Aust. J. Phys., 1979, **32**, 503–10

Kinematic Diffusion of Scalar Quantities in Turbulent Velocity Fields

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Abstract

From Kraichnan's direct interaction approximation the normal mode equations are set up for a scalar quantity diffusing kinematically under a turbulent velocity field which is statistically homogeneous and stationary. It is demonstrated: (i) that the mean scalar field responds only to the symmetric part of the velocity turbulence tensor; (ii) that the Kraichnan equation describing the normal mode behaviour is a singular nonlinear integral equation; (iii) that for velocity turbulence which is switched on and off infinitely rapidly the normal modes of the mean scalar field decay in time at a rate which is always greater than that obtaining in the absence of the turbulent velocity field. The motivation underlying these calculations is the problem of particle transport in turbulent astrophysical situations such as the interstellar medium. In such cases the effective Reynolds number is normally large compared with unity, so that expansion approximations for small Reynolds number are apparently not completely free of error.

1. Introduction

There is increasing evidence that it is rather common in astrophysical situations to have significant turbulence in which r.m.s. fluctuations are of the order of, or larger than, mean values. For instance, observations of both the solar wind and the interstellar medium indicate that the r.m.s. fluctuations in the magnetic field $\langle \delta B^2 \rangle^{\frac{1}{2}}$ are of the order of the mean field magnitude $\langle B \rangle$. To date there has been very little work done on self-consistent turbulence problems in astrophysics; attention is normally restricted to kinematic problems. In a kinematic, but turbulent, system it is customary to deal with a system which is influenced by a turbulent quantity but whose evolution does not in turn influence this turbulent quantity. For example, in kinematic cosmic ray modulation studies the cosmic rays are influenced by the turbulent Alfvén waves in the solar wind but do not in turn influence the waves.

One of the conventional techniques for handling kinematic turbulence problems is to use expansion schemes which assume that the turbulence is small in some sense (e.g. in cosmic ray diffusion theory it is assumed that $\langle \delta \boldsymbol{B}^2 \rangle \ll \langle \boldsymbol{B} \rangle^2$ even though solar wind measurements indicate that $\langle \delta \boldsymbol{B}^2 \rangle \approx \langle \boldsymbol{B} \rangle^2$). Over the years the question of the validity of such expansion schemes has led to some interesting debates in the astrophysical literature.

It is becoming apparent that in order to obtain trustworthy answers for astrophysical problems where the r.m.s. turbulence is at least as large as mean quantities some other form of approximation must be found (see Frisch (1968) for an elegant and excellent appraisal of the general situation).

The purpose of this paper is to illustrate a method of handling problems involving strong turbulence known as Kraichnan's direct interaction approximation (DIA) which does not suffer from any deficiencies due to expansion approximations. Kraichnan (1961) has shown that the DIA describes, in a statistically exact manner, an ensemble of possible physical systems. The only question which remains unanswered *in toto* to date is whether the ensemble of systems described by the DIA is, in fact, always the ensemble provided by nature. To the author's knowledge this question has not yet been completely settled.

The advantage, then, of Kraichnan's DIA is obvious. One is guaranteed, ahead of any detailed calculations, that the answers one gets to a turbulence problem are not only correct in a mathematical sense but are also physically permissible, since the DIA ensemble is physically possible. And Frisch (1968) has also remarked that 'the exact model solutions are approximate solutions of the true turbulence problem' for all values of the parameters involved. The main disadvantage of Kraichnan's DIA is that it normally leads to nonlinear singular integral equations which have to be solved before the normal modes of the ensemble average system are obtained. Nevertheless, in view of the fact that most astrophysical systems involving turbulence possess a level of turbulence which cannot be regarded as 'small' in any sense, it seems worth while to illustrate the manner in which Kraichnan's DIA applies to a simple problem. The more complex problems which occur in real astrophysically turbulent situations can then be investigated along similar lines using similar techniques.

In addition, since Kraichnan (1961) has already spelled out the general method of obtaining the nonlinear DIA equations from the full equations, the present development need only be brief, and the interested reader is referred to Kraichnan's paper for an appreciation of the details of the process.

The simple example chosen here in order to illustrate Kraichnan's DIA is the kinematic diffusion of a scalar quantity in a turbulent velocity field. The problem should be viewed as an educative device which not only illustrates a mathematical method of handling the DIA equations but which also gives physical insight into the role of turbulent velocity fields in causing diffusive behaviour in astrophysical situations.

2. Basic Equations

Consider the evolution of a scalar field $\psi(\boldsymbol{x},t)$ in an infinite medium of constant diffusivity η, which medium also possesses an incompressible turbulent velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ with $\nabla\cdot\boldsymbol{v}=0$. Let ψ evolve according to the relation

$$\partial\psi/\partial t + v_i\,\partial\psi/\partial x_i = \eta\,\nabla^2\psi\,. \tag{1}$$

Under Kraichnan's DIA the equation describing the ensemble average Green's function $G(\boldsymbol{x},t\,|\,\boldsymbol{x}',t')$ is

$$\frac{\partial G}{\partial t} - \eta\nabla^2 G = \delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t') + \frac{\partial}{\partial x_j}\left(\int_{t'}^{t} \mathrm{d}t''\, G(\boldsymbol{x},t\,|\,\boldsymbol{x}'',t'')\,\frac{\partial G}{\partial x_i''}(\boldsymbol{x}'',t''\,|\,\boldsymbol{x}',t') \times\langle v_j(\boldsymbol{x},t)\,v_i(\boldsymbol{x}'',t'')\rangle\,\mathrm{d}^3\boldsymbol{x}''\right), \tag{2}$$

with the requirement

$$G(\boldsymbol{x},t\,|\,\boldsymbol{x}',t') = 0 \qquad \text{for} \qquad t<t', \tag{3}$$

so that only forward going (in time) Green's functions are obtained. In addition, $\langle v_i v_j \rangle$ is the two-point, two-time correlation tensor of the turbulent velocity field. It is now relatively easy to obtain from equation (2) the equation satisfied by the mean scalar field $\langle \psi(\boldsymbol{x}, t) \rangle \equiv \Psi(\boldsymbol{x}, t)$.

We note that the lower limit on the t'' integral in equation (2) can be replaced by $-\infty$, since the requirement (3) ensures that no difference arises in the integral. On multiplying equation (2) by an arbitrary function $S(\boldsymbol{x}', t')$ and integrating the result over all values of $\boldsymbol{x}'$ and t', with the definition

$$\Psi(\boldsymbol{x}, t) = \int \mathrm{d}^3\boldsymbol{x}'\,\mathrm{d}t'\; G(\boldsymbol{x}, t \mid \boldsymbol{x}', t')\, S(\boldsymbol{x}', t'), \tag{4}$$

we obtain

$$\frac{\partial \Psi}{\partial t} - \eta \nabla^2 \Psi = S(\boldsymbol{x}, t) + \frac{\partial}{\partial x_j}\left(\int_{-\infty}^{t} \mathrm{d}t''\; G(\boldsymbol{x}, t \mid \boldsymbol{x}'', t'') \frac{\partial \Psi(\boldsymbol{x}'', t'')}{\partial x_i''} \times \langle v_i(\boldsymbol{x}'', t'')\, v_j(\boldsymbol{x}, t) \rangle\, \mathrm{d}^3\boldsymbol{x}'' \right). \tag{5}$$

Thus $S(\boldsymbol{x}, t)$ is the source which controls the evolution of Ψ through the Green's function G so that equation (5) is an initial value problem for Ψ given $S(\boldsymbol{x}, t)$. The problem can, of course, be converted into a normal-mode one in the standard manner. Firstly we ignore $S(\boldsymbol{x}, t)$ in equation (5). This leaves a homogeneous linear equation for Ψ given that we can obtain G from equation (2). The homogeneous equation will have a solution if, and only if, a dispersion relation is satisfied. Having obtained the dispersion relation we can then pose the initial value problem (5) as a linear superposition of the normal modes of the homogeneous problem. This technique is widely used, for example in solving the linearized Vlasov equation in plasma physics (see e.g. Montgomery and Tidman 1964), where the Landau-damped modes are obtained without reference to their initial values. The coefficients of the modes are, of course, controlled by the initial values but the dispersion relation is controlled by the homogeneous equation. For our purpose it suffices to consider the Green's function (2) together with the normal mode equation for Ψ (i.e. equation (5) with S set to zero).

Before we can progress further with equations (2) and (5) some knowledge of the functional behaviour of $\langle v_i(\boldsymbol{x}, t)\, v_j(\boldsymbol{x}', t') \rangle$ is required. For the remainder of this paper we shall suppose that the velocity turbulence is both homogeneous and stationary when

$$\langle v_i(\boldsymbol{x}', t')\, v_j(\boldsymbol{x}, t) \rangle = R_{ij}(\boldsymbol{x} - \boldsymbol{x}', t - t'). \tag{6}$$

It then follows by inspection of equation (2) that the mean Green's function must also be homogeneous and stationary:

$$G(\boldsymbol{x}, t \mid \boldsymbol{x}', t') = G(\boldsymbol{x} - \boldsymbol{x}', t - t'). \tag{7}$$

Now write

$$[R_{ij}(\boldsymbol{x}, t),\ G(\boldsymbol{x}, t),\ \Psi(\boldsymbol{x}, t)] = \int \mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega\, \exp\{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\}\, [R_{ij}(\boldsymbol{k}, \omega),\ G(\boldsymbol{k}, \omega),\ \Psi(\boldsymbol{k}, \omega)]. \tag{8}$$

Then from equations (2) and (5) we obtain

$$G(\boldsymbol{k},\omega)\,(\eta k^2-\mathrm{i}\omega)$$

$$= (2\pi)^{-4} - (2\pi)^4\, G(\boldsymbol{k},\omega)\, k_i\, k_j \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\; G(\boldsymbol{k}',\omega')\, R_{ij}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'), \tag{9}$$

$$\Psi(\boldsymbol{k},\omega)\,(\eta k^2-\mathrm{i}\omega)$$

$$= -(2\pi)^4\, \Psi(\boldsymbol{k},\omega)\, k_i\, k_j \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\; G(\boldsymbol{k}',\omega')\, R_{ij}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'). \tag{10}$$

The normal modes of the homogeneous equation (10) are then given by

$$\eta k^2 - \mathrm{i}\omega = -(2\pi)^4 k_i\, k_j \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\; G(\boldsymbol{k}',\omega')\, R_{ij}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'). \tag{11}$$

When the normal modes of the homogeneous equation for $\Psi(\boldsymbol{k},\omega)$ are satisfied at some ω (which is, in general, complex) for real k it follows by inspection of equation (9) that at dispersion (i.e. when equation (11) is satisfied) we have $1/G(\boldsymbol{k},\omega) = 0$.

It is opportune to capitalize on the behaviour of G at dispersion by changing variables in equation (9). Write

$$(2\pi)^4 g\,(\boldsymbol{k},\omega) = 1/G(\boldsymbol{k},\omega)$$

to obtain

$$g(\boldsymbol{k},\omega) = \eta k^2 - \mathrm{i}\omega + k_i\, k_j \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\; R_{ij}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')/g(\boldsymbol{k}',\omega'), \tag{12}$$

with the dispersion relation (11) being given indirectly through the complex values of ω for $\boldsymbol{k}$ fixed at which

$$g\,(\boldsymbol{k},\omega) = 0. \tag{13}$$

Before proceeding with the evaluation of the dispersion relation arising from equation (12) there are two points worth noting. Firstly R_{ij} is multiplied by $k_i k_j$ in equation (12). Thus only the symmetric part of R_{ij} (namely $R_{ij}^{(\mathrm{s})} = \frac{1}{2}(R_{ij}+R_{ji})$) enters the integral equation (12). Secondly equation (12) for $g\,(\boldsymbol{k},\omega)$ is a nonlinear singular integral equation (the nonlinearity is obvious by inspection; the fact that it is singular arises because we require $g\,(\boldsymbol{k},\omega)$ to vanish for some real k and complex ω in order that the homogeneous equation for Ψ have normal modes).

Before we investigate equation (12) in depth it is advantageous to change variables once more. Write

$$g\,(\boldsymbol{k},\omega) = \eta k^2\, \Phi(\boldsymbol{k},\omega). \tag{14}$$

Further let $R_{ij}(\boldsymbol{x},t)$ have a scale length L and an 'intensity' v^2. Then write $k \to kL$ and $\omega \to \omega\eta L^2$ so that in dimensionless form we have

$$\Phi(\boldsymbol{k},\omega) = 1 - \mathrm{i}\omega k^{-2} + R^2 k^{-2} k_i\, k_j \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\; \{k'^2\, \Phi(\boldsymbol{k}',\omega')\}^{-1}$$

$$\times R_{ij}^{(\mathrm{s})}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'), \tag{15}$$

where the Reynolds number R is defined by $R = Lv/\eta$. The dispersion relation for the normal modes of Ψ is then given by ω values satisfying the constraint condition

$$\Phi(\boldsymbol{k},\omega) = 0. \tag{16}$$

Now when the velocity turbulence is incompressible, homogeneous and isotropic, but not necessarily mirror-symmetric, we have (Batchelor 1950)

$$R_{ij}(\boldsymbol{k},\omega) = E(k,\omega)(\delta_{ij}-k_i k_j k^{-2}) + \mathrm{i}\,\varepsilon_{ijl}k_l k^{-1} H(k), \tag{17}$$

where, by Cramér's (1940) theorem,

$$E(k,\omega) \geqslant 0, \qquad -E(k,\omega) \leqslant H(k) \leqslant E(k,\omega), \tag{18}$$

for all real k and ω.

Note once again that only the symmetric part of R_{ij} enters equation (15); the helical component of R_{ij}, namely $\mathrm{i}\,\varepsilon_{ijl}k_l k^{-1} H(k)$, has no effect at all on the normal modes of $\Psi(\boldsymbol{k},\omega)$ nor on the mean Green's function. In other words, in so far as the mean field Ψ is concerned, the behaviour of Ψ in a turbulent, isotropic and mirror-symmetric ($H = 0$) velocity field is indistinguishable from its behaviour in a turbulent, isotropic but non-mirror symmetric ($H \neq 0$) velocity field.

When equation (17) is substituted into (15) we obtain

$$\Phi(\boldsymbol{k},\omega) = 1-\mathrm{i}\omega k^{-2} + R^2 \int \mathrm{d}^3\boldsymbol{k}'\,\mathrm{d}\omega'\, E(|\boldsymbol{k}-\boldsymbol{k}'|,\omega-\omega') \times |\boldsymbol{k}-\boldsymbol{k}'|^{-2}\left(1-\frac{(\boldsymbol{k}.\boldsymbol{k}')^2}{(kk')^2}\right)\left(\Phi(\boldsymbol{k}',\omega')\right)^{-1}, \tag{19}$$

with the dispersion relation given through those values of ω for which $\Phi(\boldsymbol{k},\omega) = 0$.

Some progress can be made with equation (19) before any further information about $E(k,\omega)$ is required. Firstly it is clear by inspection that

$$\Phi(\boldsymbol{k},\omega) = \Phi(k,\omega), \tag{20}$$

so that equation (19) takes on the form

$$\Phi(k,\omega) = 1-\mathrm{i}\omega k^{-2} + R^2 \int_0^\infty \mathrm{d}\kappa\,\kappa^2\, J(k,\kappa,\omega-\omega')\left(\Phi(\kappa,\omega')\right)^{-1}\mathrm{d}\omega', \tag{21}$$

where

$$J(k,\kappa,\omega-\omega') = 2\pi \int_{-1}^{+1} \frac{\mathrm{d}\mu\,(1-\mu^2)}{k^2+\kappa^2+2k\kappa\mu} E\left((k^2+\kappa^2+2k\kappa\mu)^{\frac{1}{2}},\, \omega-\omega'\right). \tag{22}$$

The dispersion relation $\Phi(k,\omega) = 0$ is now to be determined from solutions to Kraichnan's (1961) equation (21).* However, the number of solutions to the Kraichnan DIA equations in general (and not just for this problem) is very small. The structure and functional form of the solutions depend on the form assumed for $E(k,\omega)$. In this investigation equation (21) has been found to be too difficult to

* We take the liberty here of calling equation (21) (or equation 19) Kraichnan's equation, for it follows directly from his DIA after some simple, and fairly obvious, substitutions.

solve in complete generality for arbitrary forms of $E(k,\omega)$, but there does exist a simple choice for $E(k,\omega)$ for which exact solutions to equation (21) can be found: if $R_{ij}(\boldsymbol{x},t) = \delta(t)\,R_{ij}(\boldsymbol{x})$ then $E(k,\omega) = E(k)$. We shall consider this case throughout the remainder of this paper. Since $R_{ij}(\boldsymbol{x},t)$ has a velocity turbulence spectrum that is correlated only over infinitesimal time scales we shall call it the 'sudden' limit.

3. Dispersion under the Sudden Limit

When $E(k,\omega) = E(k)$, equation (21) reduces to

$$\Phi(k,\omega) = 1 - \mathrm{i}\omega k^{-2} + R^2 \int_0^\infty \kappa^2 J(k,\kappa)\,\mathrm{d}\kappa \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{\Phi(\kappa,\omega')}, \tag{23}$$

with

$$J(k,\kappa) = 2\pi \int_{-1}^{+1} \frac{\mathrm{d}\mu\,(1-\mu^2)}{k^2+\kappa^2+2k\kappa\mu}\, E\big((k^2+\kappa^2+2k\kappa\mu)^{\frac{1}{2}}\big). \tag{24}$$

Some care has to be exercised with the ω' integral in equation (23). It *cannot* be completed by contour integration in an arbitrary domain of the complex ω' plane. This is because the Green's function $G(\boldsymbol{x},t)$ vanishes for $t<0$. Accordingly the path of integration in the ω' plane must be chosen *above* all zeros and branch cuts (if any) of $\Phi(\kappa,\omega')$, with closure in the lower half complex ω' plane.

Equation (23) can be solved by invoking the following ansatz. Suppose that $\Phi(k,\omega)$ has only one zero in the complex ω plane so that we can write

$$\Phi(k,\omega) = a(k)\,\{\omega - \Omega(k)\}. \tag{25}$$

Then

$$\int_{-\infty}^{\infty} (\Phi(\kappa,\omega))^{-1}\,\mathrm{d}\omega' = -\mathrm{i}\pi/a(\kappa). \tag{26}$$

Thus when equations (25) and (26) are substituted into (23) we obtain

$$a(k)\,\{\omega - \Omega(k)\} = 1 - \mathrm{i}\omega k^{-2} - R^2\,\mathrm{i}\pi \int_0^\infty \kappa^2 \frac{J(k,\kappa)}{a(\kappa)}\,\mathrm{d}\kappa. \tag{27}$$

But if the assumed form (25) is indeed a solution of equation (23) the coefficients of powers of ω in equation (27) must be identical. Then

$$a(k) = -\mathrm{i}k^{-2}$$

and

$$\Omega(k) = -\mathrm{i}k^2\left(1 + \pi R^2 \int_0^\infty \kappa^4 J(k,\kappa)\,\mathrm{d}\kappa\right). \tag{28}$$

Now we have $J(k,\kappa) \geqslant 0$ for all real k and κ. Thus the normal modes of the mean field are given through

$$\omega = \Omega(k) \equiv -\mathrm{i}k^2\left(1 + \pi R^2 \int_0^\infty \kappa^4 J(k,\kappa)\,\mathrm{d}\kappa\right). \tag{29}$$

Since $\Omega(k) = -\mathrm{i}A(k)$ with $A(k) > 0$, and since the normal mode dependence was chosen to be $\exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x} - \mathrm{i}\omega t)$, it follows that the temporal dependence of the normal modes is $\exp\{-A(k)t\}$, so that all modes decay with a decay time $\tau(k)$ defined by

$$\tau^{-1} = k^2\left(1 + \pi R^2 \int_0^\infty \kappa^4 J(k,\kappa)\,\mathrm{d}\kappa\right). \tag{30}$$

Thus the decay of the normal modes of the mean field is *faster* than in the absence of the turbulent velocity field (when $R^2 = 0$ the decay is at a rate k^2).

It is perhaps of interest to exhibit the explicit dependence of τ^{-1} on k. From the definition of $J(k,\kappa)$ we have

$$\begin{aligned}\int_0^\infty \kappa^4 J(k,\kappa)\,\mathrm{d}\kappa &\equiv \int \mathrm{d}^3\boldsymbol{\kappa}\, E(\kappa)\{1-(\boldsymbol{k}.\boldsymbol{\kappa})^2/k^2\kappa^2\} \\ &= \tfrac{8}{3}\pi \int_0^\infty \kappa^2\,\mathrm{d}\kappa\, E(\kappa),\end{aligned} \tag{31}$$

so that the decay rate has the dependence $\tau^{-1} \propto k^2$; which also occurs in a null velocity situation ($R = 0$). What changes is the constant of proportionality, as can be seen from equations (30) and (31). In making these remarks it should be borne in mind that we are using the sudden limit for the velocity turbulence.

4. Discussion and Conclusions

(*a*) *Comments on the Calculation*

In this paper we have obtained the statistically exact normal modes of a mean scalar field undergoing convection in a turbulent velocity field and otherwise diffusing. The general singular integral equation (15) obtained from Kraichnan's DIA describes in a statistically exact manner (through the zeros of $\Phi(k,\omega)$) the normal mode dispersion relation of the mean scalar field for *arbitrary* values of the Reynolds number R. Several points follow directly from equation (15). Firstly only the symmetric part of the velocity turbulence tensor enters the problem. Thus as far as the mean field is concerned there is no evolutionary difference in a turbulent velocity field which is mirror-symmetric or helical. Secondly, as was remarked in the Introduction, equation (15) is a nonlinear singular integral equation; to date we have been unable to find its general solution for arbitrary functional forms of $J(k,\kappa,\omega)$. Thirdly, for the particular case of velocity turbulence that is switched on and off infinitely rapidly we have seen that the normal modes of the mean scalar field decay at a rate which is always *faster* than the decay rate obtaining in the absence of the turbulent convection. And the results obtained in Section 3 are valid for *arbitrary* values of the Reynolds number $R \equiv Lv/\eta$.

(*b*) *Comments pertaining to Astrophysics*

It is recognized, of course, that the simple problem considered here is perhaps not precisely the problem of, say, charged particle diffusion through HI and HII regions. These regions presumably contain rather involved and convoluted magnetic fields as well as turbulent velocity fields. But that has not been our main point. We set out to demonstrate the ease with which the statistically exact DIA equations can be applied to problems in astrophysics which contain, or are suspected of containing, a high degree of turbulence—the point being that for *all* values of the parameters involved the DIA results are exact and not approximate. Accordingly the results have the advantage of being solutions that can be used as templates against which approximate solutions can be gauged for accuracy. The DIA equations possess the further advantage that they describe an ensemble of physically possible dynamical systems, so that the results obtained apply with certainty to real physical processes.

It is considered that the prescription spelled out here, not only for obtaining the DIA equations applying to any given astrophysical problem but also for solving the equations under simple limits on the temporal structure of the turbulence, is worth a more detailed investigation than has been given. In particular it would seem very likely that the present simple problem is indicative of a method for solving a rather wide class of turbulence problems in astrophysical situations by analytic investigation. In fact the simple problem studied here was designed to illustrate this likelihood.

Acknowledgments

I am grateful to Mr H. C. Minnett, Chief, and to Dr B. J. Robinson, Cosmic Group leader, for the courtesies afforded me during my tenure of a Senior Visiting Scientist appointment at the Division of Radiophysics, CSIRO.

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Manuscript received 9 March 1979