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Nonlinear Double-diffusive Convection in a Low Prandtl Number Fluid

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Abstract

Nonlinear double-diffusive convection is studied using the modal equations of cellular convection. The boundary layer method is used by assuming a large Rayleigh number R for a fluid of low Prandtl number σ, and different ranges of the diffusivity ratio τ and the solute Rayleigh number R_s. The heat and solute fluxes are found to increase with $R\sigma$ and decrease with R_s. The effect of the solute is stabilizing, although the convection in a fluid with large σ is less affected by the solute concentration. The flow is shown to have a solute layer which thickens as σ, R, τ^{-1} or R_s^{-1} decreases. It is proved that it is only for this layer that the solute affects the boundary layer structure.

1. Introduction

Considerable progress has been made over the past 10 years in studying the convective motion of fluids in which there are gradients of two properties. The motion depends strongly for its driving mechanism on the different diffusive properties associated with the stabilizing and destabilizing forces. Double-diffusive convection is important in many areas of geophysics, astrophysics and engineering, and has been observed in nature. An example in astrophysics is the helium-rich core of some stars, in which the fluid is heated from below and is transported upwards by double-diffusive convection. (For a more detailed discussion of double-diffusive phenomena and their applications, see Turner (1973, 1974).)

The present paper studies nonlinear double-diffusive convection at small Prandtl number under the so-called modal equations of the equations for momentum, heat and solute. Briefly, these equations are constructed by expanding the fluctuating quantities in a complete set of functions of the horizontal coordinates and then truncating the expansion. The single-mode equations are derived simply by retaining only the first term in the expansion. A more detailed discussion of these equations and their derivations is given by Gough *et al.* (1975); earlier the same system of equations was derived differently by Roberts (1966) using a procedure proposed by Glansdorff and Prigogine (1964). Numerical computations of the single-mode equations for thermal convection have recently been made by Toomre *et al.* (1977), and they obtained good agreement with the asymptotic results of Gough *et al.* (1975).

Recently, Riahi (1978) studied nonlinear double-diffusive convection under the so-called mean field equations of the equations for momentum, heat and solute.

These equations are derived by ignoring the interaction between the fluctuating quantities but retaining the interaction between the mean and the fluctuating quantities. The single-mode mean field equations are identical with the modal equations (equations (7) below) when the parameter C is set to zero. Since the equation of motion in the mean field approximation ($C = 0$) becomes identical with the equation of motion in the limit of infinite Prandtl number σ, it must be expected that the mean field theory provides a most realistic description in the case of large σ. The heat transport predicted by the mean field equations (Riahi 1978) does not depend on σ. This deficiency renders the mean field equations unsuitable for use in, for example, stellar convection theory. The modal equations for $C \neq 0$ include some representation of the missing nonlinear interactions of fluctuating quantities in the mean field equations, and have the advantage that they restore the Prandtl number as a parameter of the problem. In their studies of cellular convection, Gough *et al.* (1975) compared their solutions based on the modal equations with the known solutions of the full equations for moderate Rayleigh numbers (Malkus and Veronis 1958; Schlüter *et al.* 1965). They found that accuracy is restricted to planforms for which $C \neq 0$, and otherwise to $\sigma \gg 1$. Therefore, the modal equations (for $C \neq 0$) are particularly suitable for the study of cellular convection in a low Prandtl number fluid.

On the basis of the postulate first proposed by Malkus (1954) we assume that the maximized heat transport F is that which is realized in the diffusive regime (defined as the regime in which the energy driving the flow comes from the component having the larger diffusivity). For the salt finger regime (the opposite case), the relevant postulate is that the flow fields tend to maximize the solute transport F_s. The success of previous studies of thermal convection based on Malkus's postulate encourages us to modify this postulate for our problem. A discussion of this postulate has been given by Lindberg (1971) and Straus (1972). In the latter paper it is found, for example, that in the salt finger case the mode which maximizes F_s lies within the waveband of the stable modes; this suggests a closed relation between the stability of a particular flow and its ability to transport salt across the layer.

The treatment in the present paper is for the steady case. Numerical studies by Veronis (1965, 1968) and Straus (1972) of the diffusive and salt finger regimes indicate that a steady state can be reached by a convective flow of finite amplutude. Of course, sufficiently strong convective flows are time dependent, but the present study aims at exploring the properties of nonlinear double-diffusive convection in the simpler case of a steady state, which may be considered as an approximation in some sense. The importance of double-diffusive convection to stellar situations, where the Prandtl number is small and the nonlinearities are strong, has motivated the present study.

2. Mathematical Formulation

We consider an infinite horizontal layer of fluid of depth d bounded above and below by two free perfectly conducting planes, maintained at temperatures T_0 and $T_0+\Delta T$ (with $\Delta T > 0$) and at solute concentrations S_0 and $S_0+\Delta S$ (with $\Delta S > 0$) respectively. It is convenient to use nondimensional variables in which lengths, velocities, time, temperature, solute and pressure are scaled respectively by: d, K_t/d, d^2/K_t, ΔT, ΔS and $\rho\nu K_t/d^2$. Here K_t is the thermometric conductivity, ρ is the mean density and ν is the kinematic viscosity. Then, with the usual Boussinesq approximation that density variations are taken into account only in the buoyancy term, the basic equations are

$$\sigma^{-1}(\partial \boldsymbol{u}/\partial t + \boldsymbol{u}.\nabla \boldsymbol{u}) = -\nabla P + (RT - R_s S^*)\boldsymbol{K} + \nabla^2 \boldsymbol{u}, \tag{1a}$$

$$\partial T^*/\partial t + \boldsymbol{u}.\nabla T^* = \nabla^2 T^*, \tag{1b}$$

$$\partial C^*/\partial t + \boldsymbol{u}.\nabla C^* = \tau \nabla^2 C^*, \qquad \nabla.\boldsymbol{u} = 0. \tag{1c, d}$$

Here $\boldsymbol{u} = (u, v, w)$ is the velocity vector, T^* is the temperature excess over T_0, C^* is the solute excess over S_0, T is the deviation of T^* from its horizontal average, S^* is the deviation of C^* from its horizontal average and P is the deviation of the pressure from its hydrostatic value. Also, $\boldsymbol{K}$ is a unit vector in the vertical direction, $\sigma = \nu/K_t$ is the Prandtl number, $R = \alpha g d^3 \Delta T / K_t \nu$ is the Rayleigh number, $R_s = \beta g d^3 \Delta S / K_t \nu$ is the so-called solute Rayleigh number and $\tau = K_s/K_t$ is the ratio of the diffusivity coefficients K_s and K_t of solute and heat respectively, with α the coefficient of thermal expansion, β the fractional change in density due to a change in the solute concentration, and g the acceleration due to gravity.

We consider only the case of steady solutions. For this case, two relations for the horizontal mean temperature and solute can be obtained by averaging equations (1b) and (1c), and integrating with respect to z:

$$\mathrm{d}\langle T^* \rangle/\mathrm{d}z = \langle WT \rangle - 1 - [\![WT]\!], \qquad \tau\, \mathrm{d}\langle C^* \rangle/\mathrm{d}z = \langle WS^* \rangle - \tau - [\![WS^*]\!]. \tag{2a, b}$$

Here, and subsequently, angle brackets denote horizontal averages and open brackets denote a further vertical averaging over the whole layer.

When the horizontal averages of equations (1a)–(1c) are subtracted from the steady forms of the original equations (1a)–(1c), the results are

$$\sigma^{-1}\big(\boldsymbol{u}.\nabla \boldsymbol{u} - \langle \boldsymbol{u}.\nabla \boldsymbol{u} \rangle\big) = -\nabla P + \big(RT - R_s S^*\big)\boldsymbol{K} + \nabla^2 \boldsymbol{u}, \tag{3a}$$

$$\nabla^2 T - w\big(\langle wT \rangle - 1 - [\![wT]\!]\big) = \boldsymbol{u}.\nabla T - \langle \boldsymbol{u}.\nabla T \rangle, \tag{3b}$$

$$\tau \nabla^2 S^* - \tau w\big(\langle wS^* \rangle - \tau - [\![wS^*]\!]\big) = \boldsymbol{u}.\nabla S^* - \langle \boldsymbol{u}.\nabla S^* \rangle, \tag{3c}$$

where equations (2a) and (2b) are used in deriving (3b) and (3c). In studying the system (3) we consider the case where the conductivities of the boundaries are far greater than that of the fluid, and we apply the so-called free boundary conditions to the velocity field. Thus we have

$$w = \partial u/\partial z = \partial v/\partial z = T = S^* = 0 \quad \text{at} \quad z = 0 \text{ and } 1. \tag{4}$$

The modal equations are derived from the system (3) by expanding the fluctuating variables in the planform functions $f_n(x, y)$:

$$(u, \;\; v, \;\; w) = \sum_n \left(a_n^{-2} \frac{\partial f_n}{\partial x} \frac{\partial W_n}{\partial z}, \;\; a_n^{-2} \frac{\partial f_n}{\partial y} \frac{\partial W_n}{\partial z}, \;\; f_n(x, y)\, W_n(z) \right), \tag{5a}$$

$$(T, \;\; S^*) = \sum_n f_n(x, y)\big(\theta_n(z), \;\; S_n^*(z)\big), \tag{5b}$$

where the planforms $f_n(x,y)$ satisfy

$$\nabla_1^2 f_n(x,y) = -a_n^2 f_n(x,y) \qquad (\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2), \tag{6}$$

for some horizontal wavenumber a_n. Functions f_n with different wavenumbers are naturally orthogonal, and here are chosen to be orthonormal. A similar expansion for P is not needed, since P can be eliminated by taking the double curl of equation (3a).

In the present study we truncate the expansions (5a) and (5b) by retaining only the first term. Accordingly, we set all the W_n, T_n and S_n^* to zero except for the triple W_1, T_1, and S_1^*, say. The indices for these variables are then no longer needed, and we may enter the expansions into equations (3) and, by multiplying by f_1 and taking horizontal averages, project out the desired equations for W, T and S^*. The reductions are straightforward and are not given here. Thus, we are led to the following governing equations:

$$(\mathrm{D}^2 - a^2)^2 W = a^2(RT - R_s S^*) + C\sigma^{-1}\{2\mathrm{D}W(\mathrm{D}^2 - a^2)W + W(\mathrm{D}^2 - a^2)\mathrm{D}W\}, \tag{7a}$$

$$(\mathrm{D}^2 - a^2)T = (WT - F - 1)W + C(2W\mathrm{D}T + T\mathrm{D}W), \tag{7b}$$

$$\tau^2(\mathrm{D}^2 - a^2)S^* = (WS^* - F_s - \tau)W + \tau C(2W\mathrm{D}S^* + S^*\mathrm{D}W). \tag{7c}$$

Here a is the horizontal wavenumber (its index is dropped for simplicity of notation); $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$; $F = [\![WT]\!]$ and $F_s = [\![WS^*]\!]$ are the heat and solute fluxes respectively; $C = \langle \frac{1}{2} f_1^3(x,y)\rangle$ is the parameter derived from the planform function $f_1(x,y)$. The constant C vanishes for rolls and rectangles, and takes the value of $6^{-\frac{1}{2}}$ for the hexagonal planform. We assume $C \neq 0$, and consider a value such as $6^{-\frac{1}{2}}$ as being a representative value of C. For $C = 0$, the equation system (7) reduces to the so-called mean field equations for thermo-solutal convection, which have been solved recently (Riahi 1978).

We now rescale the dependent variables such that

$$\omega = (FR)^{-\frac{1}{2}} W, \qquad \theta = (R/F)^{\frac{1}{2}} T, \qquad S = (FR)^{\frac{1}{2}} F_s^{-1} S^*. \tag{8a, b, c}$$

The governing differential equations then become

$$(\mathrm{D}^2 - a^2)^2\omega = a^2(\theta - KS) + C(FR)^{\frac{1}{2}}\sigma^{-1}\{2\mathrm{D}\omega(\mathrm{D}^2 - a^2)\omega + \omega(\mathrm{D}^2 - a^2)\mathrm{D}\omega\}, \tag{9a}$$

$$(\mathrm{D}^2 - a^2)\theta/FR + (1 - \omega\theta + F^{-1})\omega = C(FR)^{-\frac{1}{2}}(2\omega\mathrm{D}\theta + \theta\mathrm{D}\omega), \tag{9b}$$

$$\tau^2(\mathrm{D}^2 - a^2)S/FR + (1 - \omega S + \tau F_s^{-1})\omega = \tau C(FR)^{-\frac{1}{2}}(2\omega\mathrm{D}S + S\mathrm{D}\omega), \tag{9c}$$

where

$$K = F_s R_s/FR. \tag{10}$$

We use the following constraints to determine F and F_s:

$$F = \frac{1 - R^{-1}[\![\,|\nabla\theta|^2]\!]}{[\![(1-\omega\theta)^2]\!]}, \qquad \tau^{-1}F_s = \frac{1 - (\tau F_s/FR)[\![\,|\nabla S|^2]\!]}{[\![(1-\omega S)^2]\!]}, \tag{11a, b}$$

which are derived by multiplying equations (9b) and (9c) by θ and S respectively and taking the vertical average over the layer.

The boundary conditions (4) when combined with equation (1d), after using the equations (5) and (8), yield

$$\omega = D^2\omega = \theta = S = 0 \quad \text{at} \quad z = 0 \ \text{and} \ 1. \tag{12}$$

The equations (9) must then be solved subject to equations (11) and (12). In the following section we obtain the solutions by using the boundary layer method, treating R as a large parameter.

3. Solutions by Boundary Layer Method

(a) Case $\tau \ll 1$

The wavenumber a is supposed to be large (which can be justified *a posteriori*), so that the convection cells are narrow. The solutions can be obtained by matching asymptotic approximations in the interior and three distinct regions near each boundary. In the interior of the layer, z is of order one. It is assumed that

$$F \gg 1, \qquad F_s \gg \tau, \qquad a \ll \{(1-K)FR\sigma/C\}^{\frac{1}{4}}, \tag{13a, b, c}$$

$$a(FR)^{-1/6}\{(1-K)\sigma/C\}^{1/3} \ll 1, \qquad C = O(1). \tag{13d, e}$$

The governing equations (9a)–(9c) yield, after using these assumptions, the following equations

$$(\theta - KS) = 3C(FR)^{\frac{1}{2}}\sigma^{-1}\omega D\omega, \qquad \omega\theta = \omega S = 1. \tag{14a, b}$$

It is seen from the equations (14) that the inertial, bouyancy and convection terms are important in the interior. The equations (14) are satisfied by

$$\omega = (\sigma C^{-1} z)^{1/3}(1-K)^{1/3}(FR)^{-1/6}, \tag{15a}$$

$$S = \theta = (\sigma C^{-1} z)^{-1/3}(1-K)^{-1/3}(FR)^{1/6}, \tag{15b}$$

where the constant of integration is chosen so that ω satisfies its boundary condition at $z = 0$. Near each surface and adjacent to the interior are intermediate layers of thickness $O(a^{-1})$, in which vertical derivatives are important in the inertial term. Defining appropriate boundary layer coordinates $\xi_t = a(1-z)$ and $\xi_b = az$ for the upper (top) and lower (bottom) of these layers respectively, the equations (9) and matching conditions (matching the solutions to the corresponding solutions in the interior) yield the following equations in the upper intermediate layer

$$2\frac{d\omega}{d\xi_t}\left(\frac{d^2}{d\xi_t^2} - 1\right)\omega + \omega\left(\frac{d^2}{d\xi_t^2} - 1\right)\frac{d\omega}{d\xi_t} = 0, \qquad \omega\theta = \omega S = 1. \tag{16a, b}$$

Similarly, the equations (9) yield the following equations in the lower intermediate layer

$$(KS-\theta) = aC\sigma^{-1}(FR)^{\frac{1}{2}}\left\{2\frac{d\omega}{d\xi_b}\left(\frac{d^2}{d\xi_b^2}-1\right)\omega+\omega\left(\frac{d^2}{d\xi_b^2}-1\right)\frac{d\omega}{d\xi_b}\right\}, \tag{17a}$$

$$\omega\theta = \omega S = 1. \tag{17b}$$

The equations (16) and (17) yield the following asymptotic results

$$\omega = (\sigma/2C)^{1/3}(1-K)^{1/3}(FR)^{-1/6}(3\xi_t)^{2/3} \quad \text{as} \quad \xi_t \to 0, \tag{18a}$$

$$S = \theta = (2C/\sigma)^{1/3}(1-K)^{-1/3}(FR)^{1/6}(3\xi_t)^{-2/3} \quad \text{as} \quad \xi_t \to 0; \tag{18b}$$

$$\omega = (\sigma/Ca)^{1/3}(1-K)^{1/3}(FR)^{-1/6}\xi_b(3\ln\xi_b^{-1})^{1/3} \quad \text{as} \quad \xi_b \to 0, \tag{18c}$$

$$S = \theta = (Ca/\sigma)^{1/3}(1-K)^{-1/3}(FR)^{1/6}\xi_b^{-1}(3\ln\xi_b^{-1})^{-1/3} \quad \text{as} \quad \xi_b \to 0. \tag{18d}$$

Closer to each surface and adjacent to the intermediate layers are thermal layers, in which thermal conduction is significant in the heat equation and θ is brought to its zero boundary value. We define δ_t and δ_b as the thicknesses of the top and bottom thermal layers respectively. Also, $\eta_t = (1-z)/\delta_t$ and $\eta_b = z/\delta_b$ are defined to be the corresponding variables in these layers. We then find from the governing equations (9), after applying matching conditions (matching the solutions to the corresponding solutions in the intermediate layers), that the equations for the lower thermal layer are

$$\frac{d^4\omega}{d\eta_b^4} = \frac{C}{\sigma}\left(2\frac{d\omega}{d\eta_b}\frac{d^2\omega}{d\eta_b^2}+\omega\frac{d^3\omega}{d\eta_b^3}\right)\delta_b(FR)^{\frac{1}{2}}, \tag{19a}$$

$$\frac{1}{FR\,\delta_b^2}\frac{d^2\theta}{d\eta_b^2}+(1-\omega\theta)\omega = C\left(2\omega\frac{d\theta}{d\eta_b}+\theta\frac{d\omega}{d\eta_b}\right)\frac{1}{\delta_b(FR)^{\frac{1}{2}}}, \tag{19b}$$

$$\omega S = 1. \tag{19c}$$

Derivation of the equations (19) requires the following conditions:

$$FRA_b^2\delta_b^2 = 1, \qquad a\delta_b \ll 1, \qquad a^2\delta_b^4 \ll (C/\sigma)A_b^2, \tag{20a, b, c}$$

where

$$A_b = (\sigma/Ca)^{1/3}(1-K)^{1/3}(FR)^{-1/6}a\delta_b\{3\ln(1/a\delta_b)\}^{1/3}. \tag{20d}$$

Similarly, the equations for the upper thermal layer are

$$\frac{d^4\omega}{d\eta_t^4} = -\frac{C}{\sigma}\left(2\frac{d\omega}{d\eta_t}\frac{d^2\omega}{d\eta_t^2}+\omega\frac{d^3\omega}{d\eta_t^3}\right)\delta_t(FR)^{\frac{1}{2}}, \tag{21a}$$

$$\frac{1}{FR\,\delta_t^2}\frac{d^2\theta}{d\eta_t^2}+(1-\omega\theta)\omega = -C\left(2\omega\frac{d\theta}{d\eta_t}+\theta\frac{d\omega}{d\eta_t}\right)\frac{1}{\delta_t(FR)^{\frac{1}{2}}}, \tag{21b}$$

$$\omega S = 1, \tag{21c}$$

where it is found necessary to have the following conditions:

$$FRA_t^2\delta_t^2 = 1, \qquad a\delta_t \ll 1, \qquad a^2\delta_t^4 \ll (C/\sigma)A_t^2, \tag{22a, b, c}$$

$$A_t = (\sigma/2C)^{1/3}(1-K)^{1/3}(FR)^{-1/6}(3a\delta_t)^{2/3}. \tag{22d}$$

The solutions to the equations (19) satisfying the boundary conditions (12) for ω and θ are

$$\omega = A_b\eta_b, \qquad S = A_b^{-1}\eta_b^{-1}, \tag{23a, b}$$

$$\theta = \frac{C^{\frac{1}{2}}\eta_b}{2A_b}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}C\eta_b^2(1-t)\right)\,\mathrm{d}t, \qquad \mu^2 = 1+C^{-2}. \tag{23c, d}$$

Similarly, we find the following solutions for the upper thermal layer:

$$\omega = A_t\eta_t, \qquad S = A_t^{-1}\eta_t^{-1}, \tag{24a, b}$$

$$\theta = -\frac{C^{\frac{1}{2}}\eta_t}{2A_t}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}C\eta_t^2(t-1)\right)\,\mathrm{d}t. \tag{24c}$$

There are further layers closer to each boundary, in which solute conduction is significant in the solute equation and S is brought to its zero boundary value. Let us suppose that these thinner layers are of thickness $\varepsilon_b \ll \delta_b$ near the lower boundary and of thickness $\varepsilon_t \ll \delta_t$ near the upper boundary, and define the corresponding appropriate coordinates as $\zeta_b = z/\varepsilon_b$ and $\zeta_t = (1-z)/\varepsilon_t$. The governing equations and matching conditions then give the following equations in the lower solute layer:

$$\frac{\mathrm{d}^4\omega}{\mathrm{d}\zeta_b^4} = \frac{C\tau}{\sigma}\left(2\frac{\mathrm{d}\omega}{\mathrm{d}\zeta_b}\frac{\mathrm{d}^2\omega}{\mathrm{d}\zeta_b^2}+\omega\frac{\mathrm{d}^3\omega}{\mathrm{d}\zeta_b^3}\right)\varepsilon_b(FR)^{\frac{1}{2}}, \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}\eta_b^2} = 0, \tag{25a, b}$$

$$\frac{1}{FR\varepsilon_b^2}\frac{\mathrm{d}^2S}{\mathrm{d}\zeta_b^2}+(1-\omega S)\omega = C\left(2\omega\frac{\mathrm{d}S}{\mathrm{d}\zeta_b}+S\frac{\mathrm{d}\omega}{\mathrm{d}\zeta_b}\right)\frac{1}{\varepsilon_b(FR)^{\frac{1}{2}}}. \tag{25c}$$

Derivation of the equations (25) requires the following condition:

$$FRA_b^2\varepsilon_b^4 = \delta_b^2\tau^2, \tag{26}$$

where A_b is already defined by equation (20d). Similarly, the equations in the upper solute layer are

$$\frac{\mathrm{d}^4\omega}{\mathrm{d}\zeta_t^4} = -\frac{C\tau}{\sigma}\left(2\frac{\mathrm{d}\omega}{\mathrm{d}\zeta_t}\frac{\mathrm{d}^2\omega}{\mathrm{d}\zeta_t^2}+\omega\frac{\mathrm{d}^3\omega}{\mathrm{d}\zeta_t^3}\right)\varepsilon_t(FR)^{\frac{1}{2}}, \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}\eta_t^2} = 0, \tag{27a, b}$$

$$\frac{1}{FR\varepsilon_t^2}\frac{\mathrm{d}^2S}{\mathrm{d}\zeta_t^2}+(1-\omega S)\omega = -C\left(2\omega\frac{\mathrm{d}S}{\mathrm{d}\zeta_t}+S\frac{\mathrm{d}\omega}{\mathrm{d}\zeta_t}\right)\frac{1}{\varepsilon_t(FR)^{\frac{1}{2}}}. \tag{27c}$$

Derivation of the equations (27) requires

$$FRA_t^2\varepsilon_t^4 = \delta_t^2\tau^2, \tag{28}$$

where A_t is defined by equation (22d).

The solutions to the equations (25) satisfying the boundary conditions (12) are

$$\omega = B_b \zeta_b, \qquad \theta = (h_b/B_b)\eta_b, \tag{29a, b}$$

$$S = \frac{C^{\frac{1}{2}}\zeta_b}{2B_b} \int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}C\zeta_b^2(1-t)\right) \mathrm{d}t, \tag{29c}$$

where

$$B_b = (\varepsilon_b/\delta_b)A_b. \tag{29d}$$

Similarly, the solutions in the upper solute layer are

$$\omega = B_t \zeta_t, \qquad \theta = (h_t/B_t)\eta_t, \tag{30a, b}$$

$$S = -\frac{C^{\frac{1}{2}}\zeta_t}{2B_t} \int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}C\zeta_t^2(t-1)\right) \mathrm{d}t, \tag{30c}$$

where

$$B_t = (\varepsilon_t/\delta_t)A_t, \tag{30d}$$

and h_t and h_b are constants of order one whose numerical values are not needed here. Thus the solutions for velocity and temperature are very similar to the corresponding solutions for thermal convection in the first three regions, and are essentially unchanged in the solute layer. The solute concentration has substantially the same form as the temperature in the interior and a^{-1} layer, is essentially unchanged in the temperature layer, and has a similar form to the temperature in the solute layer.

To determine F and F_s we must evaluate the expressions $[\![|\nabla\theta|^2]\!]$, $[\![|\nabla S|^2]\!]$, $[\![(1-\omega\theta)^2]\!]$ and $[\![(1-\omega S)^2]\!]$ in the equations (11). Within the boundary layer approximation, using the results obtained above and keeping only the leading-order terms, we find that

$$[\![|\nabla\theta|^2]\!] = 3a^2(FR)^{\frac{1}{3}}\{(1-K)\sigma/C\}^{-\frac{2}{3}} + \delta_b^{-1}A_b^{-2}I_1, \tag{31a}$$

$$[\![|\nabla S|^2]\!] = 3a^2(FR)^{\frac{1}{3}}\{(1-K)\sigma/C\}^{-\frac{2}{3}} + \varepsilon_b^{-1}B_b^{-2}J_1, \tag{31b}$$

$$[\![(1-\omega\theta)^2]\!] = \delta_b I_2, \qquad [\![(1-\omega S)^2]\!] = \varepsilon_b J_2, \tag{31c, d}$$

where I_1 and I_2 are the integrals

$$\int_0^\infty (\mathrm{d}\theta/\mathrm{d}\eta_b)^2\,\mathrm{d}\eta_b \qquad \text{and} \qquad \int_0^\infty (1-\eta_b\theta)^2\,\mathrm{d}\eta_b$$

in the lower thermal layer respectively. Similarly J_1 and J_2 are the integrals

$$\int_0^\infty (\mathrm{d}s/\mathrm{d}\zeta_b)^2\,\mathrm{d}\zeta_b \qquad \text{and} \qquad \int_0^\infty (1-\zeta_b S)^2\,\mathrm{d}\zeta_b$$

in the lower solute layer respectively. Use of the equations (31) in (11) and maximization of the heat flux with respect to the wavenumber a yield the following results (see Chan (1971) for more details on the maximization procedure):

$$a = (\tfrac{1}{3})^{1/2}(32)^{1/32}\gamma^{15/32}(1-\gamma)^{-3/16}\{R\sigma C^{-1}(1-K)\}^{9/32}(\ln R\sigma)^{-1/32}I^{3/16}, \tag{32a}$$

$$\delta_t = \left(\frac{1}{48}\right)^{1/5}\left(\frac{I}{1-\gamma}\right)^{3/20}\left(\frac{32}{\gamma}\right)^{9/40}\{R\sigma C^{-1}(1-K)\}^{-3/8}(\ln R\sigma)^{-1/40}, \tag{32b}$$

$$\delta_b = \left(\frac{I}{1-\gamma}\right)^{1/8}\left(\frac{32}{\gamma}\right)^{3/16}\{R\sigma C^{-1}(1-K)\}^{-5/16}(\ln R\sigma)^{-3/16}, \tag{32c}$$

$$\varepsilon_t = \zeta_t \tau^{1/2}, \qquad \varepsilon_b = \delta_b \tau^{1/2}, \tag{32d, e}$$

$$F = \left(\frac{1-\gamma}{I}\right)^{9/8}\left(\frac{\gamma}{32}\right)^{3/16}\{R\sigma C^{-1}(1-K)\}^{5/16}(\ln R\sigma)^{3/16}, \tag{32f}$$

$$F_s = \frac{F\tau^{1/2}}{1+\gamma(\tau^{3/2}-1)}, \qquad K = \frac{R_s \tau^{1/2}}{R\{1+\gamma(\tau^{3/2}-1)\}}. \tag{32g, h}$$

Here γ is a root of the equation

$$G \equiv \left(1 - \frac{R_s \tau^{1/2}}{R\{1+\gamma(\tau^{3/2}-1)\}}\right)(1-7\gamma) + \frac{5\gamma(1-\gamma)R_s\tau^{1/2}(\tau^{3/2}-1)}{3R\{1+\gamma(\tau^{3/2}-1)\}^2} = 0, \tag{32i}$$

and it is found that

$$I = I_1 + I_2 = J_1 + J_2 = 1\cdot062(1+C^2)^{\frac{1}{4}}. \tag{32j}$$

Various assumptions, including the conditions (13), (20) and (22), lead us to the following conditions for the validity of the solutions:

$$(\ln R)^{11/3} R^{-1} \ll \sigma(1-K) \ll (R^{-1}\ln R)^{1/9}, \qquad 1-K \gg (\ln R\sigma)^{-1}, \tag{33a, b}$$

$$R_s \tau^{1/2} R^{-1}\{1+\gamma(\tau^{3/2}-1)\}^{-1} < 1, \qquad \sigma \ll R^{-1/9}(\ln R)^{10/9}. \tag{33c, d}$$

It turns out that the equations (32) with the conditions (33) are valid for all possible ranges of τ. The equations (10), (32f)–(32i) and the conditions (33a)–(33c) could be further simplified using the condition $\tau \ll 1$. If we simplify equation (32i) using $\tau \ll 1$, it becomes a quadratic equation for γ with two real and positive roots. To maximize F we use the root which gives the relative maximum of F. It is easily seen that $\gamma = \frac{1}{7}$ for $K \ll 1$. For $K \approx 1$ and as $K \to 1$ in this range, F and F_s decrease rapidly. Thus, for sufficiently small $|1-K|$, nonlinear maximizing convection is inhibited entirely by the solute concentration.

(b) Case $\tau = O(1)$

It is found that, for the case $\tau = O(1)$, the solute layer merges with the thermal layer. The relations (14)–(19b), (20)–(21b), (22)–(23a), (23c)–(24a), (24c) and (31)–(33) are valid here. However, the equation for the solute concentration in the temperature layer takes the form

$$\frac{\tau^2}{FR\delta_b^2}\frac{d^2S}{d\eta_b^2} + (1-\omega S)\omega = \tau C\left(2\omega\frac{dS}{d\rho_b} + S\frac{d\omega}{d\rho_b}\right)\frac{1}{\delta_b (FR)^{\frac{1}{2}}}, \tag{34a}$$

$$\frac{\tau^2}{FR\,\delta_t^2}\frac{d^2S}{d\eta_t^2}+(1-\omega S)\omega = -\tau C\left(2\omega\frac{dS}{d\rho_t}+S\frac{d\omega}{d\rho_t}\right)\frac{1}{\delta_t(FR)^{\frac{1}{2}}} \tag{34b}$$

in the lower and upper layers respectively. The solutions to the equations (34) satisfying the boundary conditions (12) are

$$S = \frac{C^{\frac{1}{2}}\eta_b}{2\tau A_b}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\big(C\eta_b^2(1-t)/2\tau\big)\,dt, \tag{35a}$$

$$S = -\frac{C^{\frac{1}{2}}\eta_t}{2\tau A_t}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\big(C\eta_t^2(t-1)/2\tau\big)\,dt \tag{35b}$$

in the lower and upper layers respectively. In particular, for $\tau = 1$ we find from equation (32i) that $\gamma = \frac{1}{7}$ as expected, since the problem can be reparameterized to be equivalent to a singly diffusive case in which $\gamma = \frac{1}{7}$. For $\tau > 1$, we have $G \geqslant 0$ for $\gamma = \frac{1}{7}$ and $G < 0$ for $\gamma = 1$ (since $K < 1$). Hence there is always one positive root between these two values of γ. Similarly, for $\tau < 1$, there is always one positive root for γ in the interval $(0, \frac{1}{7})$. When there is more than one valid positive root we choose that which gives the relative maximum of F.

(*c*) *Case* $\tau \gg 1$

For $\tau \gg 1$ we have $\varepsilon_t \gg \delta_t$ and $\varepsilon_b \gg \delta_b$. The relations (14)–(18), (32) and (33) are valid here. However, we have the following equations for the solute layer after using the equations (9) and the matching conditions: in the lower layer, equations (25a), (25c) and $\omega\theta = 1$; in the upper layer, equations (27a), (27c) and $\omega\theta = 1$. The solutions to these equations are found to be: in the lower layer,

$$\omega = B_b\zeta_b, \qquad \theta = B_b^{-1}\zeta_b^{-1}, \tag{36a, b}$$

$$S = \frac{C^{\frac{1}{2}}\zeta_b}{2B_b}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\big(\tfrac{1}{2}C\zeta_b^2(1-t)\big)\,dt, \tag{36c}$$

where

$$B_b = (\sigma/Ca)^{1/3}(1-K)^{1/3}(FR)^{-1/6}a\varepsilon_b\{3\ln(1/a\varepsilon_b)\}^{1/3}; \tag{36d}$$

in the upper layer,

$$\omega = B_t\zeta_t, \qquad \theta = B_t^{-1}\zeta_t^{-1}, \tag{36e, f}$$

$$S = -\frac{C^{\frac{1}{2}}\zeta_t}{2A_t}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\big(\tfrac{1}{2}C\zeta_t^2(t-1)\big)\,dt, \tag{36g}$$

where

$$B_t = (\sigma/2C)^{1/3}(1-K)^{1/3}(FR)^{-1/6}(3a\varepsilon_t)^{2/3}. \tag{36h}$$

We obtain similar equations for the thermal layer: in the lower layer, equations (19a), (19b) and $d^2S/d\zeta_b^2 = 0$; in the upper layer, equations (21a), (21b) and $d^2S/d\zeta_t^2 = 0$. The solutions to these equations are found to be: in the lower layer,

$$\omega = A_b\eta_b, \qquad S = (h_b/A_b)\zeta_b, \tag{37a, b}$$

$$\theta = \frac{C^{\frac{1}{2}}\eta_b}{2A_b}\int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}C\eta_b^2(1-t)\right) \, dt, \tag{37c}$$

where

$$A_b = (\delta_b/\varepsilon_b)B_b; \tag{37d}$$

in the upper layer,

$$\omega = A_t \eta_t, \qquad S = (h_t/A_t)\zeta_t, \tag{37e, f}$$

$$\theta = -\frac{C^{\frac{1}{2}}\eta_t}{2A_t}\int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}C\eta_t^2(t-1)\right) \, dt, \tag{37g}$$

where

$$A_t = (\delta_t/\varepsilon_t)B_t. \tag{37h}$$

The boundary layer structure is now valid if

$$\tau \ll \{R\sigma(1-K)\}^{1/16}(\ln R\sigma)^{7/16}. \tag{38}$$

Otherwise, the solute layer merges with the intermediate layer, and the boundary layer structure breaks down. The relations (10), (32f)–(32i) and (33a)–(33c) could be further simplified using the condition $\tau \gg 1$. If we simplify equation (32i) using $\tau \gg 1$, it yields the following solution

$$\gamma = \frac{1}{14}\left[1 + \frac{16R_s}{3\tau R} + \left\{\left(1+\frac{16R_s}{3\tau R}\right)^2 + \frac{56R_s}{3\tau R}\right\}^{\frac{1}{2}}\right], \tag{39}$$

which is easily seen to be in the range $\frac{1}{7} \leqslant \gamma < 1$.

4. Discussion

The boundary layer analysis given in the previous section has shown that, for given R, R_s and σ, the heat and solute fluxes are continuous functions of τ. For $\tau \gg 1$ or $\tau \ll 1$, the solute and temperature have different boundary layers. It is found from the equations (32) that the relation $F \approx \delta_b^{-1}$ holds for the strongly convective case ($K \ll 1$), as in thermal convection problems at high R. However, for $K \approx 1$ the relation between F and δ_b also depends on K. Detailed calculations of F and F_s indicate that δ_b has essentially the unique role of fixing and determining F. By analogy, ε_b should have the role of fixing and determining F_s in the salt finger regime. It is noted from the equations (32) that δ_t and δ_b depend on $R\sigma$ and K, and that τ and R_s are not free parameters. However, ε_t and ε_b depend strongly on τ as well as on $R\sigma$ and K. If $\tau \approx 1$, $\delta_t \approx \varepsilon_t$, $\delta_b \approx \varepsilon_b$ and either $\tau \gg 1$ or $\tau \ll 1$, the solute concentration has a layer distinct from that of the temperature. This is expected since ε_t and ε_b appear whenever the solute conduction term in the solute equation becomes important. For example, when $\tau \gg 1$, as we approach the boundary $z = 0$ from the interior, the solute and thermal conduction terms become important successively. The former is $O(\tau)$, and hence we have $\varepsilon_b \gg \delta_b$.

It is clear that the solutions are possible for $K < 1$. For $K \ll 1$, the stabilizing effect of solute is unimportant. As $K \to 1$, however, a, F and F_s decrease rapidly and the maximizing convection is suppressed entirely by the solute concentration.

Care must also be exercised here since, for the nonlinear regime, our problem is valid only asymptotically away from its stability regime.

The present study is restricted to the case of a single horizontal wavenumber. It is known (Busse 1969; Chan 1971, 1974; Riahi 1977, 1978) that multi-wave solutions are often possible, and these allow greater heat and solute fluxes than single-wave solutions. Riahi (1978) studied nonlinear thermosolutal convection under the mean field approximation and through the multi-boundary-layer theory of Busse (1969). He showed, in particular, that the solute concentration affects the boundary layer structure only in the last mode (the mode having the smallest horizontal length scale) for the range $\tau \gg 1$ or $\tau \ll 1$. This result supports his conjecture that the layering problem of double-diffusive convection (Turner 1974) is a higher-mode phenomenon. The multi-modal analysis of the nonlinear thermal convection for $C \neq 0$ poses serious analytical difficulties and has not yet been solved.

An important result of our present analysis is that the fluxes F and F_s increase with $\sigma(1-K)$. Thus the convection in a fluid with larger σ is less affected by the stabilizing effects of solute than one with smaller σ. Our basic model can be transformed easily to the case, in which temperature and solute are higher at the top, by interchanging the role of temperature and solute. The results are then applicable after replacing F, F_s, R, R_s, τ, δ and ε by F_s, F, R_s/τ, R/τ, $1/\tau$, ε and δ respectively.

To what extent does the boundary layer structure considered here resemble that actually existing in a laboratory or a natural situation? A detailed correspondence is not to be expected, since the present model is highly idealized, e.g. it assumes a horizontally infinite layer of fluid and statistical homogeneity in the horizontal planes. The present boundary layer structure, similar to that originally discussed by Howard (1963) in his study of heat transport by turbulent convection, is independent of the horizontal variations and is based essentially on the following discussion. It can be seen from the normalization relations (8) that $[\![\omega\theta]\!] = [\![\omega S]\!] = 1$. To maximize F we find from the constraints (11) that we need to have $\omega\theta$ and ωS, which are zero at the boundaries, grow rapidly to the value 1. The region in which $\omega\theta$ differs appreciably from 1 is called the thermal boundary layer, though this is not necessarily the boundary layer for ω and θ individually, but only in their product (Howard 1963). Similarly, the region in which ωS differs appreciably from 1 is called the solute boundary layer, though this also is not necessarily the boundary layer in ω and S separately, but only in their product.

The dynamic balances derived for the boundary layers, however, provide the elements for the understanding of the convective motion and the transport processes. Furthermore, the role of the Prandtl number σ and other parameters of the problem can be illuminated by the boundary layer analysis which is of considerable interest. The essential validity of the boundary layer structure developed by Gough *et al.* (1975) for the simpler problem of singly diffusive convection, which is also based on the modal equations, has been tested and confirmed by the numerical studies of Toomre *et al.* (1977). The latter considered mostly hexagonal planforms, and their results are in agreement with the results of the boundary layer method.

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