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Bounds for the Centre-of-mass-corrected Energy-weighted Sum Rules for Nuclear Form Factors and Multipole Moments

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Abstract

Energy-weighted angular-momentum-projected sum rules for form factors and multipole moments of arbitrary states are corrected for centre-of-mass motion. Cumbersome expressions are obtained for the general case, but comparatively simple bounds have been obtained for particular cases. The results show that the corrections can only be important for light nuclei or at large momentum transfers.

1. Introduction

Energy-weighted sum rules (EWSR) have proven very useful in nuclear physics. Transition form factors, transition charge densities, static quadrupole moments and other nuclear properties have been calculated (Kao and Fallieros 1970; Deal and Fallieros 1973; Ui and Tsukamoto 1974; Reinhard and Drechsel 1975; Tassie 1975; Koo and Tassie 1976, 1978, 1979; Noble 1978; Koo 1979), and good agreement has been obtained with experiment. In addition the application of the EWSR to the nucleus has given an understanding of the hydrodynamical model (Tassie 1956), as indicated by the work of Deal and Fallieros (1973) and Suzuki and Rowe (1976), and has also yielded (in the appropriate limits) matrix element relations of the harmonic vibrational model (Koo 1978). However, most of these calculations have been done for the laboratory frame, where the centre-of-mass (c.m.) motion of the nucleus will affect the results. Accordingly the effects of the c.m. motion on the EWSR must be appropriately calculated since only intrinsic excitations of the nucleus are of interest.

The problem of spurious excitations of the nucleus due to the neglect or improper treatment of the c.m. motion is well known, and one obvious way of avoiding such spurious contributions to the EWSR is to specify all the relevant transition operators in the rest frame (or the c.m. frame) of the nucleus. In the present paper we discuss the effects of the c.m. motion on the EWSR, giving examples for specific sum rules wherever possible. We use the method of Deal (1972) and extend it to apply to the generalized angular-momentum-projected sum rules for arbitrary multipolarities and transitions between arbitrary states (Reinhard and Drechsel 1975; Koo 1978; Koo and Tassie 1979).

It should be noted that non-exchange forces which are independent of momentum or are at most linear in momentum dependence, such as the first-order spin–orbit force, do not contribute to the EWSR. However, more-complicated forces can contribute to the EWSR; in particular, exchange forces make important contributions to the isovector EWSR but do not contribute to the isoscalar EWSR. For the above reasons, the treatment of c.m. corrections to the EWSR (neglecting consideration of the nuclear forces) has many applications. In the present paper, for simplicity, we completely neglect consideration of the nuclear forces.

To make the paper reasonably complete and self contained, we include in Section 2 a brief derivation of the EWSR in the laboratory frame of reference, pointing out the assumptions and definitions used. At the same time we state formulae that will be useful for comparisons with the c.m.-corrected EWSR of Section 3. Maximal and minimal bounds are derived in Section 4, where some specific examples are given for a brief appraisal of the importance of the c.m. corrections. Model-independent and exact evaluations of the c.m. correction to some specific EWSR are discussed, and their consequences are briefly mentioned in Section 5. This paper is not intended as a complete and final treatment of c.m. corrections, but is mainly intended to provide a justification for their neglect in our other papers (Koo and Tassie 1978, 1979; Koo 1979).

2. Laboratory Frame Sum Rules

Consider a system of A particles and operators $G(\alpha)$ that can be described by sums of single-particle operators. Let

$$G(\alpha) = \sum_{i=1}^{A} g_\alpha(r_i) . \tag{1}$$

From the well-known commutation relation

$$[p_{j\mu}, x_{k\nu}] = -\mathrm{i}\hbar\delta_{kj}\,\delta_{\mu\nu}, \tag{2}$$

where k and j are the particle labels, and μ and ν are the component labels of the coordinates, we obtain, by neglecting the potential energy operator of the system,

$$[G(\alpha), [H, G(\beta)]] = (\hbar^2/m) \sum_i \nabla_i g_\alpha(r_i) \cdot \nabla_i g_\beta(r_i) . \tag{3}$$

Linear EWSR can now be obtained in a standard manner by taking the matrix element of equation (3) between two eigenstates of the Hamiltonian and then inserting a complete set of states $|n\rangle$ within the left-hand side.

The form factor operator is

$$F(q) = \sum_k \varepsilon_k \exp(-\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}_k), \tag{4}$$

where $\varepsilon_k = \frac{1}{2}$ for an isoscalar operator, $\varepsilon_k = \frac{1}{2}\tau_{k3}$ for an isovector operator and $\varepsilon_k = \frac{1}{2}(1+\tau_{k3})$ for a charged operator. Replacing $G(\alpha)$ and $G(\beta)$ in equation (3) by the form factor operator yields the relation

$$[F^\dagger(q_1), [H, F(q_2)]] = (\hbar^2/m)\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 \sum_k \varepsilon_k^2 \exp(-\mathrm{i}(\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot \boldsymbol{r}_k) . \tag{5}$$

Taking the matrix element of equation (5) between the state $|i\rangle$ and inserting a complete set of states $|n\rangle$ within the left-hand side yields the progenitor sum rule (Noble 1971)

$$\sum_n \omega_{ni}\langle n|F(q_1)|i\rangle^*\langle n|F(q_2)|i\rangle = (\hbar^2/2m)\,\boldsymbol{q}_1\cdot\boldsymbol{q}_2\left\langle i\left|\sum_k \varepsilon_k^2\exp(-\mathrm{i}(\boldsymbol{q}_2-\boldsymbol{q}_1)\cdot\boldsymbol{r}_k)\right|i\right\rangle, \tag{6}$$

where $\omega_{ni} = E_n - E_i$ is the excitation energy from state $|i\rangle$ to state $|n\rangle$.

The generalized angular-momentum-projected EWSR for multipole operators may be derived by coupling the standard double commutator of the Hamiltonian H and two multipole operators to overall angular momentum L, and then evaluating it between two states of spins J_i and J_f. Thus for the multipole form factor operator

$$F^l_m(q) = \sum_k \varepsilon_k\, j_l(qr_k)\,\mathrm{Y}_{lm}(\Omega_k) \tag{7}$$

we define the coupling scheme

$$[F^{l_1}(q_1),[H,F^{l_2}(q_2)]]^L_M = \sum_{m_1m_2}(l_1\,m_1\,l_2\,m_2\,|\,LM)[F^{l_1}_{m_1}(q_1),[H,F^{l_2}_{m_2}(q_2)]], \tag{8}$$

so that by doing the necessary angular-momentum-recoupling algebra, the generalized multipole form factor sum rule is (Reinhard and Drechsel 1975; Koo and Tassie 1979)

$$\hat{L}\sum_n\left[\omega_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}F^{l_1}_{fn}(q_1)\,F^{l_2}_{ni}(q_2) + (-)^{l_1-l_2+L}\omega_{nf}\begin{Bmatrix}J_f & J_i & L\\ l_1 & l_2 & J_n\end{Bmatrix}F^{l_2}_{fn}(q_2)\,F^{l_1}_{ni}(q_1)\right]$$
$$= (-)^{J_i+J_f}\,\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}\int r^2\,\mathrm{d}r\,\rho^L_{fi}(r)\,2\left\{\frac{\mathrm{d}j_{l_1}(q_1r)}{\mathrm{d}r}\frac{\mathrm{d}j_{l_2}(q_2r)}{\mathrm{d}r}\right.$$
$$\left.+r^{-2}\{l_1(l_1+1)+l_2(l_2+1)-L(L+1)\}\,j_{l_1}(q_1r)\,j_{l_2}(q_2r)\right\}, \tag{9}$$

where $\hat{L} = (2L+1)^{\frac{1}{2}}$. The parameter ξ is a number depending on the isospin characters of the two multipole operators used: when both operators are isoscalar or isovector, $\xi = \frac{1}{4}$ and $\rho^L_{fi}(r)$ is the isoscalar transition density; when both the operators are charged operators, $\xi = 1$ and $\rho^L_{fi}(r)$ is the transition charge density. The reduced matrix element is defined by (Edmonds 1968)

$$\langle J_f M_f|F^l_m(q)|J_i M_i\rangle = (-)^{J_f-M_f}\begin{pmatrix}J_f & l & J_i\\ -M_f & m & M_i\end{pmatrix}F^l_{fi}(q). \tag{10}$$

Other generalized multipole sum rules may be obtained in the same way, but the multipole moment and multipole density sum rules may be more easily obtained from equation (9) by equating terms of same powers of q_1 or q_2 and by taking the appropriate Fourier–Bessel transform. The results are

$$\hat{L}\sum_n\left[\omega_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}Q_{fn}^{l_1\alpha}F_{ni}^{l_2}(q_2)+(-)^{l_1-l_2+L}\omega_{nf}\begin{Bmatrix}J_f & J_i & L\\ l_1 & l_2 & J_n\end{Bmatrix}F_{fn}^{l_2}(q_2)\,Q_{ni}^{l_1\alpha}\right]$$

$$=(-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}\int r^{l_1+2\alpha}\rho_{fi}^L(r)\,\mathrm{d}r$$

$$\times\{l_1(l_1+1)+l_2(l_2+1)-L(L+1)+2(l_1+2\alpha)r\,\mathrm{d}/\mathrm{d}r\}\,j_{l_2}(q_2r), \tag{11}$$

where

$$Q_m^{l\alpha}=\sum_k\varepsilon_k r_k{}^{l+2\alpha}\,\mathrm{Y}_{lm}(\Omega_k)$$

is the generalized multipole moment operator;

$$\hat{L}\sum_n\left[\omega_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}Q_{fn}^{l_1\alpha}Q_{ni}^{l_2\beta}+(-)^{l_1-l_2+L}\omega_{nf}\begin{Bmatrix}J_f & J_i & L\\ l_1 & l_2 & J_n\end{Bmatrix}Q_{fn}^{l_2\beta}\,Q_{ni}^{l_1\alpha}\right]$$

$$=(-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}$$

$$\times\{2(l_1+2\alpha)(l_2+2\beta)+l_1(l_1+1)+l_2(l_2+1)-L(L+1)\}\int r^{l_1+l_2+2\alpha+2\beta}\rho_{fi}^L(r)\,\mathrm{d}r; \tag{12}$$

$$\hat{L}\sum_n\left[\omega_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}Q_{fn}^{l_1\alpha}\rho_{ni}^{l_2}(r)+(-)^{l_1-l_2+L}\omega_{nf}\begin{Bmatrix}J_f & J_i & L\\ l_1 & l_2 & J_n\end{Bmatrix}\rho_{fn}^{l_2}(r)\,Q_{ni}^{l_1\alpha}\right]$$

$$=(-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}r^{l_1+2\alpha-2}$$

$$\times\{4\alpha(2l_1+2\alpha+1)+l_1(l_1+1)-l_2(l_2+1)+L(L+1)+2(l_1+2\alpha)\,r\,\mathrm{d}/\mathrm{d}r\}\rho_{fi}^L(r). \tag{13}$$

3. EWSR Corrected for CM Motion

Because it is necessary to sum over nuclear states of spin J_n, where J_n is the total angular momentum in the rest frame $\tilde{S}$ of the nucleus, the sum rules must be obtained in the frame $\tilde{S}$. In the frame $\tilde{S}$, the nucleons are described by the intrinsic coordinates

$$\tilde{\boldsymbol{r}}_i=\boldsymbol{r}_i-\boldsymbol{R},\qquad \tilde{\boldsymbol{p}}_i=\boldsymbol{p}_i-A^{-1}\boldsymbol{P}, \tag{14}$$

satisfying

$$\sum_{i=1}^{A}\tilde{\boldsymbol{r}}_i=0,\qquad \sum_{i=1}^{A}\tilde{\boldsymbol{p}}_i=0, \tag{15}$$

where nucleonic coordinates without tilde accents are defined in the laboratory frame S, while

$$\boldsymbol{R}=A^{-1}\sum_{i=1}^{A}\boldsymbol{r}_i\qquad\text{and}\qquad \boldsymbol{P}=\sum_{i=1}^{A}\boldsymbol{p}_i \tag{16}$$

are respectively the position and the total momentum of the centre of mass of the nucleus in S.

Since the nucleonic coordinates in $\tilde{S}$ are not independent, the commutation relation corresponding to equation (2) is modified. By using equations (2), (14) and (16), the commutation relations for the intrinsic coordinates are (Deal 1972)

$$[\tilde{p}_{k\mu}, \tilde{r}_{j\nu}] = -\mathrm{i}\hbar\delta_{\mu\nu}[\delta_{kj} - A^{-1}], \tag{17}$$

which has a term that depends on two-particle correlation. The double commutator for the intrinsic coordinates corresponding to equation (3) then becomes

$$\sum_{ijk} [\tilde{g}_\alpha(\tilde{r}_i), [\tilde{\nabla}_j^2, \tilde{g}_\beta(\tilde{r}_k)]] = -2 \sum_{ik} \tilde{\nabla}_i \tilde{g}_\alpha(\tilde{r}_i) \cdot \tilde{\nabla}_k g_\beta(\tilde{r}_k) [\delta_{ik} - A^{-1}], \tag{18}$$

where the operators with tilde accents are defined in the frame $\tilde{S}$.

For the form factor operator defined by equation (4), we have (using equation 18)

$$\begin{aligned}
&[\tilde{F}^\dagger(q_1), [\tilde{H}, \tilde{F}(q_2)]] \\
&\quad = (\hbar^2/m)\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 \sum_{jk} \varepsilon_j \varepsilon_k \exp(-\mathrm{i}\boldsymbol{q}_2 \cdot \tilde{\boldsymbol{r}}_j + \mathrm{i}\boldsymbol{q}_1 \cdot \tilde{\boldsymbol{r}}_k) [\delta_{jk} - A^{-1}] \\
&\quad = (\hbar^2/m)\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 \left(\sum_k \varepsilon_k^2 \exp(-\mathrm{i}(\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot \boldsymbol{r}_k) - A^{-1} \tilde{F}^\dagger(q_1) \tilde{F}(q_2) \right).
\end{aligned} \tag{19}$$

Taking the expectation value of equation (19) between the state $|i\rangle$ we obtain the progenitor sum rule of Noble (1971) corrected for c.m. motion:

$$\begin{aligned}
&\sum_n \{\tilde{\omega}_{ni} + (\hbar^2/2mA)\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2\} \langle n | \tilde{F}(q_1) | i \rangle^* \langle n | \tilde{F}(q_2) | i \rangle \\
&\qquad = (\hbar^2/2m)\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 \left\langle i \middle| \sum_k \varepsilon_k^2 \exp(-\mathrm{i}(\boldsymbol{q}_2 - \boldsymbol{q}_1) \cdot \boldsymbol{r}_k) \middle| i \right\rangle
\end{aligned} \tag{20}$$

which, when compared with equation (6), acquires a correction term dependent upon the momentum transfers and A.

Equation (20) yields the result of Deal (1972) when $|i\rangle$ is the ground state $|0\rangle$ with spin zero. By choosing $q_1 = q_2$ we obtain

$$\begin{aligned}
\sum_n \{\tilde{\omega}_{n0} + (\hbar^2/2mA) q^2\} \langle n | \tilde{F}(q) | i \rangle^* \langle n | \tilde{F}(q) | i \rangle &= (\hbar^2/2m) q^2 \left\langle 0 \middle| \sum_k \varepsilon_k^2 \middle| 0 \right\rangle \\
&= (\hbar^2/2m) q^2 B,
\end{aligned} \tag{21}$$

where $B = Z$ when $\varepsilon_k = \frac{1}{2}(1 + \tau_{k3})$, and $B = \frac{1}{4}A$ when $\varepsilon_k = \frac{1}{2}$ or $\frac{1}{2}\tau_{k3}$. Note that $\{\tilde{\omega}_{n0} + (\hbar^2/2mA)q^2\}$ is the total energy of the recoiling nucleus. Examining equation (21), we can see that the contribution of the c.m. correction term will be appreciable for large momentum transfers.

For the multipole form factor EWSR corrected for the c.m. motion, we make use of equations (4), (8) and (18) to obtain

$$\begin{aligned}
&[\tilde{F}^{l_1}(q_1), [\tilde{H}, \tilde{F}^{l_2}(q_2)]]_M^L \\
&= \sum_{m_1 m_2} \sum_{jk} (l_1 m_1 l_2 m_2 | L M)\, \tilde{\nabla}_j j_{l_1}(q_1 \tilde{r}_j) \mathrm{Y}_{l_1 m_1}(\tilde{\Omega}_j) \cdot \tilde{\nabla}_k j_{l_2}(q_2 \tilde{r}_k) \mathrm{Y}_{l_2 m_2}(\tilde{\Omega}_k) [\delta_{jk} - A^{-1}].
\end{aligned} \tag{22}$$

In a way analogous to that of obtaining equation (9), we get

$$\hat{L}\sum_{n}\left[\tilde{\omega}_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}\tilde{F}^{l_1}_{fn}(q_1)\,\tilde{F}^{l_2}_{ni}(q_2)+(-)^{l_1-l_2+L}\tilde{\omega}_{nf}\begin{Bmatrix}J_f & J_i & L_n\\ l_1 & l_2 & J_n\end{Bmatrix}\tilde{F}^{l_2}_{fn}(q_2)\,\tilde{F}^{l_1}_{ni}(q_1)\right]$$

$$+(-)^{J_i+J_f}(\hbar^2/mA)\,q_1 q_2\sum_{J_1J_2}(-)^{-J_1-l_2}\,\Phi_{J_1l_1}\Phi_{J_2l_2}$$

$$\times\begin{Bmatrix}J_f & J_i & L\\ J_2 & J_1 & J_n\end{Bmatrix}\begin{Bmatrix}J_2 & J_1 & L\\ l_1 & l_2 & 1\end{Bmatrix}\tilde{F}^{J_1}_{fn}(q_1)\,\tilde{F}^{J_2}_{ni}(q_2)$$

$$=(-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\,\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}\int\tilde{r}^2\,\mathrm{d}\tilde{r}\,\tilde{\rho}^L_{fi}(\tilde{r})\,2\left\{\frac{\mathrm{d}j_{l_1}(q_1\tilde{r})}{\mathrm{d}\tilde{r}}\frac{\mathrm{d}j_{l_2}(q_2\tilde{r})}{\mathrm{d}\tilde{r}}\right.$$

$$\left.+\tilde{r}^{-2}\{l_1(l_1+1)+l_2(l_2+1)-L(L+1)\}j_{l_1}(q_1\tilde{r})j_{l_2}(q_2\tilde{r})\right\},\qquad(23)$$

where

$$\Phi_{Jl}=\{\tfrac{1}{2}(J+l+1)\}^{\frac{1}{2}}(\delta_{Jl+1}+\delta_{Jl-1}).$$

Equation (23) should be compared with equation (9), and again an additional term is obtained. This term will contribute significantly to the multipole form factor sum rules at large q. Since $J_1=l_1\pm 1$ and $J_2=l_2\pm 1$, the multipolarities of the form factors in the correction term are either one unit greater or less than the corresponding operators in the principal sum and so, for any given $|f\rangle$ and $|i\rangle$, the intermediate states involved in the correction term are different from that of the principal sum.

Other multipole c.m.-corrected sum rules involving the generalized multipole moment $Q^{l\alpha}_m$ and the multipole density can be directly derived from equation (23). By noting that

$$j_l(qr)=\sum_{\alpha=0}^{\infty}(-)^{\alpha}(qr)^{l+2\alpha}\{2^{\alpha}\alpha!\,(2l+2\alpha+1)!!\}^{-1},\qquad(24)$$

$$F^l_m(q)=\sum_{\alpha=0}^{\infty}(-)^{\alpha}q^{l+2\alpha}\{2^{\alpha}\alpha!\,(2l+2\alpha+1)!!\}^{-1}\,Q^{l\alpha}_m,\qquad(25)$$

we can, by repeated substitution of equations (24) and (25) into equation (23), followed by equating coefficients of terms with equal powers of q_1 and q_2, obtain a mixed multipole moment–multipole form factor EWSR and a pure multipole moment EWSR.

Substituting once, and then equating coefficients of terms with the same power of q_1, we obtain

$$\hat{L}\sum_{n}\left[\tilde{\omega}_{ni}\begin{Bmatrix}J_f & J_i & L\\ l_2 & l_1 & J_n\end{Bmatrix}\tilde{Q}^{l_1\alpha}_{fn}\,\tilde{F}^{l_2}_{ni}(q_2)+(-)^{l_1-l_2+L}\tilde{\omega}_{nf}\begin{Bmatrix}J_f & J_i & L\\ l_1 & l_2 & J_n\end{Bmatrix}\tilde{F}^{l_2}_{fn}(q_2)\,\tilde{Q}^{l_1\alpha}_{ni}\right]$$

$$+(\hbar^2/mA)q_2\sum_{J_1J_2}(-)^{J_i+J_f-J_1-l_2}\,\Phi_{J_1l_1}\Phi_{J_2l_2}\begin{Bmatrix}J_f & J_i & L\\ J_2 & J_1 & J_n\end{Bmatrix}\begin{Bmatrix}J_2 & J_1 & L\\ l_1 & l_2 & 1\end{Bmatrix}\tilde{F}^{J_2}_{fn}(q_2)\,\tilde{Q}^{J_1\,\phi(\alpha)}_{ni}$$

$$=(-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\,\hat{l}_1\,\hat{l}_2\begin{pmatrix}L & l_2 & l_1\\ 0 & 0 & 0\end{pmatrix}\int\tilde{r}^{l_1+2\alpha}\,\tilde{\rho}^L_{fi}(\tilde{r})\,\mathrm{d}\tilde{r}$$

$$\times\{l_1(l_1+1)+l_2(l_2+1)-L(L+1)+2(l_1+2\alpha)\tilde{r}\,\mathrm{d}/\mathrm{d}\tilde{r}\}j_{l_2}(q_2\tilde{r}),\qquad(26)$$

where

$$\tilde{Q}_{ni}^{J_1\,\phi(\alpha)} = (2l_1+2\alpha+1)\tilde{Q}_{ni}^{l_1-1\,\alpha} \quad \text{for} \quad J_1 = l_1-1$$
$$= -2\alpha\tilde{Q}_{ni}^{l_1+1\,\alpha-1} \quad J_1 = l_1+1 .$$

Equation (26) should be compared with equation (11), and large contribution of the c.m. correction term is expected at large q. Substituting equations (24) and (25) into equation (26) and similarly equating the coefficients of terms with the same power of q_2, we obtain the c.m.-corrected generalized pure multipole moment EWSR

$$\hat{L}\sum\left[\tilde{\omega}_{ni}\begin{Bmatrix} J_f & J_i & L \\ l_2 & l_1 & J_n \end{Bmatrix}\tilde{Q}_{fn}^{l_1\alpha}\tilde{Q}_{ni}^{l_2\beta} + (-)^{l_1-l_2+L}\tilde{\omega}_{nf}\begin{Bmatrix} J_f & J_i & L \\ l_1 & l_2 & J_n \end{Bmatrix}\tilde{Q}_{fn}^{l_2\beta}\tilde{Q}_{ni}^{l_1\alpha}\right]$$
$$+(\hbar^2/mA)\sum_{J_1J_2}(-)^{J_i+J_f-J_1-l_2}\Phi_{J_1l_1}\Phi_{J_2l_2}\begin{Bmatrix} J_f & J_i & L \\ J_2 & J_1 & J_n \end{Bmatrix}\begin{Bmatrix} J_2 & J_1 & L \\ l_1 & l_2 & 1 \end{Bmatrix}\tilde{Q}_{fn}^{J_2\,\phi(\beta)}\tilde{Q}_{ni}^{J_1\,\phi(\alpha)}$$
$$= (-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\hat{l}_2\begin{pmatrix} L & l_2 & l_1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times\{2(l_1+2\alpha)(l_2+2\beta)+l_1(l_1+1)+l_2(l_2+1)-L(L+1)\}\int\tilde{r}^{l_1+l_2+2\alpha+2\beta}\tilde{\rho}_{fi}^{L}(\tilde{r})\,\mathrm{d}\tilde{r}. \tag{27}$$

Comparison with equation (12) shows that the contribution of the c.m. motion correction is expected to be significant for light nuclei. For $\alpha = \beta = 0$, and choosing $|J_i\rangle = |J_f\rangle = |0\rangle$, we obtain a result equivalent to that of Deal (1972).

Assuming that the sum rule (26) is uniformly convergent, and using the transformation

$$\rho_{fn}^{l}(r) = (2/\pi)\int F_{fn}^{l}(q)j_l(qr)q^2\,\mathrm{d}q, \tag{28}$$

we obtain the c.m.-corrected density sum rule:

$$\hat{L}\sum\left[\tilde{\omega}_{ni}\begin{Bmatrix} J_f & J_i & L \\ l_2 & l_1 & J_n \end{Bmatrix}\tilde{Q}_{fn}^{l_1\alpha}\tilde{\rho}_{ni}^{l_2}(\tilde{r}) + (-)^{l_1-l_2+L}\tilde{\omega}_{nf}\begin{Bmatrix} J_f & J_i & L \\ l_1 & l_2 & J_n \end{Bmatrix}\tilde{\rho}_{fn}^{l_2}(\tilde{r})\tilde{Q}_{ni}^{l_1\alpha}\right]$$
$$+(\hbar^2/mA)\sum_{J_1J_2}(-)^{J_i+J_f-J_1-l_2}\Phi_{J_1l_1}\Phi_{J_2l_2}\begin{Bmatrix} J_f & J_i & L \\ J_2 & J_1 & J_n \end{Bmatrix}\begin{Bmatrix} J_2 & J_1 & L \\ l_1 & l_2 & 1 \end{Bmatrix}\tilde{r}^{-1}\tilde{\rho}_{fn}^{J_2}(\tilde{r})\tilde{Q}_{ni}^{J_1\,\phi(\alpha)}$$
$$= (-)^{J_i+J_f}\xi(\hbar^2/2m)(4\pi)^{-\frac{1}{2}}\hat{l}_1\hat{l}_2\begin{pmatrix} L & l_2 & l_1 \\ 0 & 0 & 0 \end{pmatrix}\tilde{r}^{l_1+2\alpha-2}$$
$$\times\{4\alpha(2l_1+2\alpha+1)+l_1(l_1+1)-l_2(l_2+1)+L(L+1)+2(l_1+2\alpha)\tilde{r}\,\mathrm{d}/\mathrm{d}\tilde{r}\}\,\tilde{\rho}_{fi}^{L}(\tilde{r}), \tag{29}$$

where

$$\tilde{\rho}_{fn}^{J_2}(\tilde{r}) = \{(l_2-1)+\tilde{r}\,\mathrm{d}/\mathrm{d}\tilde{r}\}\,\tilde{\rho}_{fn}^{l_2-1}(\tilde{r}) \quad \text{for} \quad J_2 = l_2-1$$
$$= \{(l_2+1)+(1+\tilde{r}\,\mathrm{d}/\mathrm{d}\tilde{r})\}\,\tilde{\rho}_{fn}^{l_2+1}(\tilde{r}) \quad J_2 = l_2+1 .$$

Equation (29) should be compared with the uncorrected EWSR equation (13). The factor of $\tilde{r}^{-1}$ in the correction term on the left-hand side of equation (29) underlies the importance of this term for regions near the centre of the nucleus, where the most difficult problems and ambiguities are encountered as to the behaviour of the charge densities.

4. Bounds on the CM-corrected Sum Rules

Although the c.m. correction term in the sum rules cannot be evaluated explicitly in general, its contribution can be better appreciated by establishing bounds for the sum rules. Qualitatively its contributions are expected to be large at large q and for small A. Since the m dependence of the multipole operators in equation (7) is purely geometrical, we can choose the subscript m to be zero in the appropriate quantization axis. We also restrict our discussion to even–even nuclei and consider the case $\alpha = \beta$, in which the states $|J_i\rangle$ and $|J_f\rangle$ are both taken to be the ground state $|0\rangle$ with zero spin.

(a) *Maximal Bounds*

For any operator Q we have

$$\langle s|Q^\dagger|n\rangle\langle n|Q|s\rangle \geqslant 0, \tag{30}$$

and this inequality leads to maximal bounds for certain sum rules. From equation (21), the usual progenitor sum rule value for $q_1 = q_2$ is a maximal bound:

$$\sum_n \tilde{\omega}_{n0}\langle n|\tilde{F}(q)|i\rangle^*\langle n|\tilde{F}(q)|i\rangle \leqslant (\hbar^2/2m)q^2 B. \tag{31}$$

Similarly a maximal bound for the multipole form factor sum rule for $q_1 = q_2$ and $|J_i\rangle = |J_f\rangle = |0\rangle$ is

$$\sum_n \tilde{\omega}_{n0}\langle n|\tilde{F}_0^l(q)|0\rangle^*\langle n|\tilde{F}_0^l(q)|0\rangle \leqslant \xi(\hbar^2q^2/2m)(2l+1)^{-1}\int \tilde{r}^2\tilde{\rho}(\tilde{r})\,\mathrm{d}\tilde{r}\sum_j \Phi_{Jl}^2\{j_l(q\tilde{r})\}^2, \tag{32}$$

and from equation (27), a maximal bound for the corresponding multipole moment sum rule is

$$\sum_n \tilde{\omega}_{n0}\langle n|Q_0^{l\alpha}|0\rangle^*\langle n|Q_0^{l\alpha}|0\rangle \leqslant (\hbar^2/8\pi m)\{(l+2\alpha)^2+l(l+1)\}\left\langle 0\left|\sum_i \varepsilon_i^2\tilde{r}_i^{2l+4\alpha-2}\right|0\right\rangle. \tag{33}$$

The usual sum rule values merely provide upper bounds for the sums of the energy-weighted transition strengths. The experimental observation of over exhaustions of the bounds would therefore imply the effects of velocity-dependent forces and of exchange forces which have been neglected.

(b) *Minimal Bounds*

For minimal bounds we note that

$$(\tilde{E}_{\min}^l)^{-1}\langle 0|\tilde{Q}^\dagger\tilde{H}\tilde{Q}|0\rangle \geqslant \langle 0|\tilde{Q}^\dagger\tilde{Q}|0\rangle, \tag{34}$$

where $\tilde{E}^l_{\min}$ is the energy of the lowest state of the Hamiltonian operating on the Hilbert space of multipolarity l, that is,

$$\tilde{H}|n\rangle = \tilde{E}^l_n|n\rangle, \qquad \tilde{H}|0\rangle = 0.$$

Substituting the inequality (34) into equation (27), with $\alpha = \beta$, $|J_i\rangle = |J_f\rangle = |0\rangle$ we get

$$\sum_n [\tilde{E}^l_n\langle n|\tilde{Q}^{l\alpha}_0|0\rangle^*\langle n|\tilde{Q}^{l\alpha}_0|0\rangle$$
$$+(\hbar^2/2mA)(2l+1)\{l(2l+2\alpha+1)^2(\tilde{E}^{l-1}_{\min})^{-1}\tilde{E}^{l-1}_n\langle n|\tilde{Q}^{l-1\,\alpha}_0|0\rangle^*\langle n|\tilde{Q}^{l-1\,\alpha}_0|0\rangle$$
$$+4\alpha^2(l+1)(\tilde{E}^{l+1}_{\min})^{-1}\tilde{E}^{l+1}_n\langle n|\tilde{Q}^{l+1\,\alpha-1}_0|0\rangle^*\langle n|\tilde{Q}^{l+1\,\alpha-1}_0|0\rangle\}]$$
$$\geqslant (\hbar^2/8\pi m)\{(l+2\alpha)^2+l(l+1)\}\left\langle 0\left|\sum_i \varepsilon_i^2\tilde{r}_i^{2l+4\alpha-2}\right|0\right\rangle. \tag{35}$$

Since the second and the third term on the left-hand side of equation (35) now have the structures of EWSR, we can substitute their maximal bounds and obtain

$$\sum_n \tilde{E}^l_n\langle n|\tilde{Q}^{l\alpha}_0|0\rangle^*\langle n|\tilde{Q}^{l\alpha}_0|0\rangle$$
$$\geqslant (\hbar^2/8\pi m)\Bigg(\{(l+2\alpha)^2+l(l+1)\}\left\langle 0\left|\sum_i \varepsilon_i^2\tilde{r}_i^{2l+4\alpha-2}\right|0\right\rangle$$
$$-(\hbar^2/2mA)(2l+1)^{-1}\left\langle 0\left|\sum_i \varepsilon_i^2\tilde{r}_i^{2l+4\alpha-4}\right|0\right\rangle$$
$$\times\big[l(2l+2\alpha+1)^2\{(l+2\alpha-1)^2+l(l-1)\}(\tilde{E}^{l-1}_{\min})^{-1}$$
$$+4\alpha^2(l+1)\{(l+2\alpha-1)^2+(l+1)(l+2)\}(\tilde{E}^{l+1}_{\min})^{-1}\big]\Bigg). \tag{36}$$

The true values of the 2^l-pole sum rules then lie somewhere between the maximal and the minimal bound. Similarly a minimal bound for the form factor sum rule is

$$\sum \tilde{\omega}_{n0}\langle n|\tilde{F}(q)|0\rangle^*\langle n|\tilde{F}(q)|0\rangle \geqslant (\hbar^2/2m)q^2B(1-\hbar^2q^2/2mA\tilde{E}_{\min}), \tag{37}$$

where $\tilde{E}_{\min}$ is the energy of the first-excited state of the nucleus. For the pure multipole form factor sum rule, a minimal bound is

$$\sum_n \tilde{E}^l_n\langle n|\tilde{F}^l_0(q)|0\rangle^*\langle n|\tilde{F}^l_0(q)|0\rangle$$
$$\geqslant \xi(\hbar^2/2m)(2l+1)^{-1}q^2\int \tilde{r}^2\,\mathrm{d}\tilde{r}\,\tilde{\rho}(\tilde{r})$$
$$\times\sum_J \Phi^2_{Jl}\left(\{j_J(q\tilde{r})\}^2-\frac{\hbar^2}{2mA}\frac{q^2}{2J+1}\frac{1}{\tilde{E}^J_{\min}}\sum_{J'}\Phi^2_{J'J}\{j_{J'}(q\tilde{r})\}^2\right). \tag{38}$$

For illustration, we explicitly consider E2 transitions for inequality (36) which becomes, with $\alpha = 0$ and $\varepsilon_i = \frac{1}{2}(1+\tau_{i3})$,

$$\sum_n \tilde{\omega}_{n0}|\langle n|\tilde{Q}^{20}_0|0\rangle|^2 \geqslant (10\hbar^2/8\pi m)Ze\{\langle\tilde{r}^2\rangle - 62\cdot 3(A\tilde{E}^1_{\min})^{-1}\}, \tag{39}$$

where $\langle\tilde{r}^2\rangle$ is the mean square charge radius of the ground state and $\tilde{E}^1_{\rm min}$ is the energy of the lowest 1^- state. Similarly inequality (38), for E2 transitions, becomes

$$\sum_n \tilde{\omega}_{n0}\,|\langle n|\tilde{F}^2_0(q)|0\rangle|^2$$

$$\geqslant (\hbar^2/10m)q^2\int \tilde{r}^2\,\mathrm{d}\tilde{r}\,\tilde{\rho}(\tilde{r})\{3\{j_3(q\tilde{r})\}^2+2\{j_1(q\tilde{r})\}^2$$

$$-(\hbar^2/2mA)q^2\big((7\tilde{E}^3_{\rm min})^{-1}[4\{j_4(q\tilde{r})^2+3(j_2(q\tilde{r})\}^2]$$

$$-(3\tilde{E}^1_{\rm min})^{-1}[2\{j_2(q\tilde{r})\}^2+\{j_0(q\tilde{r})\}^2]\big)\}, \quad (40)$$

where $\tilde{E}^3_{\rm min}$ is the energy of the lowest 3^- state. Corrections for other multipoles can be similarly derived.

For a particular nucleus, the bounds imposed can be evaluated given the charge density and the energy levels.

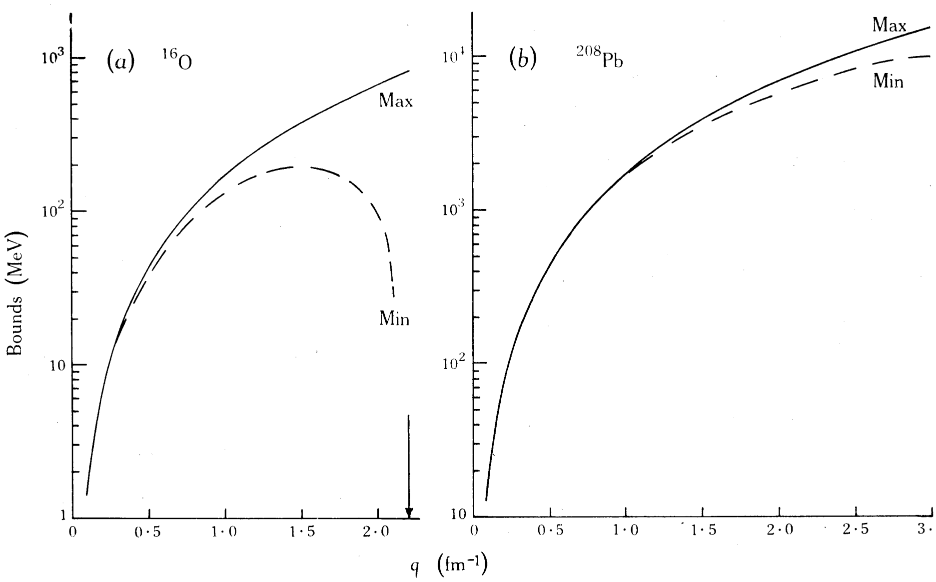


Fig. 1. Maximal and minimal bounds imposed by the c.m. motion of the nucleus for the form factor sum rule

$$\sum_n \tilde{E}_n|\langle n|\tilde{F}(q)|0\rangle|^2$$

for (*a*) ^{16}O and (*b*) ^{208}Pb.

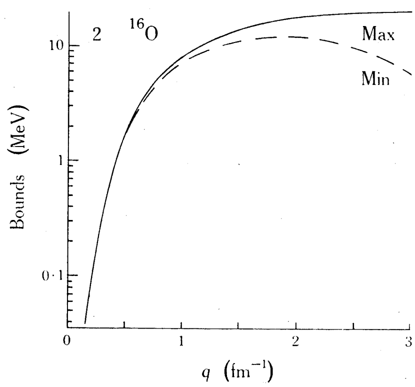
Figs 1*a* and 1*b* display the maximal and minimal bounds for the progenitor sum rule (equations 31 and 39) and show that the effect of the c.m. correction is practically negligible at low momentum transfers ($<0\cdot5$ fm^{-1} for ^{16}O and $<1\cdot0$ fm^{-1} for ^{208}Pb), but may become significant at larger momentum transfers. The effect of the c.m. correction is obviously less important for ^{208}Pb than for ^{16}O.

Fig 2 shows the bounds for the quadrupole form factor sum rule (equations 32 and 40) of ^{16}O as a function of q. A similar conclusion to that of the progenitor sum rule can be drawn. Although not shown here, the c.m. effect becomes less significant as A increases.

Fig. 3 shows the bounds for the quadrupole moment sum rule as a function of A. The difference between the maximal and minimal bounds is more than 10% for nuclei with $A < 15$, while for nuclei with $A > 15$ the difference is less significant.

5. Specific CM-corrected Sum Rules

In certain cases, the c.m. correction term can be evaluated explicitly without any model assumption, and exact sum rules are then obtained. We discuss some of these cases below.



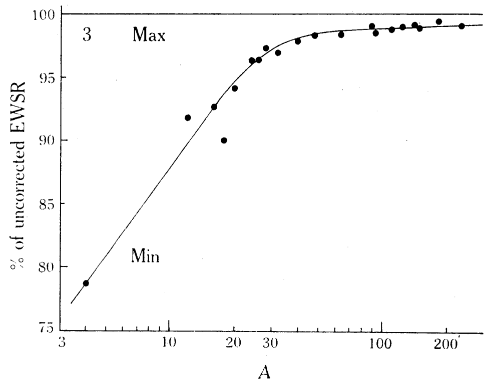


Fig. 2 (*left*). Maximal and minimal bounds imposed by the c.m. motion of the nucleus for the quadrupole form factor sum rule

$$\sum_n \tilde{E}_n |\langle n | \tilde{F}_0^2(q) | 0\rangle|^2$$

for ^{16}O. The charge distribution used is obtained from Sick and McCarthy (1970).

Fig. 3 (*right*). Maximal and minimal bounds imposed by the c.m. motion of the nucleus for the quadrupole moment sum rule

$$\sum_n \tilde{E}_n |\langle n | \tilde{Q}_0^{20} | 0\rangle|^2 .$$

The points are calculated using experimentally determined energy levels and mean square radii. The line depicting the minimal bound is merely to guide the eye.

The usual nuclear analogue of the isoscalar Thomas–Reiche–Kuhn sum rule has a nonzero sum rule value, and is hence inconsistent with the result that the $\alpha = 0$ isoscalar dipole operator is zero. The resolution, however, lies in the proper treatment of the c.m. motion, and it can be easily checked, using equation (35) with $\alpha = 0$ and $l = 1$, that the isoscalar dipole sum rule is zero (Deal 1972, 1973). The corresponding isovector sum rule has a correction. It can also be easily shown that, using equation (35) with $\alpha = 0$ and $l = 1$, the isovector dipole sum rule is

$$\sum_n \tilde{\omega}_{n0} \langle n | \tilde{Q}_0^{10} | 0\rangle^* \langle n | \tilde{Q}_0^{10} | 0\rangle = \frac{\frac{3}{4}\hbar^2}{8\pi m}\frac{4NZ}{A},$$

which is a well-known result. The corresponding EWSR uncorrected for c.m. motion has the factor A in the place of $4NZ/A$, and this implies in general a small correction as $4NZ/A \approx (0{\cdot}95 \rightarrow 1{\cdot}0)A$.

For nonvanishing isoscalar dipole transitions, the null first-order ($\alpha = 0$) dipole operator implies the effect of higher-order ($\alpha = 1, 2, \ldots$) dipole operators, and these may be confused with transitions of higher multipolarities (Koo and Tassie 1978).

For instance, in a recent electron scattering study of ^{197}Au (Torizuka 1978), the isoscalar E3 giant resonance at $\sim$20 MeV excitation contributed $(147 \pm 21)\%$ to the isoscalar E3 EWSR, and this excess E3 strength might be an example of such a higher E1 contribution. It can also be shown easily, using equation (27), that the $\alpha = 1$ isoscalar monopole and the $\alpha = 0$ isoscalar quadrupole moment sum rules are the same, with or without the c.m. correction.

6. Discussion

We have obtained the c.m.-corrected generalized angular-momentum-projected EWSR for transitions between arbitrary states. The results are cumbersome, but comparatively simple bounds have been established for three of the sum rules. Two important points are to be noted: for heavy nuclei, as expected, the required c.m. correction is small; the c.m. correction can become progressively more important as the momentum transferred to the nucleus becomes larger, as seen in Figs 1 and 2. Although similar bounds cannot be obtained for other sum rules, we should expect similar behaviour. Thus we may with confidence neglect the c.m. correction in the calculation of nuclear structure (Koo and Tassie 1979, Koo 1979) at low momentum transfers and for $A > 30$.

It is possible that there are large c.m. corrections to form factor sum rules at large momentum transfer. However, it must be remembered that at very large momentum transfers our treatment (based on the usual assumptions of nuclear theory, namely the description of the nucleus as composed of neutrons and protons described by the Schrödinger equation) is not applicable, as the experimental inelastic electron scattering is dominated by meson production. To deal with this problem, a relativistic description must be used.

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