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The Equilibrium Statistical Mechanics of a One-dimensional Self-gravitational System

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Abstract

It is shown that the exact statistical mechanical properties of a one-dimensional self-gravitational system in the limit of an infinite number of particles are easily obtained with the aid of the virial expansion.

Rybicki (1971) studied the statistical mechanical properties of a one-dimensional self-gravitational system of $N = 2, 3, \ldots$ particles. He obtained the free energy and the one-particle distribution function for both finite and infinite N. Monaghan (1978) applied approximate methods to this system for the limiting case of infinite N. We here call attention to the fact that the properties of this system in the limit of infinite N may be calculated exactly with the aid of the 'molecular field approximation'.

The potential energy of a one-dimensional system of N particles is given by

$$\Phi\{x_j\} = \lambda \sum_{N \geqslant i > j \geqslant 1} |x_i - x_j|, \tag{1}$$

where x_j is the coordinate of the jth particle and $\{x_j\}$ stands for the set of $x_1, x_2, \ldots, x_N$. The configurational part F of the free energy is given in terms of the one-particle distribution function $\rho^{(1)}(x)$ by

$$\beta F = \int \rho^{(1)}(x) \left[\ln\left(\rho^{(1)}(x)\right) - 1\right] \mathrm{d}x - \tfrac{1}{2} \iint \rho^{(1)}(x)\, \rho^{(1)}(x') \left[\exp\left(-\lambda\beta\,|x - x'|\right) - 1\right] \mathrm{d}x\, \mathrm{d}x' - \Sigma, \tag{2}$$

where Σ is the sum of all the more than singly connected diagrams of vertices and simple bonds connecting them. Here each diagram of n vertices represents an n-fold integral with respect to the variables $x_1, x_2, \ldots, x_n$, the integrand being the product of (i) the one-particle distribution function $\rho^{(1)}(x_i)$ for the ith vertex, (ii) the factor $\exp(-\lambda\beta\,|x_i - x_j|) - 1$ for a bond connecting the ith and jth vertices and (iii) the inverse of the symmetry number of the diagram (Morita 1959; Morita and Hiroike 1961). The integrations are taken over the volume of the system. In addition, we have $\beta = 1/k_{\mathrm{B}} T$ as usual, while $\rho^{(1)}(x)$ is so normalized that

$$\int \rho^{(1)}(x)\, \mathrm{d}x = N.$$

By transforming to reduced coordinates $\xi_j = x_j/L$, where $L = (N\beta\lambda)^{-1}$, we convert equation (1) to

$$\beta\Phi\{L\xi_j\} = N^{-1} \sum_{N \geqslant i > j \geqslant 1} |\xi_i - \xi_j| . \tag{3}$$

We now define the functions

$$\bar{\rho}^{(1)}(\xi) = L\rho^{(1)}(L\xi) \qquad \text{and} \qquad \bar{\rho}^{(2)}(\xi, \xi') = L^2 \rho^{(2)}(L\xi, L\xi'),$$

which play the role of the one-particle and two-particle distribution functions respectively. The normalization is such that

$$\int \bar{\rho}^{(1)}(\xi)\, \mathrm{d}\xi = N. \tag{4}$$

Equation (2) then takes the form

$$\beta F = \int \bar{\rho}^{(1)}(\xi) \left[\ln\left(\bar{\rho}^{(1)}(\xi)/L\right) - 1\right] \mathrm{d}\xi$$

$$-\tfrac{1}{2} \iint \bar{\rho}^{(1)}(\xi)\, \bar{\rho}^{(1)}(\xi') [\exp(-N^{-1}|\xi - \xi'|) - 1]\, \mathrm{d}\xi\, \mathrm{d}\xi' - \Sigma. \tag{5}$$

In Σ, a diagram of n vertices represents an n-fold integral with respect to the variables $\xi_1, \xi_2, \ldots, \xi_n$, the integrand being the product of (i) the transformed one-particle distribution function $\bar{\rho}^{(1)}(\xi_i)$ for the ith vertex, (ii) the factor $\exp(-N^{-1}|\xi_i - \xi_j|) - 1$ for a bond connecting the ith and jth vertices and (iii) the inverse of the symmetry number of the diagram. If we make the assumption that $\bar{\rho}^{(1)}(\xi)$ takes nonzero values of order $O(N)$ only within a range of order $O(N^0)$ from an arbitrary fixed point as $N \to \infty$, then a term represented by a diagram of m vertices and n bonds will be of order $O(N^{m-n})$. Since $n \geqslant m$ holds for all diagrams represented by the sum Σ in equation (5), we estimate this term to be of order $O(N^0)$; the situation is similar to the case of the Husimi–Temperley model (Katsura 1963). The $\bar{\rho}^{(1)}(\xi)$ is so determined as to make the right-hand side of equation (5) stationary under the subsidiary condition (4) (Morita and Hiroike 1961), so that we have

$$\ln\left(\bar{\rho}^{(1)}(\xi)/L\right) = -N^{-1} \int \bar{\rho}^{(1)}(\xi')\, |\xi - \xi'|\, \mathrm{d}\xi' + \beta\mu + \ldots . \tag{6}$$

In the limit as $N \to \infty$, the integration limits tend to $\pm\infty$. The continuation dots (ellipses) in equation (6) denote terms which vanish in this limit. If we consider the limiting case and thereby ignore such terms, the integral equation (6) becomes solvable by a two-fold differentiation with respect to ξ, to give the solution

$$\bar{\rho}^{(1)}(\xi) = N \exp(\xi - \xi_0)/[\exp(\xi - \xi_0) + 1]^2 = \tfrac{1}{4} N \operatorname{sech}^2(\tfrac{1}{2}|\xi - \xi_0|), \tag{7}$$

where ξ_0 is an arbitrary constant. This result justifies our assumption that $\bar{\rho}^{(1)}(\xi)$ takes nonzero values of order $O(N)$ only within a range of order $O(N^0)$ from an arbitrary fixed point (in this case ξ_0) as $N \to \infty$. Hence equation (7) for $\bar{\rho}^{(1)}(\xi)$ is exact for the system under consideration. In fact, if we take the position of the range centre ξ_0 to be the origin, we see that the exact solution has already been obtained

by Rybicki (1971). Also, Monaghan (1978) has noted that this solution satisfies the Debye–Hückel equation, i.e. equation (6) excluding the terms represented by the ellipses. Substitution of the expression (7) for $\bar{\rho}^{(1)}(\xi)$ into equation (5) yields

$$\beta F = N[\ln(N/L) - 2] + \ldots, \tag{8}$$

where the ellipses represent terms of order $o(N)$. This result is equivalent to that given by Rybicki (1971).

From the standpoint of the virial expansion, the pair distribution function $\bar{\rho}^{(2)}(\xi, \xi')$ may be expressed in terms of the individual one-particle distribution functions by (see e.g. equation (4.3) of Morita (1959) and equations (4.9), (4.13) and (4.14) of Morita and Hiroike (1961))

$$\bar{\rho}^{(2)}(\xi, \xi') = \bar{\rho}^{(1)}(\xi)\bar{\rho}^{(1)}(\xi')\exp(-N^{-1}|\xi - \xi'| + \ldots), \tag{9}$$

where the terms indicated by the ellipses are of order $O(N^{-1})$ and so are negligible. Thus, in the limit $N \to \infty$, equation (9) becomes, exactly,

$$\bar{\rho}^{(2)}(\xi, \xi') = \bar{\rho}^{(1)}(\xi)\bar{\rho}^{(1)}(\xi'), \tag{10}$$

a result conjectured by Monaghan (1978) from an approximate calculation.

Finally, we comment on this one-dimensional result in relation to the situation in higher dimensions. In the two- and three-dimensional cases, the gravitational potentials are respectively the logarithm and the inverse of the distance between two particles, and these potentials tend to $-\infty$ as the distance of separation is reduced. For the limiting state in which all particles collapse to a single point, the free energy of the classical system then is $-\infty$, and this system is not stable, as is well known (see e.g. Ruelle 1969).

References

Katsura, S. (1963). *Adv. Phys.* **12**, 391.
Monaghan, J. J. (1978). *Aust. J. Phys.* **31**, 95.
Morita, T. (1959). *Prog. Theor. Phys.* **21**, 501.
Morita, T., and Hiroike, K. (1961). *Prog. Theor. Phys.* **25**, 537.
Ruelle, D. (1969). 'Statistical Mechanics: Rigorous Results' (Benjamin: New York).
Rybicki, G. B. (1971). *Astrophys. Space Sci.* **14**, 56.

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