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# Existence and Stability of Circular Orbits in a Schwarzschild Field with Nonvanishing Cosmological Constant

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*Abstract*

The Schwarzschild metric with $\Lambda > 0$ allows circular orbits only within the annular region $3m < r_0 < (3m/\Lambda)^{\frac{1}{2}}$. For $m < \frac{1}{3}\Lambda^{-\frac{1}{2}}$ this region includes a narrower annular zone, throughout which circular orbits are stable but outside of which they are unstable.

The existence and stability of circular orbits in a Schwarzschild field in the absence of the cosmological constant $\Lambda$ have been considered by Darwin (1959) and Mielnik and Plebański (1962). However, there have been various suggestions that plausible cosmological models should include the cosmological constant (see e.g. Gunn and Tinsley 1975; Lake 1977), and with this in mind I examine here the effect of taking $\Lambda > 0$ upon the existence and stability of circular orbits.

In the presence of the cosmological constant, the metric describing the Schwarzschild field takes the form

$$\mathrm{d}s^2 = \chi c^2 \,\mathrm{d}t^2 - \chi^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\theta^2, \tag{1a}$$

where

$$\chi = 1 - 2mr^{-1} - \tfrac{1}{3}\Lambda r^2. \tag{1b}$$

The geodesic equations for equatorial orbits in this field yield

$$\mathrm{d}t/\mathrm{d}s = \chi^{-1}p, \qquad r^2\,\mathrm{d}\theta/\mathrm{d}s = h, \qquad 1 = \chi c^2 \dot{t}^2 - \chi^{-1}\dot{r}^2 - r^2\dot{\phi}^2, \tag{2}$$

where $p$ and $h$ are parameters related to the energy and angular momentum of an orbiting test particle (Felice 1968). These equations in turn yield the following relation governing equatorial orbits

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{m}{h^2} + 3mu^2 - \frac{\Lambda}{3h^2}u^{-3}, \tag{3}$$

where we have transformed the radial coordinate to $u = r^{-1}$. Equation (3) is identical with the relation given by Adler and Bazin (1975) apart from the presence of the term involving $\Lambda$.

We now consider circular orbits $u = u_0$, with a small radial perturbation of $\varepsilon$ on $u_0$. Equation (3) then becomes

$$\frac{d^2\varepsilon}{d\phi^2} + u_0 + \varepsilon = \frac{m}{h^2} + 3m(u_0+\varepsilon)^2 - \frac{\Lambda}{3h^2}(u_0+\varepsilon)^{-3}, \tag{4}$$

from which we obtain

$$d^2\varepsilon/d\phi^2 + \varepsilon(1 - 6mu_0 - \Lambda/h^2u_0^4) = 0, \tag{5}$$

with

$$h^2 = (3mu_0^3 - \Lambda)/3u_0^4(1-3mu_0), \tag{6}$$

having retained first-order perturbation terms only. For equation (6) to have a positive real root $h$, we obtain the following condition for the existence of circular orbits

$$3m < r_0 < r^*, \qquad \text{where} \qquad r^* = (3m/\Lambda)^{\frac{1}{3}}, \tag{7a, b}$$

bearing in mind that we require $\chi_0 > 0$. Note that the first of the inequalities (7a) agrees with Darwin (1959).

The stability of circular orbits in the region given by (7a) requires that (see equation 5)

$$1 - 6mu_0 - \Lambda/h^2u_0^4 > 0, \tag{8}$$

or

$$F(u_0) > 0, \tag{9a}$$

where

$$F(u_0) = -18m^2u_0^4 + 3mu_0^3 + 15mu_0\Lambda - 4\Lambda. \tag{9b}$$

Now $F(u_0) = 0$ has two positive roots $1/\alpha$ and $1/\beta$, with $\alpha > \beta$, and so circular orbits are stable in the region

$$\beta < r_0 < \alpha < r^*.$$

For $m < \frac{1}{3}\Lambda^{-\frac{1}{2}}$, we have $1/\alpha > u^*$ (where $u^*$ is the reciprocal of $r^*$) because $F(u^*) < 0$. Thus, for a given $m$ and a postulated $\Lambda$, it is possible to calculate $\alpha$ and $\beta$ by numerical methods. Finally, if $\Lambda$ is allowed to approach zero, we recover the results of Darwin (1959) for the Schwarzschild field in the absence of the cosmological constant: namely, that the region $3m < r_0 < \infty$ permits circular orbits and that the region $6m < r_0 < \infty$ permits stable circular orbits.

## References

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