{f}} |\boldsymbol{r}+\boldsymbol{r}'|). \tag{11}$$

The validity of this substitution of the spherical Hankel function $\mathrm{h}_{l_{\mathrm{f}}}$ for $u_{l_{\mathrm{f}}}$ depends on the presence of strong absorption when the nuclear cores overlap so that contributions from the transfer function for small $\boldsymbol{r}$ are insignificant in the integral (9). Braun-Munzinger and Harney (1974) have given a careful discussion of the conditions that must hold in order for the approximation to be acceptable.

With the help of an addition theorem for h_l, the final bound state becomes

$$\phi_{\mathrm{f}}^* = (4\pi)^{\frac{1}{2}} N_{\mathrm{f}} \sum_{l_1 m_1 l_2 m_2} \mathrm{i}^{l_2+l_1-l_{\mathrm{f}}} (-1)^{-m_2} \hat{l}_1 \hat{l}_{\mathrm{f}} \hat{l}_2$$
$$\times \begin{pmatrix} l_1 & l_2 & l_{\mathrm{f}} \\ m_1 & -m_2 & m_{\mathrm{f}} \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_{\mathrm{f}} \\ 0 & 0 & 0 \end{pmatrix} \mathrm{h}_{l_1}^{(1)*}(\mathrm{i}\alpha_{\mathrm{f}} r)\, \mathrm{j}_{l_2}^*(\mathrm{i}\alpha_{\mathrm{f}} r')\, \mathrm{Y}_{l_1}^{m_1}(\hat{\boldsymbol{r}})\, \mathrm{Y}_{l_2}^{m_2*}(\hat{\boldsymbol{r}}'). \tag{12}$$

When the recoil phase factor is expanded in partial waves and the expressions for ϕ_{i} and ϕ_{f}^* are substituted, the transfer function simplifies to

$$G_{lm}(\boldsymbol{r}) = 4\pi N_{\mathrm{f}} (-1)^{j_{\mathrm{f}}+s} \hat{J}_{\mathrm{a}} \hat{l} \hat{l}_{\mathrm{f}} \begin{Bmatrix} l_{\mathrm{f}} & s & j_{\mathrm{f}} \\ j_{\mathrm{i}} & l & l_{\mathrm{i}} \end{Bmatrix}$$
$$\times \sum_{l_1 m_1 L M} \mathrm{i}^{L+l_{\mathrm{i}}+l_{\mathrm{f}}+l_1} (-1)^{L+l_1-l} \hat{l}_1 \hat{L} \begin{pmatrix} l_1 & l & L \\ m_1 & m & M \end{pmatrix}$$
$$\times \gamma_l(l_1, L)\, \mathrm{h}_{l_1}^{(1)*}(\mathrm{i}\alpha_{\mathrm{f}} r)\, \mathrm{Y}_{l_1}^{m_1}(\hat{\boldsymbol{r}})\, \mathrm{Y}_L^M(\hat{\boldsymbol{Q}}), \tag{13}$$

where the coefficients $\gamma_l(l_1, L)$ are given by

$$\gamma_l(l_1, L) = (-1)^L \sum_{l_2} \hat{l}_2^2 \hat{l}_i \begin{pmatrix} l_1 & l_f & l_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_2 & L & l_i \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} l_1 & l_f & l_2 \\ l_i & L & l \end{Bmatrix} A(l_2 L), \tag{14}$$

with

$$A(l_2 L) = \mathrm{i}^{l_2} \int_0^\infty \mathrm{d}r'\, r'^2 \mathrm{j}_L(Qr')\, u_{l_i}(r')\, V_{\mathrm{bc}}(r')\, \mathrm{j}_{l_2}^*(\mathrm{i}\alpha_{\mathrm{f}} r'). \tag{15}$$

4. Eikonal Model for Scattering States

The scattering states $\chi_{\mathrm{i}}^{(+)}$ and $\chi_{\mathrm{f}}^{(-)}$ are usually generated from optical model potentials found by fitting elastic scattering data. Although such potentials may give the correct asymptotic form of the elastic scattering wavefunction, the wavefunction of the colliding nuclei may not be represented adequately for small internuclear distances. However, one important characteristic of the optical model potentials used in heavy ion scattering is their very strong absorptive parts, which have larger radii than the real parts of the potentials. This is a fortunate circumstance since the most poorly understood part of the scattering wavefunction where the nuclei overlap contributes minimally to the transfer amplitude (9).

The integrand in equation (9) also decreases rapidly for large internuclear separations because of the exponential decay of the transfer function, arising from the bound state factors. Thus the main contribution to the transfer amplitude comes from a limited range in the coordinate r corresponding to grazing collisions of the nuclei.

In this paper an eikonal model (McCarthy and Pursey 1961), which has been applied successfully to other nuclear (Kazacs and Smith 1975) and atomic scattering (Furness and McCarthy 1973) problems is used to represent the distorted waves $\chi_{\mathrm{i}}^{(+)}$ and $\chi_{\mathrm{f}}^{(-)}$ in the transfer region:

$$\chi_{\mathrm{i}}^{(+)}(\boldsymbol{k},\boldsymbol{r}) = A_{\mathrm{i}}(r)\exp\{\mathrm{i}f_{\mathrm{i}}(r)\boldsymbol{k}.\boldsymbol{r}\}, \qquad \chi_{\mathrm{f}}^{(-)}(\boldsymbol{k},\boldsymbol{r}) = A_{\mathrm{f}}(r)\exp\{\mathrm{i}f_{\mathrm{f}}^*(r)\boldsymbol{k}.\boldsymbol{r}\}. \tag{16}$$

The effect of the real part of the optical potential is expressed through the real part f^{R} of the function f which modifies the wave number. The imaginary part f^{I} of f describes the attenuation of the scattered wave at backward angles and the real function $A(r)$, which vanishes for small r, accounts for the strong absorption when the nuclei overlap.

In order to test the validity of the model wavefunctions in the transfer region, they have been compared with polar plots of optical model wavefunctions used in typical DWBA calculations of transfer amplitudes. As an example, an optical model wavefunction for elastic scattering of ^{14}N on ^{12}C at 78 MeV is shown in Fig. 1. The parameter values for the potential are those of Set 3 of von Oertzen *et al.* (1970). The radial wavefunction $R_l(r)$ for each partial wave l was obtained using the DWBA code DWUCK and a small program was written to produce the magnitude $|\psi|$ and the phase ϕ of the elastic wavefunction,

$$\psi(r,\theta) = |\psi|\exp\mathrm{i}\phi = \sum_l \mathrm{i}^l(2l+1)\,\mathrm{P}_l(\cos\theta)\,R_l(r),$$

from R_l and the Legendre polynomials P_l, as functions of the elastic scattering angle θ and the internuclear distance r.

It is seen from Fig. 1 that the optical model wavefunctions resemble plane waves quite closely in the transfer region. Most distortion occurs at backward angles and within the complex well. Inside the dashed line of the figure the magnitude of the wavefunction is less than 10% of its maximum value, so that the distorted part of the scattering wavefunction is relatively unimportant.

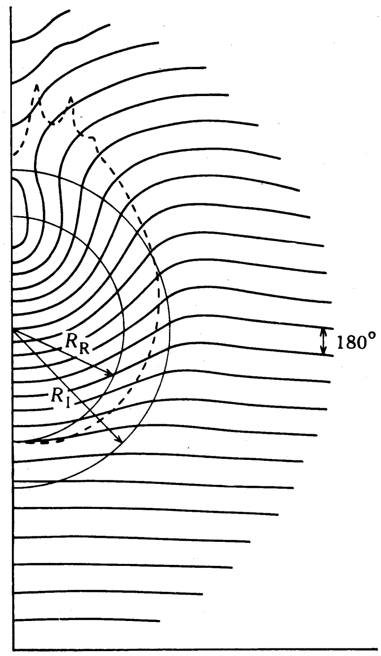

Fig. 1. Polar plot of contours of equal phase of an optical model wavefunction for the elastic scattering of ^{14}N on ^{12}C at an incident energy E of 78 MeV. The parameter values for the potential are:
$V_R = -100$ MeV, $V_I = -38\cdot5$ MeV,
$R_R = 4\cdot21$ fm, $R_I = 5\cdot90$ fm,
$a_R = 0\cdot77$ fm, $a_I = 0\cdot26$ fm.
Inside the dashed line the magnitude of the wavefunction is less than 10% of that of the incident plane wave.

The functions $f(r)$ and $A(r)$ may be obtained from the optical model wavefunction ψ by expanding, for fixed r, the logarithm of the magnitude of ψ and the phase of ψ in Legendre polynomials of $u = \cos\theta$:

$$-(kr)^{-1}\ln|\psi| = a_0 + a_1 u + a_2 P_2(u) + \ldots,$$

$$(kr)^{-1}\phi = b_0 + b_1 u + b_2 P_2(u) + \ldots.$$

The coefficients a_i and b_i are determined by

$$(2i+1)a_i = 2(kr)^{-1}\int_{-1}^{1} P_i(u)\ln|\psi|\,du, \qquad (2i+1)b_i = 2(kr)^{-1}\int_{-1}^{1} P_i(u)\,\phi\,du,$$

whence

$$A(r) = \exp\{-kra_0(r)\}, \qquad f^{I}(r) = a_1(r), \quad f^{R} = b_1(r).$$

The functions $A(r)$ and $f(r)$ for the wavefunction of Fig. 1 are plotted in Fig. 2.

When the representation (16) of the scattered waves is substituted in the expressions (5), the transfer amplitude β_{lm} takes the form

$$\beta_{lm} = \int \mathrm{d}^3 r\, A_\mathrm{i}(r)\, A_\mathrm{f}(rA/B) \exp(\mathrm{i}\boldsymbol{P}.\boldsymbol{r})\, G_{lm}(\boldsymbol{r}), \tag{17}$$

with the complex vector

$$\boldsymbol{P} = f_\mathrm{i}\boldsymbol{k}_\mathrm{i} - (A/B) f_\mathrm{f}\boldsymbol{k}_\mathrm{f}. \tag{18}$$

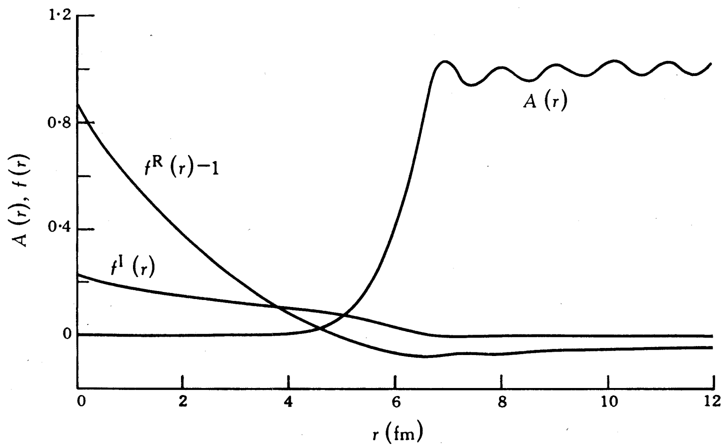

Fig. 2. Values of the parameters $A(r)$ and $f(r)$ of the model wavefunctions of equations (16) as functions of the distance r between the nuclei ^{14}N and ^{12}C for elastic scattering at an incident energy of 78 MeV. The results are obtained by averaging the values given by the optical model wavefunction of Fig. 1 over the scattering angle.

The recoil momentum of equation (7) becomes

$$\boldsymbol{Q} = (c/a) f_\mathrm{i}\boldsymbol{k}_\mathrm{i} + (c/B) f_\mathrm{f}\boldsymbol{k}_\mathrm{f}. \tag{19}$$

After substitution of the transfer function (13) in the amplitude (17), the angular integrations may be performed, to yield

$$\beta_{lm} = (4\pi)^2 N_\mathrm{f} (-1)^{j_\mathrm{f}+s} \hat{\jmath}_\mathrm{a} \hat{l} \hat{l}_\mathrm{f} \begin{Bmatrix} l_\mathrm{f} & s & j_\mathrm{f} \\ j_\mathrm{i} & l & l_\mathrm{i} \end{Bmatrix}$$
$$\times \sum_{l_1 m_1 L M} \mathrm{i}^{L+l_\mathrm{i}+l_\mathrm{f}+l_1} (-1)^{L+l_1-l} \hat{l}_1 \hat{L} \begin{pmatrix} l_1 & l & L \\ m_1 & m & M \end{pmatrix} \gamma_l(l_1, L)\, F_{l_1}(P)\, \mathrm{Y}_{l_1}^{m_1}(\hat{\boldsymbol{P}})\, \mathrm{Y}_L^{M}(\hat{\boldsymbol{Q}} \,. \tag{20}$$

The function F_{l_1} results from the radial integration,

$$F_{l_1}(P) = \int_0^\infty \mathrm{d}r\, r^2 B(r)\, \mathrm{j}_{l_1}(Pr)\, g_{l_1}(\alpha_\mathrm{f} r), \tag{21}$$

with

$$B(r) = A_\mathrm{i}(r)\, A_\mathrm{f}(rA/B), \qquad g_{l_1}(\alpha_\mathrm{f} r) = \mathrm{i}^{l_1} \mathrm{h}_{l_1}^{(1)*}(\mathrm{i}\alpha_\mathrm{f} r).$$

If it is assumed, as in the case of the reactions studied in Section 5 below, that the scattering states $\chi_{\mathrm{i}}^{(+)}$ and $\chi_{\mathrm{f}}^{(-)}$ have similar parameterizations then the spherical polar angles specifying the complex vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ are real, and all the dependence on the magnetic quantum numbers may be removed, yielding the final result,

$$\sum_{lm} |\beta_{lm}|^2 = (4\pi N_{\mathrm{f}} \hat{J}_{\mathrm{a}} \hat{l}_{\mathrm{f}})^2 \sum_{l} (-1)^l \hat{l}^2 \begin{Bmatrix} l_{\mathrm{f}} & s & j_{\mathrm{f}} \\ j_{\mathrm{i}} & l & l_{\mathrm{i}} \end{Bmatrix}$$

$$\times \sum_{Jl_1l_1'LL'} \mathrm{i}^{L-l_1-L'+l_1'} (\hat{l}_1 \hat{l}_1' \hat{L} \hat{L}' \hat{J})^2 \begin{pmatrix} l_1 & l_1' & J \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & L' & J \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} L & L' & J \\ l_1' & l_1 & l \end{Bmatrix}$$

$$\times \gamma_l(l_1, L)\, \gamma_l(l_1', L')\, F_{l_1}(P)\, F_{l_1'}^*(P)\, \mathrm{P}_J(\hat{\boldsymbol{Q}} . \hat{\boldsymbol{P}}). \quad (22)$$

The Legendre polynomial P_J here has the argument

$$\hat{\boldsymbol{Q}} . \hat{\boldsymbol{P}} = \frac{\{(c/a)\boldsymbol{k}_{\mathrm{i}} + (c/B)\boldsymbol{k}_{\mathrm{f}}\} . \{\boldsymbol{k}_{\mathrm{i}} - (A/B)\boldsymbol{k}_{\mathrm{f}}\}}{|(c/a)\boldsymbol{k}_{\mathrm{i}} + (c/B)\boldsymbol{k}_{\mathrm{f}}|\, |\boldsymbol{k}_{\mathrm{i}} - (A/B)\boldsymbol{k}_{\mathrm{f}}|} \quad (23)$$

and the coefficients γ_l, which determine the rate of convergence of the sums over the angular momenta J, L, L', l_1', l_1, are given explicitly by equations (14) and (15). Note that both P in equation (21) and Q in equation (15) are magnitudes of complex vectors. Putting $f(r) = f_{\mathrm{i}}(r) \approx f_{\mathrm{f}}(r)$, we have

$$P \approx f(r)\, \{k_{\mathrm{i}}^2 + (A^2/B^2)k_{\mathrm{f}}^2 - (2A/B)\boldsymbol{k}_{\mathrm{i}} . \boldsymbol{k}_{\mathrm{f}}\}^{\frac{1}{2}} \quad (24)$$

and

$$Q \approx f(r)\, \{(c^2/a^2)k_{\mathrm{i}}^2 + (c^2/B^2)k_{\mathrm{f}}^2 + (2c^2/aB)\boldsymbol{k}_{\mathrm{i}} . \boldsymbol{k}_{\mathrm{f}}\}^{\frac{1}{2}}. \quad (25)$$

From the $3j$ symbols of equation (22) the possible orbital angular momentum transfers are limited by the selection rules

$$|j_{\mathrm{i}} - j_{\mathrm{f}}| \leqslant l \leqslant j_{\mathrm{i}} + j_{\mathrm{f}}, \qquad |l_{\mathrm{i}} - l_{\mathrm{f}}| \leqslant l \leqslant l_{\mathrm{i}} + l_{\mathrm{f}}, \quad (26)$$

where j_{i} (j_{f}) and l_{i} (l_{f}) are the total and orbital angular momenta of the state ϕ_{i} (ϕ_{f}).

In the no-recoil approximation we have $Q = 0$, and $A(l_2 L)$ vanishes for $L \neq 0$ owing to the vanishing of the spherical Bessel function $\mathrm{j}_L(Qr')$ in equation (15). Then $l_2 = l_{\mathrm{i}}$ in the expression (14) for $\gamma_l(l_1, L)$, and $l_1 = l$ from the $3j$ symbol in equation (20) for β_{lm}. Thus, in the limit of no recoil, equation (22) becomes

$$\sum_{lm} |\beta_{lm}|^2 = (4\pi N_{\mathrm{f}} \hat{J}_{\mathrm{a}} \hat{l}_{\mathrm{f}})^2 \sum_{l} \hat{l}^2 \begin{Bmatrix} l_{\mathrm{f}} & s & j_{\mathrm{f}} \\ j_{\mathrm{i}} & l & l_{\mathrm{i}} \end{Bmatrix}^2 \gamma_l^2(l, 0)\, |F_l(P)|^2. \quad (27)$$

It is important to note that $\gamma_l(l, 0)$ now contains the $3j$ symbol

$$\begin{pmatrix} l & l_{\mathrm{f}} & l_{\mathrm{i}} \\ 0 & 0 & 0 \end{pmatrix}$$

which leads to the additional selection rule

$$(-1)^l = (-1)^{l_{\mathrm{i}} + l_{\mathrm{f}}}, \quad (28)$$

that is, $l+l_i+l_f$ must be even. When recoil is included, the additional recoil angular momentum L is transferred, allowing transfers l such that $l+l_i+l_f$ is odd or even.

A computer program evaluating the differential cross section (10) with the expression (22) for arbitrary l transfers has been written. A sharp cutoff model with diffraction radius R_0 was first adopted for the scattering states, allowing the radial integrals of equation (21) to be performed analytically. In terms of the integral

$$\mathscr{I}_l(r) = \int_r^\infty \mathrm{d}r\, r^2 \mathrm{j}_l(Pr)\, g_l(\alpha_f r) = \frac{r^2}{\alpha_f^2+P^2}\Big(\alpha_f \mathrm{j}_l(Pr)\, g_{l-1}(\alpha_f r) - P\mathrm{j}_{l-1}(Pr)\, g_l(\alpha_f r)\Big), \quad (29)$$

we have, taking $B(r)$ as a step function,

$$F_{l_1}(P) = \mathscr{I}_{l_1}(R_0). \quad (30)$$

In order to produce the correct slope of the experimental angular distributions, it was found necessary to modify equation (30) by including a more realistic representation of the absorption when the nuclei overlap. After integration by parts, equation (21) becomes

$$F_{l_1}(P) = \int_0^\infty \frac{\mathrm{d}B(r)}{\mathrm{d}r}\mathscr{I}_{l_1}(r)\,\mathrm{d}r. \quad (31)$$

A simple gaussian form was taken for $\mathrm{d}B(r)/\mathrm{d}r$, namely

$$\mathrm{d}B(r)/\mathrm{d}r = (\Delta r\sqrt{\pi})^{-1}\exp\{-(r-R_0)^2/(\Delta r)^2\},$$

which is sharply peaked at the radius R_0 with a width determined by the additional parameter Δr. This representation of $\mathrm{d}B/\mathrm{d}r$ results in an expression for $B(r)$ in terms of the error function,

$$B(r) = \tfrac{1}{2}\{1+\mathrm{erf}((r-R_0)/\Delta r)\},$$

which is a reasonable approximation to the actual $B(r)$ determined from the optical model wavefunctions. In particular, the values $R_0 = 6\cdot5$ fm and $\Delta r = 0\cdot6$ fm give quite an accurate representation of $B(r)$ associated with the optical model of Fig. 1.

In the evaluation of the integrals (15) determining the coefficients γ_l, bound state wavefunctions for a square well were used. Of course, more refined representations of the scattering states and bound states are possible within the framework of the theory.

5. Results

In this section we consider the specific reaction $^{12}C(^{14}N,{}^{13}N)^{13}C$, which has been studied extensively. The predictions of the present model are compared both with the experimental data (von Oertzen *et al.* 1970; De Vries *et al.* 1974; Nair *et al.* 1975) and full finite-range DWBA calculations (De Vries and Kubo 1973; De Vries *et al.* 1974; Nair *et al.* 1975) based on the direct evaluation of the amplitude (2).

The ground state transition for the reaction is assumed to result from the transfer of a neutron in a $1p_{1/2}$ orbital in ^{14}N to a $1p_{1/2}$ orbital in ^{13}C. From the selection rules (26) there are two possible l transfers, namely $l = 0$ and 1. The predictions of our model for incident energies of 78, 100 and 155 MeV and the different choices of parameters listed in Table 1 are shown in Fig. 3.

As a reasonable first approach to model calculations, the function $f(r)$ of equations (24) and (25) was fixed at a constant value $f^{\mathrm{R}}+\mathrm{i}f^{\mathrm{I}}$ equal to its average value in the grazing region, determined by the procedure discussed in Section 4. The parameters R_0 and Δr were varied until the best fit to the slope of the angular distribution in

Table 1. Parameter sets for model distributions in Fig. 3

Parameter set	E (MeV)	R_0 (fm)	Δr (fm)	f^{R}	f^{I}	$S_{\mathrm{i}}\,S_{\mathrm{f}}$
A	78	6·0	0·8	0·95	0·04	0·48
B	78	6·5	0·6	0·95	0·04	0·63
C	100	6·0	0·8	0·95	0·04	0·48
D	155	6·0	0·8	0·95	0·04	0·48
E	155	6·5	0·6	0·95	0·04	0·38

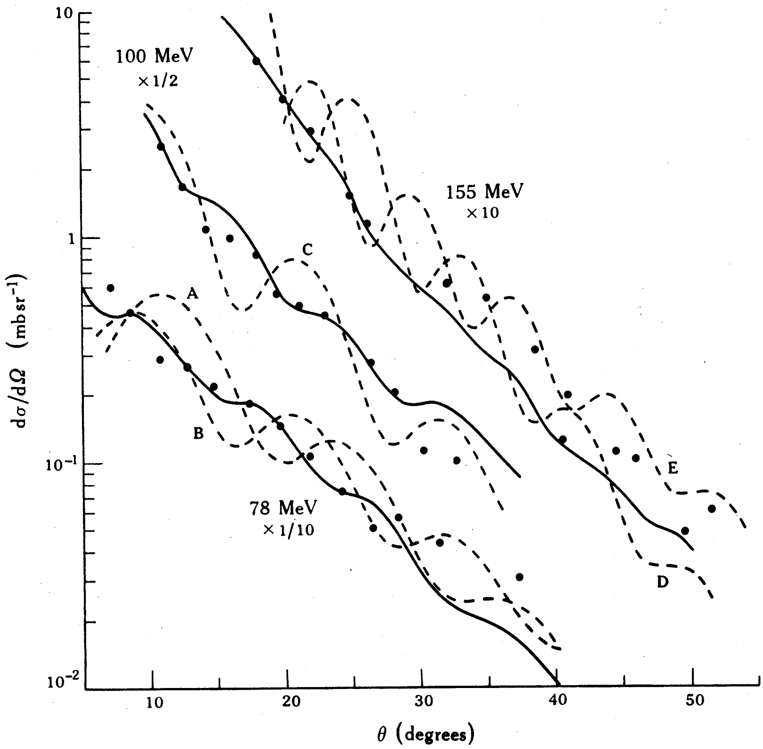

Fig. 3. Angular distributions for the reaction ^{12}C(^{14}N, ^{13}N)^{13}C(g.s.) at incident energies of 78, 100 and 155 MeV. The dashed curves are the present model fits for the parameter sets A–E of Table 1. The solid curves depict results of exact DWBA calculations at 78, 100 and 155 MeV by De Vries and Kubo (1973), De Vries *et al.* (1974) and Nair *et al.* (1975) respectively. The experimental data are from von Oertzen *et al.* (1970), De Vries *et al.* (1974) and Nair *et al.* (1975).

the ground state transition was found. This occurred for values of R_0 between 6·0 and 6·5 fm and Δr between 0·6 and 0·8 fm. For intermediate values the residual oscillations have about the same magnitude. The phase of the residual oscillations is sensitive to the radius R_0, as seen from a comparison of curves D and E in Fig. 3, but in view of the limitations of the model there seems little point in attempting to fit this fine structure. It is encouraging to find that the empirical values of Δr and R_0 obtained by fitting the experimental data are in good agreement with the theoretical values of $\Delta r = 0{\cdot}6$ fm and $R_0 = 6{\cdot}5$ fm found by fitting the function $B(r)$ of the optical model wavefunction.

Also shown in Fig. 3 are the results of exact DWBA calculations (De Vries and Kubo 1973; De Vries *et al.* 1974; Nair *et al.* 1975). The residual oscillations in the model distributions are greater in magnitude than those of the full calculations but overall there is reasonable agreement. The product spectroscopic factor $S_i\,S_f$ found from the normalization of the model distributions to the experimental data is $0{\cdot}48$ for parameter set A of Table 1 and $0{\cdot}63$ for parameter set B, which are in good agreement with the $0{\cdot}51$ result of De Vries and Kubo (1973) and $0{\cdot}5$ of Nair *et al.* (1975).

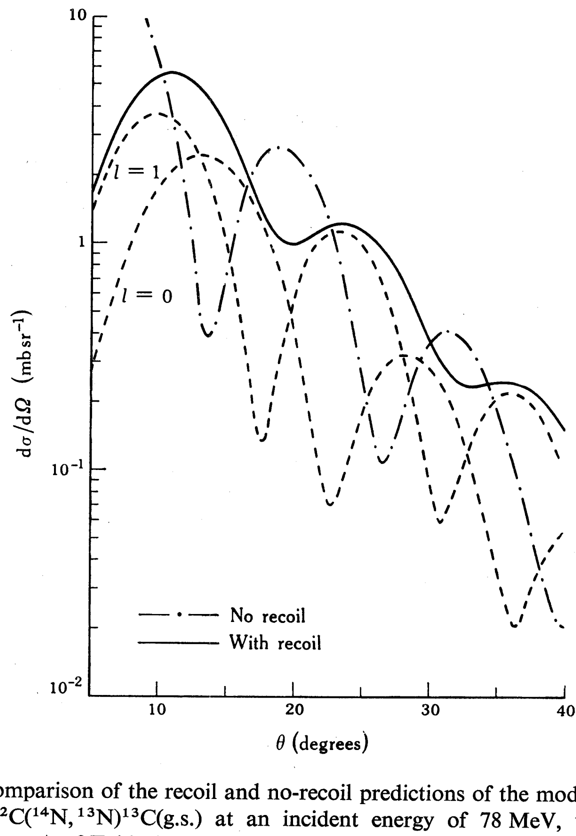

Fig. 4. Comparison of the recoil and no-recoil predictions of the model for the reaction $^{12}C(^{14}N, ^{13}N)^{13}C$(g.s.) at an incident energy of 78 MeV, using the parameter set A of Table 1. It is seen that there are marked oscillations in the distribution when recoil is ignored (dot–dash curve). The smoother distribution when recoil is included (solid curve) is the sum of the $l = 0$ and 1 contributions (dashed curves).

In the no-recoil limit, equation (28) permits $l = 0$ transfers only. The resulting distributions, which are plotted in Fig. 4, have marked oscillations in complete disagreement with the data, showing the necessity of including recoil as emphasized in previous work. The addition of the $l = 0$ and 1 components, to give the much smoother distribution when recoil is included, is also shown in Fig. 4.

An attempt has been made to fit the anomalous $1p_{1/2} \rightarrow 2s_{1/2}$ distribution (De Vries *et al.* 1974) shown in Fig. 5 with the same set of scattering state parameters used for the ground state transition. Here only $l = 0$ transfers are possible and one might expect from simple arguments (Greider 1970) that the distribution should exhibit larger oscillations than the ground state distribution, where there is a mixture of transfers,

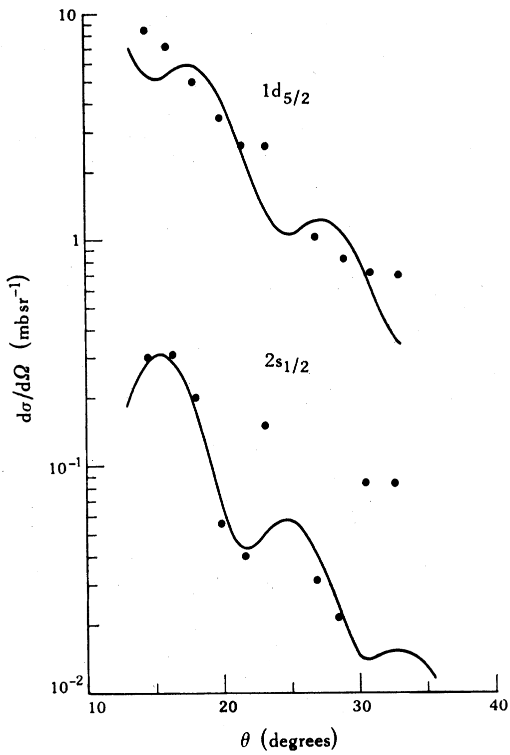

Fig. 5. Angular distributions from the model for transfer to the excited $1d_{5/2}$ (3·85 MeV) and $2s_{1/2}$ (3·09 MeV) states of ^{13}C in the reaction $^{12}C(^{14}N, ^{13}N)^{13}C^*$ at an incident energy of 100 MeV. The model parameters used are the same as set C in Table 1 for the ground state transition. The experimental data are from De Vries *et al.* (1974).

as indeed the experimental distribution does. However, the oscillations predicted by the present model are no greater than in the ground state transitions. This can be understood in the context of the model, when it is realized that there is an additional effect tending to smooth the oscillations; when recoil is included the amplitude for $l = 0$ contains an additional admixture of functions $\mathscr{I}_{l_1}$ with $l_1 > 0$.

The magnitude of the model distribution for the $1p_{1/2} \rightarrow 2s_{1/2}$ transition is at least an order of magnitude too large when compared with the experimental data. The finite-range DWBA calculation of De Vries *et al.* (1974) also gives very poor fits for this transition and underestimates the spectroscopic factor. It has been suggested that the single-step DWBA description of this transition is inadequate and some other reaction mechanism occurs.

Finally we consider the $1p_{1/2} \rightarrow 1d_{5/2}$ transition. With the scattering state parameters of the ground state transition, the slope of the experimental angular distribution is reproduced (see Fig. 5). The residual oscillations from the incomplete interference of the $l = 2$ and 3 transfers are again larger than those of the exact DWBA calculations. The spectroscopic factor for the $1d_{5/2}$ level in ^{13}C found in this case is $0 \cdot 23$, which is in fair agreement with the values of $0 \cdot 37$ found by De Vries *et al.* (1974) and $0 \cdot 57$ by Nair *et al.* (1975), all of which are smaller than the expected value of $0 \cdot 8$ given by Cohen and Kurath (1967).

6. Conclusions

A model for transfer, including the essential features of the finite-range DWBA theory, has been formulated. The model retains the possibility of a detailed examination of recoil effects and strong absorption, which are the predominant features of heavy ion induced transfer reactions at high energy.

Choice of the parameters specifying the model scattering wavefunctions has been guided by considering the form of the usual optical model wavefunctions. With reasonable values of the model parameters, the general features of the more complex DWBA calculations for the reaction $^{12}C(^{14}N, ^{13}N)^{13}C$ have been reproduced. While the ground state transition is satisfactorily understood, the predictions for the transition to the $2s_{1/2}$ state (like those of the full DWBA calculations) are not in agreement with experiment. The results for the transition to the $1d_{5/2}$ level of ^{13}C are in reasonable agreement with other calculations and experiment.

The model confirms the importance of treating the kinematics of transfer between heavy ions correctly and the dominant effect of the strong absorption when the nuclear cores interact. In view of the agreement of the model with the usual DWBA calculations it is expected that the present theory will be useful in making systematic estimates of reaction cross sections for a wide range of reactions.

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