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Relation between Polynomials Orthogonal with Respect to a Class of Gaussian Weight Functions

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Abstract

Polynomials of a three-dimensional vector $\boldsymbol{x}$, orthogonal with respect to a weight function $\sim\exp(-\frac{1}{2}x^2)$ and expressed in spherical polar coordinates, are called Burnett functions. The matrix relating the Burnett functions of $\boldsymbol{x} = \boldsymbol{\alpha}(\boldsymbol{c}-\boldsymbol{W})$ to those of $\boldsymbol{c}$ is constructed for the case where $\boldsymbol{\alpha}$ is a real nonsingular 3×3 matrix and $\boldsymbol{W}$ is a real vector. These matrices are infinite dimensional, block lower triangular representations of the group whose elements $(\boldsymbol{\alpha}, \boldsymbol{W})$ satisfy the composition law

$$(\boldsymbol{\alpha}', \boldsymbol{W}')(\boldsymbol{\alpha}, \boldsymbol{W}) = (\boldsymbol{\alpha}'\boldsymbol{\alpha}, \boldsymbol{W}+\boldsymbol{\alpha}^{-1}\boldsymbol{W}').$$

The physical origin of the problem in the kinetic theory of gases is briefly discussed.

Introduction

The primary motivation for this work is that the transformation formula derived here is needed in the calculation of matrix elements of the Boltzmann collision operator in a general basis (see the previous accompanying paper: Kumar 1980, present issue pp. 449–68). But the problem has independent mathematical interest and it may find applications elsewhere in view of the known relationship to the quantum mechanical oscillator problem. In particular, there is a connection between the transformations derived here and Talmi transformations discussed elsewhere (see e.g. Kumar 1966*a*, 1966*b*).

To understand the physical origin of the problem in the context of kinetic theory, consider the function $\bar{w}(x)$ defined by

$$\bar{w}(x) = (\det\boldsymbol{\alpha})\, w(x), \qquad w(x) = (2\pi)^{-3/2}\exp(-\tfrac{1}{2}x^2), \qquad \boldsymbol{x} = \boldsymbol{\alpha}(\boldsymbol{c}-\boldsymbol{W}). \quad \text{(1a, b, c)}$$

Here $\boldsymbol{c}$ and $\boldsymbol{W}$ are real three-dimensional vectors and $\boldsymbol{\alpha}$ is a real nonsingular 3×3 matrix. We have

$$\int \bar{w}(x)\,\mathrm{d}\boldsymbol{c} = 1, \qquad \int \boldsymbol{c}\,\bar{w}(x)\,\mathrm{d}\boldsymbol{c} = \boldsymbol{W}, \qquad \text{(2a, b)}$$

$$\int (\boldsymbol{c}-\boldsymbol{W})(\boldsymbol{c}-\boldsymbol{W})\,\bar{w}(x)\,\mathrm{d}\boldsymbol{c} = (\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha})^{-1}. \qquad \text{(2c)}$$

Hence $\bar{w}(x)$, as a function of $\boldsymbol{c}$, describes the velocity distribution of a gas of unit density, drifting with average velocity $\boldsymbol{W}$ and having a (kinetic) pressure tensor

$$\mathbf{P} = m(\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha})^{-1}, \qquad (3)$$

where m is the mass of the particles of the gas.

If a gas is known to have a drift velocity W and a pressure tensor $\mathbf{P}$ then one expects that a good representation of the velocity distribution function $f(\boldsymbol{c})$ of this gas will be in terms of polynomials $\phi^{(\nu)}(\boldsymbol{x})$ orthogonal with respect to a weight function $\bar{w}(x)$ in which the parameter $\boldsymbol{\alpha}$ is chosen to satisfy the relation (2c), with $\mathbf{P}$ and W given. Specifically, the following form is found useful:

$$f(\boldsymbol{c}) = \bar{w}(x) \sum_{\nu} \phi^{[\nu]}(\boldsymbol{x}) \mathfrak{F}^{(\nu)}, \tag{4}$$

where the $\mathfrak{F}^{(\nu)}$ are constants. Since the cross sections are expressed in terms of the velocity variables $\boldsymbol{c}$, in evaluating the collision integral with a distribution of this form it is required to make the $\boldsymbol{c}$ dependence explicit. The mathematical problem that emerges is that of expressing the polynomials $\phi^{(\nu)}(\boldsymbol{x})$ and the functions $\bar{w}(x)\,\phi^{(\nu)}(\boldsymbol{x})$ in terms of the corresponding quantities of the variable $\boldsymbol{c}$.

It may be noted that a real nonsingular matrix $\boldsymbol{\alpha}$ admits a unique decomposition in the form

$$\boldsymbol{\alpha} = \mathbf{R}_2 \mathbf{U} \mathbf{R}_1, \tag{5}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are orthogonal matrices (i.e. rotations) and $\mathbf{U}$ is a diagonal matrix. These matrices are obtained from the relations

$$\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}} = \mathbf{R}_2 \mathbf{U}^2 \mathbf{R}_2^{\mathrm{T}}, \qquad \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha} = \mathbf{R}_1^{\mathrm{T}} \mathbf{U}^2 \mathbf{R}_1. \tag{6}$$

It is then evident that, given a real symmetric positive-definite $\mathbf{P}$, the relation (3) determines $\boldsymbol{\alpha}$ only up to a rotation. It does not determine $\mathbf{R}_2$, which remains arbitrary.

If the coordinate axes for $\boldsymbol{c}$ are chosen to lie along the principal axes of the pressure tensor $\mathbf{P}$, then $\mathbf{P}$ becomes diagonal. If in addition one takes $\mathbf{R}_2$ to be the unit matrix then $\boldsymbol{\alpha}$ is also diagonal. Thus one can confine oneself to diagonal tensors $\mathbf{P}$ and $\boldsymbol{\alpha}$ in those physical situations where these choices do not entail other complications. For instance, this is the case for the motion of charged particles in a neutral gas, when the only external field present is the electric field. On the other hand, if one were to consider the interaction of two gases with differently oriented pressure tensors then only one of these can be described by a diagonal pressure tensor, since a common coordinate system is needed to describe the velocities in a collision. Also if other vectors, for example magnetic fields, are present then the rotations may bring in other complications in relation to those vectors.

One is thus lead to consider nondiagonal $\mathbf{P}$ and $\boldsymbol{\alpha}$. If one does not make the special assumption that $\mathbf{R}_2$ is equal to the unit matrix, then $\boldsymbol{\alpha}$ can be any real nonsingular matrix. This allows one to make use of the group property discussed below. It should be noted that the advantage of putting $\mathbf{R}_2 = \mathbf{I}$, or taking $\boldsymbol{\alpha}$ to be diagonal, appears only at the level of numerical calculations. There is no simplification at the formal level. In fact, the specialization entails a loss of symmetry which may be troublesome at the formal level.

Notation and Preliminaries

The notation used here for vectors and matrices is standard. All three-dimensional vectors are real. The unit vector in the direction of a vector $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$; thus $\hat{\boldsymbol{a}} = \boldsymbol{a}/a$. The cartesian components of $\boldsymbol{a}$ are (a_1, a_2, a_3) and its spherical components are

$$a_{\pm} = (\pm\mathrm{i}\,a_1 + a_2)/\sqrt{2}, \qquad a_0 = \mathrm{i}\,a_3, \tag{7}$$

For spherical harmonics $\mathrm{Y}_m^{(l)}$, rotation matrices $\mathfrak{D}_{mm'}^{(l)}$ and corresponding irreducible tensors we use the definitions and conventions of Fano and Racah (1959), except that Y is used in place of the Gothic $\mathfrak{Y}$. A spherical harmonic may be written as a sum over products of spherical components:

$$a^{2n+l}\,\mathrm{Y}_m^{[l]}(\hat{\boldsymbol{a}}) = \sum_{r=0}^{[\frac{1}{2}l]} A_{lmr}\, a^{2(n+r)}\, a_0^{l-2r-|m|}\, a_{\sigma(m)}^{|m|}. \tag{8a}$$

$$A_{lmr} = \left(\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right)^{\frac{1}{2}} \frac{(-)^{l-|m|}}{2^{l-\frac{1}{2}|m|}}\, \frac{(2l-2r)!}{r!\,(l-r)!\,(l-2r-|m|)!}, \tag{8b}$$

with $\sigma(m) = \mathrm{sgn}(m)$. Conversely, a product of spherical components may be written as a sum over spherical harmonics. Since we have

$$a_+^r\, a_-^s = (a_+\, a_-)^p\, a_{\sigma(m)}^{|m|}, \qquad 2p = r+s-|m|, \quad m = r-s, \tag{9}$$

and $a_+\, a_- = \frac{1}{2}(a^2+a_0^2)$ from equations (7), we need only the relation

$$a_0^{l-|m|}\, a_{\sigma(m)}^{|m|} = \sum_{r=0}^{[\frac{1}{2}l]} B_{lmr}\, a^l\, \mathrm{Y}_m^{[l-2r]}(\hat{\boldsymbol{a}}), \tag{10a}$$

$$B_{lmr} = (-)^r \frac{(l-|m|)!}{2^{\frac{1}{2}|m|}}\, \frac{(2l-4r+1)}{(2l-2r+1)!}\, \frac{(2r+1)\,!}{(2r+1)!!} \left(\frac{4\pi}{2l-4r+1}\, \frac{(l-2r+|m|)!}{(l-2r-|m|)!}\right)^{\frac{1}{2}}. \tag{10b}$$

We use the so-called plane wave expansion in the form of the identity

$$\exp(2\boldsymbol{a}.\boldsymbol{b}) = \sum_{v=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} N_{vl}^2\, \chi_m^{(vl)}(\boldsymbol{a})\, \chi_m^{[vl]}(\boldsymbol{b}), \tag{11a}$$

$$N_{vl}^2 = 2\pi^{3/2}\,\Gamma(v+1)/\Gamma(v+l+\tfrac{3}{2}), \tag{11b}$$

$$\chi_m^{(vl)}(\boldsymbol{a}) = \{(-)^v/v!\}\, a^{2v+l}\, \mathrm{Y}_m^{(l)}(\hat{\boldsymbol{a}}). \tag{11c}$$

The convention of parentheses and square brackets for indices is due to Fano and Racah (1959) and is summarized for the present purposes by

$$\mathrm{Y}_m^{[l]*} = \mathrm{Y}_m^{(l)}, \qquad \chi_m^{[vl]*} = \chi_m^{(vl)}. \tag{12}$$

The Burnett functions $\phi^{[\mathbf{v}]}(\boldsymbol{x})$ are defined by the generating function

$$G(\boldsymbol{a},\boldsymbol{x}) \equiv \exp(-a^2 + 2\boldsymbol{a}.\boldsymbol{x}/\surd 2) = \sum_{\mathbf{v}} N_{vl}\, \chi^{(\mathbf{v})}(\boldsymbol{a})\, \phi^{[\mathbf{v}]}(\boldsymbol{x}), \tag{13}$$

with $\mathbf{v} \equiv (v,l,m)$, i.e. the above sum is an abbreviation for the type of sum occurring in equation (11a). The orthogonality relation takes the form

$$\int w(x)\, \phi^{(\mathbf{v})}(\boldsymbol{x})\, \phi^{[\mathbf{v}']}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \delta_{\mathbf{v}\mathbf{v}'} \equiv \delta_{vv'}\, \delta_{ll'}\, \delta_{mm'}. \tag{14}$$

Note that from equations (1)

$$w(x)\, \mathrm{d}\boldsymbol{x} = \overline{w}(x)\, \mathrm{d}\boldsymbol{c}. \tag{15}$$

The explicit form of the Burnett functions is

$$\phi^{[\mathbf{v}]}(\boldsymbol{x}) \equiv \phi_m^{[vl]}(\boldsymbol{x}) = R_{vl}(x)\, \mathrm{Y}_m^{[l]}(\hat{\boldsymbol{x}}), \tag{16a}$$

$$R_{vl}(x) = N_{vl}(x/\surd 2)^l\, \mathrm{S}_{l+\frac{1}{2}}^{(v)}(\tfrac{1}{2}x^2), \tag{16b}$$

where the $S_{l+\frac{1}{2}}^{(\nu)}$ are Sonine polynomials. Under complex conjugation, parity operations and rotations, the polynomials (16a) transform as:

$$\phi_m^{[\nu l]*} = \phi_m^{(\nu l)}, \qquad \phi_m^{(\nu l)}(-x) = (-)^l \phi_m^{(\nu l)}(x), \tag{17a, b}$$

$$\phi_m^{(\nu l)}(\mathbf{R}x) = \sum_{m'=-l}^{l} \mathfrak{D}_{mm'}^{(l)}(\mathbf{R})\, \phi_{m'}^{(\nu l)}(x). \tag{17c}$$

All these properties arise from the spherical harmonic part in equation (16a), the radial part $R_{\nu l}$ not being affected by these transformations.

More information about the use of Burnett functions and irreducible tensors in kinetic theory may be found in earlier papers (Kumar 1966*a*, 1966*b*, 1967). The notation used here is consistent with these papers except for a slight change in the definition of the generating function (13). Equations (8) and (10) may be derived along the lines suggested by Kumar (1966*a*).

Group Structure and Transformation Matrix

If x and x' are related to c by equations of the form (1c) then they are related to each other by an equation of the same form. We write

$$x = (\alpha, W)\, x' \equiv \alpha(x' - W). \tag{18}$$

The α and W of this equation will, of course, have different values from those in equation (1c). The transformations (α, W), with α a real nonsingular matrix and W a real vector, constitute a group.† The unit element of this group is $(\mathbf{I}, 0)$ with $\mathbf{I}$ the unit matrix. The group multiplication follows from successive applications of equation (18):

$$(\alpha', W')(\alpha, W) = (\alpha'\alpha, W + \alpha^{-1} W'). \tag{19}$$

The inverse of the element (α, W) is $(\alpha^{-1}, -\alpha W)$. From equations (19) and (5) we have

$$(\alpha, W) = (\alpha, 0)(\mathbf{I}, W) = (\mathbf{R}_2, 0)(\mathbf{U}, 0)(\mathbf{R}_1, 0)(\mathbf{I}, W). \tag{20}$$

The element $(\mathbf{I}, W)$ is a translation, the $(\mathbf{R}_i, 0)$, $i = 1, 2$, are rotations and $(\mathbf{U}, 0)$ is a scaling of the axes. The parity operation is represented by $(-\mathbf{I}, 0)$ and is included in the group. The effect of the parity operation and rotations on the functions $\phi^{(\nu)}(x)$ are given by equations (17b) and (17c).

In view of the linearity of equation (18), the polynomials $\phi^{(\nu)}(x)$ are linearly related to the polynomials $\phi^{(\nu')}(x')$. The relationship is expressed by the transformation matrix $A(\alpha, W)$ as:

$$\phi^{(\nu)}(x) = \sum_{\nu'} [A(\alpha, W)]^{\nu\nu'} \phi^{(\nu')}(x'). \tag{21}$$

Successive transformations (18) lead to matrix multiplication of the corresponding A's on the left-hand side. The group multiplication is thus represented by matrix

† This is the group of affine transformations in three dimensions. Its geometric applications are well known. For the reasons given in the Introduction, we are interested in the connection between some specific polynomial systems rather than in this group or its representation theory.

multiplications and the matrices $A(\alpha, W)$ form a representation of the group defined by equation (19). Hence from equation (20) we have

$$A(\alpha, W) = D(\mathbf{R}_2)\, A(\mathbf{U}, 0)\, D(\mathbf{R}_1)\, A(\mathbf{I}, W), \tag{22}$$

where, in view of the relation (17c)

$$[A(\mathbf{R}_2, 0)]^{\mathbf{v}\mathbf{v}'} \equiv [D(\mathbf{R}_2)]^{\mathbf{v}\mathbf{v}'} = \delta_{\nu\nu'}\,\delta_{ll'}\,\mathfrak{D}^{(l)}_{mm'}(\mathbf{R}_2). \tag{23}$$

An integral representation of the matrix element is obtained by using the orthogonality relation (14) with integration over $\boldsymbol{c}$ according to equation (15):

$$[A(\alpha, W)]^{\mathbf{v}\mathbf{v}'} = \int \phi^{(\mathbf{v})}(\boldsymbol{x})\, \phi^{[\mathbf{v}']}(\boldsymbol{x}')\, \bar{w}(x')\, \mathrm{d}\boldsymbol{c}. \tag{24}$$

The inverse of equation (21) similarly gives

$$[A^{-1}(\alpha, W)]^{\mathbf{v}'\mathbf{v}} = \int \phi^{(\mathbf{v}')}(\boldsymbol{x}')\, \phi^{[\mathbf{v}]}(\boldsymbol{x})\, \bar{w}(x)\, \mathrm{d}\boldsymbol{c}. \tag{25}$$

Application of equation (21) to $\phi^{(\mathbf{v}')}(\boldsymbol{x}')$ with the inverse of (18) verifies the relation

$$A^{-1}(\alpha, W) = A(\alpha^{-1}, -\alpha W). \tag{26}$$

Also, using equations (25) and (15) one may verify the relation

$$\bar{w}(x)\, \phi^{[\mathbf{v}]}(\boldsymbol{x}) = \sum_{\mathbf{v}'} \bar{w}(x')\, \phi^{[\mathbf{v}']}(\boldsymbol{x}')\, [A^{-1}(\alpha, W)]^{\mathbf{v}'\mathbf{v}}. \tag{27}$$

Note that normalized weight functions (1a) are used. One multiplies both sides by $\phi^{(\mathbf{v})}(\boldsymbol{x}')$ and integrates with respect to $\boldsymbol{c}$, using the relations (14) and (15), to verify equation (25).

The polynomial $\phi^{(\mathbf{v})}(\boldsymbol{x})$ is of degree $2\nu + l$ in $\boldsymbol{x}$ and therefore also in $\boldsymbol{x}'$. It follows that on the right-hand side of equation (21) the coefficient of $\phi^{(\mathbf{v}')}(\boldsymbol{x}')$ must vanish whenever we have $2\nu' + l' > 2\nu + l$. That is to say, if the transformation matrix A is arranged in blocks labelled by $p = 2\nu + l$, $p' = 2\nu' + l'$, then all elements in the blocks with $p' > p$ must vanish. This also follows from equation (24) and the orthogonality properties of the polynomials. The matrix is thus block lower triangular. Note that the rotational part of the matrix, equation (23), always occurs in the diagonal blocks and is not a lower triangular matrix in m indices. The inverse matrix A^{-1} is also block lower triangular.

We are thus dealing with infinite dimensional, block lower triangular representations of the group in question. In the following sections we complete the construction of the transformation matrix (i.e. of this representation) by deriving explicit formulae for the scaling and translation matrices.

Scaling Transformation: the Matrix $A(\mathbf{U}, 0)$

The general formula for the matrix element is the main result of this section. We first state this formula and then give an outline of its derivation. This is followed by two subsections where simpler formulae for two special cases, most likely to be used in applications, are given.

In the general case

$$\mathbf{U} = \mathrm{diag}(u_1, u_2, u_3) \tag{28}$$

and all the u_i, $i = 1, 2, 3$, are different and nonvanishing. From the u_i we form the quantities

$$u_\pm = \tfrac{1}{2}(u_2 \pm u_1), \tag{29a}$$

$$v_1 = u_-^2 + u_+^2 = \tfrac{1}{2}(u_1^2 + u_2^2), \tag{29b}$$

$$v_2 = 2u_+ u_- = \tfrac{1}{2}(u_2^2 - u_1^2), \tag{29c}$$

$$v_3 = u_-^2 + u_+^2 - u_3^2 = \tfrac{1}{2}(u_1^2 + u_2^2 - 2u_3^2). \tag{29d}$$

In terms of these, the required matrix element for $(2v+l) > (2v'+l')$ is given by

$$\begin{aligned}[A(\mathbf{U}, \mathit{0})]^{vlm,v'l'm'} &= \frac{(-)^{v+v'+n}}{n!}\left(\frac{v!\,\Gamma(v+l+\frac{3}{2})}{v'!\,\Gamma(v'+l'+\frac{3}{2})}\right)^{\frac{1}{2}} \\ &\quad \times \sum (-)^{n'} \binom{n}{n'} \binom{s}{s_1, s_2, s_3, s_4} \binom{|m'|}{t} \binom{p}{q} 2^{-p} \\ &\quad \times v_1^{s_4} v_2^{(s_1+s_2)} v_3^{s_3} (u_{-\sigma(m')})^{|m'|-t} (u_{\sigma(m')})^t \\ &\quad \times u_3^{l'-2r'-|m'|} A_{l'm'r} B_{Lmr},\end{aligned} \tag{30}$$

where the symbol B_{Lmr} was defined by equation (10b), A_{lmr} by equation (8b) and the remaining two symbols are the binomial and multinomial coefficients

$$\binom{n}{n'} = \frac{n!}{n'!(n-n')!}, \qquad \binom{s}{s_1, s_2, s_3, s_4} = \frac{s!}{s_1!\,s_2!\,s_3!\,s_4!}, \tag{31}$$

with $s = s_1 + s_2 + s_3 + s_4$. All the subscripts and superscripts take the values $0, 1, 2, \ldots$. Of these the following are defined in terms of the others, and basically serve as abbreviations for longer expressions:

$$r = \tfrac{1}{2}(L-l), \qquad L = 2(s_1 + s_2 + s_3 - r' - q) + l', \tag{32a, b}$$

$$s = n' + v' + r', \qquad p = s_1 + s_2 + \tfrac{1}{2}(|m'| - |m|), \tag{32c, d}$$

$$2n = 2v + l - (2v' + l'). \tag{32e}$$

The sum in equation (30) is over the following ranges of the other indices:

$$n' \text{ from } 0 \text{ to } n, \qquad r' \text{ from } 0 \text{ to } [\tfrac{1}{2}l'], \tag{33a, b}$$

$$s_i \text{ from } 0 \text{ to } s = s_1 + s_2 + s_3 + s_4, \tag{33c}$$

$$t \text{ from } 0 \text{ to } |m'|, \qquad q \text{ from } 0 \text{ to } p. \tag{33d, e}$$

The values taken by s_1, s_2 and t are further restricted by the requirement

$$m + |m'| = 2(s_1 - s_2 + t). \tag{34}$$

This restriction also has the consequence that when $m+|m'|$ is odd no values of s_1, s_2 and t can satisfy the relation and the matrix element (30) must vanish. It also vanishes if $2v+l$ is less than $2v'+l'$, since in that case there is no allowable value of n satisfying equation (32e):

$$[A(\mathbf{U},0)]^{\mathbf{v},\mathbf{v}'} = 0 \quad \text{if} \quad m+|m'| \text{ is odd} \tag{35a}$$

$$\text{or} \quad \text{if} \quad (2v+l) < (2v'+l'). \tag{35b}$$

The statement of the formula for the matrix element is now complete. The observation (35) may also be understood by inspection from the integral formula below.

The derivation starts from equation (24). Without loss of generality we put $\boldsymbol{x} = \mathbf{U}\boldsymbol{c}$ and $\boldsymbol{x}' = \boldsymbol{c}$, so that

$$[A(\mathbf{U},0)]^{\mathbf{v},\mathbf{v}'} = \int \phi^{(\mathbf{v})}(\mathbf{U}\boldsymbol{c})\,\phi^{[\mathbf{v}']}(\boldsymbol{c})\,w(c)\,\mathrm{d}\boldsymbol{c}. \tag{36}$$

From the generating function of equation (13) this is equal to the coefficient of $N_{v'l'}\chi^{(\mathbf{v}')}(\boldsymbol{a})\,N_{vl}\chi^{[\mathbf{v}]}(\boldsymbol{b})$ in $\mathscr{I}$, where

$$\mathscr{I} = \int G(\boldsymbol{b},\mathbf{U}\boldsymbol{c})\,G(\boldsymbol{a},\boldsymbol{c})\,w(c)\,\mathrm{d}\boldsymbol{c} = \exp\{-(a^2+b^2-(\boldsymbol{a}+\boldsymbol{b}.\mathbf{U})^2)\}. \tag{37}$$

Introducing the vector

$$\boldsymbol{\beta} = (\boldsymbol{b}.\mathbf{U})/b = \hat{\boldsymbol{b}}.\mathbf{U}, \tag{38}$$

one then has

$$\mathscr{I} = \exp\{-b^2(1-\beta^2)+2b\boldsymbol{a}.\boldsymbol{\beta}\}. \tag{39}$$

One now uses equations (11) to expand the second part of the exponential followed by an expansion in powers of b. The required coefficient is obtained by inspection for the $\boldsymbol{a}$-dependent part and powers of b, and by using the orthogonality of spherical harmonics of $\hat{\boldsymbol{b}}$. At this stage one has

$$[A(\mathbf{U},0)]^{\mathbf{v},\mathbf{v}'} = \frac{(-)^{n+v+v'}\,v!\,N_{v'l'}}{n!\,v'!\,N_{vl}} \int \mathrm{Y}_m^{(l)}(\hat{\boldsymbol{b}})\,(1-\beta^2)^n\,\chi^{[\mathbf{v}']}(\boldsymbol{\beta})\,\mathrm{d}\hat{\boldsymbol{b}}, \tag{40}$$

where the integration is over the solid angles of the vector $\boldsymbol{b}$, and n is given by equation (32e). All the complication arises from the need to express $(1-\beta^2)^n\chi^{[\mathbf{v}']}(\boldsymbol{\beta})$ in terms of the spherical harmonics of $\hat{\boldsymbol{b}}$.

The first binomial in equation (30) arises from expanding $(1-\beta^2)^n$. One then uses equations (8) to express the entire $\boldsymbol{\beta}$ dependence in powers of β, β_0 and $\beta_{\sigma(m')}$, where $\sigma(m')$ is the sign of m'. Using equations (38) and (29) one has

$$\begin{pmatrix}\beta_+\\ \beta_-\\ \beta_0\end{pmatrix} = \begin{pmatrix}u_+ & u_- & 0\\ u_- & u_+ & 0\\ 0 & 0 & u_3\end{pmatrix}\begin{pmatrix}\hat{b}_+\\ \hat{b}_-\\ \hat{b}_0\end{pmatrix}, \tag{41a}$$

$$\beta_{\sigma(m)} = (u_{\sigma(m)}\hat{b}_+ + u_{-\sigma(m)}\hat{b}_-), \tag{41b}$$

$$\beta^2 = v_1 + v_2(\hat{b}_+^2+\hat{b}_-^2) + v_3\hat{b}_0^2. \tag{41c}$$

The multinomial in equation (30) arises from β^{2s}, and the second binomial from $\beta_{\sigma(m)}^{|m|}$. One now has a sum over products of powers of b_0, b_+ and b_-. These are converted to sums over spherical harmonics using equations (9) and (10). The evaluation of the integral in equation (40) then completes the derivation of the result (30). The definitions (32a)–(32d) and the restrictions (33) and (34) develop in carrying out these steps. The normalization factors are simplified using equation (11b).

We now turn to the two special cases which are needed for applications to the problems in swarm experiments (see Kumar 1980).

(i) *Isotropic Case*

In this case all the axes are scaled by the same factor and the matrix **U** is a multiple of unity:

$$\mathbf{U} = u\mathbf{I}. \tag{42}$$

We have $\boldsymbol{x} = u\boldsymbol{c}$, so that the spherical harmonic part of the Burnett function is unaffected (equations 16). The matrix may be evaluated using equation (21) and the scaling property of the Sonine (i.e. associated Laguerre) polynomials (see p. 192 of Erdélyi *et al.* 1953):

$$[A(u\mathbf{I},0)]^{\nu,\nu'} = [A_l(u)]^{\nu\nu'}\delta_{ll'}\delta_{mm'}, \tag{43a}$$

$$[A_l(u)]^{\nu\nu'} = \frac{1}{(\nu-\nu')!}\left(\frac{\nu!\,\Gamma(\nu+l+\frac{3}{2})}{\nu'!\,\Gamma(\nu'+l+\frac{3}{2})}\right)^{\frac{1}{2}} u^{2\nu'+l}\,(1-u^2)^{\nu-\nu'} \quad \text{for} \quad \nu \geqslant \nu', \tag{43b}$$

$$= 0 \qquad\qquad \nu < \nu'. \tag{43c}$$

This may also be verified from equation (30) by noting that in the present case

$$u_+ = u_3 = u, \qquad u_- = 0, \qquad v_1 = u^2, \qquad v_2 = v_3 = 0. \tag{44}$$

The vanishing of these constants restricts the sum so that we have $s_1 = s_2 = s_3 = p = q = 0$, $L = l'-2r'$ and $2r = l'-l-2r'$. The restriction $m = m'$ emerges from here, and $l = l'$ from the relation

$$\sum_{r'} A_{l'm'r'}\,B_{l'-2r',m',r} = \delta_{ll'}. \tag{45}$$

Finally, the sum over n' leads to the results (43).

(ii) *Cylindrically Symmetric Case*

When there is symmetry about the 3 axis (i.e. the z axis) we have

$$\mathbf{U} = \operatorname{diag}(u,u,u_0), \tag{46}$$

so that in equation (30)

$$u_1 = u_2 = u_+ = u, \qquad u_- = 0, \qquad u_3 = u_0, \tag{47a}$$

$$v_1 = u^2, \qquad v_2 = 0, \qquad v_3 = u^2-u_0^2. \tag{47b}$$

Since $v_2 = 0$ we have $s_1 = s_2 = 0$ and, from the condition (34), $m+|m'| = 2t$. If $\sigma(m')$ is plus, $u_- = 0$ implies $|m'| = t$ so that $m = m'$. If $\sigma(m')$ is minus, $u_- = 0$

implies $t = 0$ so that $m = -|m'| = m'$. From the condition (32d) then $p = q = 0$ and equation (30) simplifies to

$$[A(\mathbf{U},0)]^{v,v'} = \delta_{mm'}\frac{(-)^{v+v'+n}}{n!}\left(\frac{v!\,\Gamma(v+l+\frac{3}{2})}{v'!\,\Gamma(v'+l'+\frac{3}{2})}\right)^{\frac{1}{2}}$$

$$\times\sum(-)^{n'}\binom{n}{n'}\binom{s}{s'}(u^2-u_0^2)^{s-s'}u_0^{l'-2r'-|m'|}u^{2s+|m|}A_{l'm'r'}B_{Lmr}. \quad (48)$$

where $L = 2(s'-r')$, $2r = L-l$, $s = n'+v'+r'$ and the summation goes over n' from 0 to n, over r' from 0 to $[\frac{1}{2}l']$ and over s' from 0 to s. The index n has the fixed value given by $2n = 2v+l-2v'-l'$. The matrix element vanishes for $n < 0$.

Translations: the Matrix $A(\mathbf{I}, \boldsymbol{W})$

With $\boldsymbol{x} = \boldsymbol{c}-\boldsymbol{W}$ and $\boldsymbol{x}' = \boldsymbol{c}$, equation (21) becomes

$$\phi^{(v)}(\boldsymbol{c}-\boldsymbol{W}) = \sum_{v'}[A(\mathbf{I},\boldsymbol{W})]^{vv'}\phi^{(v')}(\boldsymbol{c}). \quad (49)$$

The integral representation is

$$[A(\mathbf{I},\boldsymbol{W})]^{vv'} = \int\phi^{(v)}(\boldsymbol{c}-\boldsymbol{W})\,\phi^{[v']}(\boldsymbol{c})\,w(c)\,\mathrm{d}\boldsymbol{c}. \quad (50)$$

From the generating function of equation (13) this is equal to the coefficient of $N_{v'l'}\chi^{(v')}(\boldsymbol{a})\,N_{vl}\chi^{(v)}(\boldsymbol{b})$ in $\mathscr{I}'$, where

$$\mathscr{I}' = \int G(\boldsymbol{b},\boldsymbol{c}-\boldsymbol{W})\,G(\boldsymbol{a},\boldsymbol{c})\,w(c)\,\mathrm{d}\boldsymbol{c} = \exp(2\boldsymbol{a}.\boldsymbol{b}+\sqrt{2}\,\boldsymbol{b}.\boldsymbol{W}). \quad (51)$$

The required coefficient is found by using equation (11a) and the addition theorem of spherical harmonics to express the product of the two $\chi(\boldsymbol{b})$'s in terms of one $\chi(\boldsymbol{b})$. The result for $(2v+l) \geqslant (2v'+l')$ is

$$[A(\mathbf{I},\boldsymbol{W})]^{vv'} = \sum_{l''=|l-l'|}^{l+l'}\frac{(-)^{v+v'+v''}v!\,N_{v''l''}^2}{v'!\,v''!\,N_{vl}N_{v'l'}}$$

$$\times\sigma(l'l''l)(l'm'l''m''\,|\,lm)\,\chi^{(v'')}(\boldsymbol{W}/\sqrt{2}), \quad (52\text{a})$$

where $m'' = m'-m$, $2v'' = 2v+l-2v'-l'-l''$, $(l'm'l''m''\,|\,lm)$ is a Wigner coefficient and

$$\sigma(l'l''l) = \mathrm{i}^{l'+l''-l}\{(2l'+1)(2l''+1)/4\pi(2l+1)\}^{\frac{1}{2}}(l'0l''0\,|\,l0). \quad (52\text{b})$$

These coefficients arise from the addition theorem of spherical harmonics. The matrix element vanishes for $(2v+l) < (2v'+l')$.

Some Properties and Relation to Talmi Coefficients

The purpose of this section is to derive alternative formulae for the scaling and translation matrices (equations 56 and 58 below), in terms of Talmi transformation coefficients, which have been more widely studied and used earlier in kinetic theory work (see Kumar 1966*a*, 1966*b*; 1967). The connection pointed out here has roots in the overlap between the underlying group structures. From equation (31) of Kumar (1967) we have

$$\phi^{(\mathbf{v}_1)}(\alpha_1 \mathbf{c}_1) = \sum_{N,\mathbf{v}} T\big((\Gamma)N,(\gamma)\mathbf{v}\,|\,(\alpha_1)\mathbf{v}_1,(\alpha_2)\mathbf{0}\big)\,\phi^{(N)}(\Gamma \mathbf{G})\,\phi^{(\mathbf{v})}(\gamma \mathbf{g}), \tag{53a}$$

where $T(...)$ is the Talmi coefficient,

$$\mathbf{c}_1 = \mathbf{G}+(\alpha_2/\Gamma)^2\mathbf{g}, \tag{53b}$$

and the quantities Γ and γ are defined in terms of the parameters α_1 and α_2, which can take arbitrary positive values:

$$\Gamma^2 = \alpha_1^2+\alpha_2^2, \qquad \gamma^2 = (\alpha_1\alpha_2/\Gamma)^2. \tag{53c}$$

Comparing equation (53a) with (21) and making the necessary identifications, we find

$$[A(\alpha_1 \mathbf{I}, \mathbf{W})]^{\mathbf{v}_1\mathbf{v}_1'} = \sum_{N,\mathbf{v}} T\big((\Gamma)N,(\gamma)\mathbf{v}\,|\,(\alpha_1)\mathbf{v}_1,(\alpha_2)\mathbf{0}\big) \times[A(\Gamma\mathbf{I},\mathbf{0})]^{N\mathbf{v}_1'}\,\phi^{(\mathbf{v})}\big(-(\alpha_1\Gamma/\alpha_2)\mathbf{W}\big), \tag{54}$$

or, using the inverse and composition relations for the A matrices,

$$[A\big((\alpha_1/\Gamma)\mathbf{I},\mathbf{W}\big)]^{\mathbf{v}_1 N} = \sum_{\mathbf{v}} T\big((\Gamma)N,(\gamma)\mathbf{v}\,|\,(\alpha_1)\mathbf{v}_1,(\alpha_2)\mathbf{0}\big)\,\phi^{(\mathbf{v})}\big(-(\alpha_1\Gamma/\alpha_2)\mathbf{W}\big). \tag{55}$$

An interesting feature of this formula is the way it depends on the two parameters α_1 and α_2.

The isotropic scaling matrix is obtained from above by putting $\mathbf{W} = \mathbf{0}$:

$$[A\big((\alpha_1/\Gamma)\mathbf{I},\mathbf{0}\big)]^{\mathbf{v}_1 N} = \sum_{\mathbf{v}} T\big((\Gamma)N,(\gamma)\mathbf{v}\,|\,(\alpha_1)\mathbf{v}_1,(\alpha_2)\mathbf{0}\big)\,\phi^{(\mathbf{v})}(\mathbf{0}). \tag{56}$$

If the explicit form for the T coefficient is used along with

$$\phi^{(\mathbf{v})}(\mathbf{0}) = \delta_{l0}\,\delta_{m0}\sqrt{(4\pi)}\,N_{\nu 0}^{-1}, \tag{57}$$

one can recover equations (43).

Putting $\alpha_1 = 1$ in equation (54), one obtains an alternative to equation (52a) for the translation matrix:

$$[A(\mathbf{I}, \mathbf{W})]^{\mathbf{v}_1\mathbf{v}_1'} = \sum_{N,\mathbf{v}} T\big((\Gamma)N,(\gamma)\mathbf{v}\,|\,(\alpha_1)\mathbf{v}_1,(\alpha_2)\mathbf{0}\big) \times[A(\Gamma\mathbf{I},\mathbf{0})]^{N_1\mathbf{v}_1'}\,\phi^{(\mathbf{v})}\big(-(\Gamma/\alpha_2)\mathbf{W}\big). \tag{58}$$

Putting $\mathbf{x}' = \mathbf{W} = 0$ in equation (21), we observe that

$$\phi^{(\mathbf{v})}(0) = \sum_{\mathbf{v}'} [A(\boldsymbol{\alpha},\mathbf{0})]^{\mathbf{v}\mathbf{v}'}\,\phi^{(\mathbf{v}')}(\mathbf{0}). \tag{59}$$

This shows that this particular A matrix for any $\boldsymbol{\alpha}$ has a unit eigenvalue, with $\phi^{(\mathbf{v})}(\mathbf{0})$ the corresponding eigenfunction. Equation (56) is then unaltered by an insertion of the form (59), that is,

$$[A\big((\alpha_1/\Gamma)\mathbf{I},\mathbf{0}\big)]^{\mathbf{v}_1 N} = \sum_{\mathbf{v}\mathbf{v}'} T(...)\,[A(\boldsymbol{\alpha}',\mathbf{0})]^{\mathbf{v}\mathbf{v}'}\,\phi^{(\mathbf{v}')}(0). \tag{60}$$

This is a nontrivial alteration of the sum. In particular, the left-hand side must always vanish for $(2\nu_1+l_1) < 2N+L$ for arbitrary $\boldsymbol{\alpha}'$.

Summary

We have the system of polynomials $\{\phi^{(\nu)}(\boldsymbol{x})\}$, defined by equations (13) and (16), orthogonal with respect to the weight function $\bar{w}(x)$ defined by the equations (1). The transformation matrix $A(\boldsymbol{\alpha}, \boldsymbol{W})$ relating different polynomial systems, equations (21) and (27), has the integral representation (24). With the decomposition of $\boldsymbol{\alpha}$ given by equation (5) the matrix A can be written as a product of rotation, scaling and translation matrices by equation (22).

The general formula for the scaling matrix is equation (30). It simplifies for the isotropic and cylindrically symmetric cases which are useful in applications: equations (43) and (48).

The translation matrix is given by equation (52a).

Alternative formulae for the isotropic scaling matrix and translation matrix are given by equations (56) and (58).

The physical origin of the problem in kinetic theory was given in the introduction. Other points are summarized in the titles of sections.

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