

On the Relationship between Quantum, Random and Semiclassical Electrodynamics

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Abstract

We investigate a form of random electrodynamics in which random electromagnetic fields are used in conjunction with a semiclassical treatment of the Dirac equation. We demonstrate that the S -matrix expansion of this theory is identical with the quantum electrodynamic expansion. We show how one can determine for a particular process whether or not a purely semiclassical calculation will suffice. In particular we show that the Lamb shift needs for its explanation both semiclassical interactions and random field fluctuations.

1. Introduction

Quantum electrodynamics (QED) yields remarkably accurate predictions for a host of phenomena, and in fact there is no real evidence for any discrepancy between its predictions and experiment. However, in order to arrive at a numerical result there are a large number of difficulties to be overcome by way of divergent and/or ambiguous expressions. The practical solution of these problems through renormalization is well known, as are the efforts to construct a more satisfactory theoretical edifice based on an axiomatic 'already renormalized' approach.

The situation has led to the development of various semiclassical theories based on a classical representation of the electromagnetic (EM) field and a quantum description of the atomic matter with which it interacts (Milonni 1976 and references therein). Indeed, some of this theory preceded QED since many calculations involving the interaction of radiation and matter were first carried through using semiclassical ideas. Most semiclassical calculations are free from the divergence problems of QED since the classical EM field due to a finite and continuous distribution of sources is free from these problems. However, there are significant differences between semiclassical and quantum theories; in particular Jaynes's 'neo-classical' theory (Crisp and Jaynes 1969; Stroud and Jaynes 1970) reaches conclusions which differ from those of QED in situations which are experimentally accessible. Despite this, semiclassical theories must continue to be important in many areas of application, because they lead more readily to viable approximation schemes. Consequently, the validity of semiclassical theories and their relationship to QED are matters of current interest.

One of the essential features of the quantum theory of radiation is its prediction of a fluctuating zero-point field of energy density $\frac{1}{2}\hbar\omega$ per mode. In a series of papers

Boyer (1969, 1975 and references therein) has investigated the consequences of using classical electrodynamics but taking a random field as the vacuum solution of Maxwell's equations. This random field is constructed by assuming that each mode of the solution to the homogeneous equations has a random phase which is uniformly distributed over the interval $[0, 2\pi]$. The various modes are uncorrelated. Such a theory correctly accounts for the blackbody radiation spectrum (Marshall 1965; Boyer 1969; Theimer 1971) which is generally held to be key evidence for the quantum nature of light. Indeed, by examining the 'Brownian motion' of classical point particles acted on by these random fields, several authors have developed a theory of 'random electrodynamics' and have offered derivations of the Schrödinger (Kershaw 1964; Nelson 1966; De La Peña-Auerbach 1967; Bess 1973; De La Peña-Auerbach and Cetto 1977, 1978) and Dirac (Bess 1979) equations.

Random fields have played an important role in the theory of Van der Waals forces since the pioneering work of Lifshitz (1956) and Casimir (1956, 1967). For example, to investigate the interaction between two dielectric bodies, one calculates the mutual response of the bodies to the random fields using classical or semiclassical methods, and from this the force between the bodies is extracted as that part which results from the mutual correlation of the response due to the proximity of the bodies. In practice, the information is obtained by subtracting the energy of each of the two bodies calculated in the absence of the other, all of the quantities being infinite: a process which is called renormalization in quantum electrodynamics. All of these calculations may be done using the standard methods of statistical mechanics, and one of us has shown (Davies 1971, 1972) that the two approaches are completely equivalent. The essential observation which we want to make is that the semiclassical interactions and the random fields are both needed in order to arrive at the correct conclusions.

In this paper we investigate the same basic ideas applied to problems which are commonly considered to be the domain of QED. By incorporating both the semiclassical interactions and the vacuum random field in a semiclassical framework, we show that many of the results of QED may be recovered. We use the Dirac equation for the description of electrons and positrons, and Maxwell's equations for the EM fields. Neither of the fields is second quantized. In the case of the Dirac equation, this fact is irrelevant, provided we are content to use the language of particle-hole theory and apply the appropriate antisymmetrization to the wavefunctions. For the EM field, we take the vacuum solution of Maxwell's equations to be a nonzero random field, while still regarding the fields themselves as classical. We offer no suggestion regarding the origin of the random fields, indeed it is perfectly consistent with our approach to assume that they are a manifestation of the ultimate quantum nature of the EM field. The semiclassical theory developed here is thus aimed at elucidating the validity (or otherwise) of the various semiclassical arguments which are common in practical calculations and associated questions of interpretation, and at showing how and when semiclassical arguments may be confidently used in place of QED.

In order to reduce the length of our development we will use the notation and methods of Sakurai (1967) and Akhiezer and Berestetskii (1965) including the use of natural units in conjunction with (rationalized) Heaviside-Lorentz units; reference to these two books will be abbreviated as S or AB respectively.

2. Basic Equations

In Hamiltonian form the Dirac equation is (S, p. 81)

$$i \partial \psi / \partial t = H \psi, \quad (1)$$

where

$$H = \alpha \cdot (\mathbf{p} - e\mathbf{A}) + e\phi + \beta m, \quad (2)$$

and we will use the standard representation

$$\alpha = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (3)$$

In a fully quantized calculation the EM field operators are split into external fields and interactions. Here the fields are not operators and we will split them into three parts: external, interaction and random. The usual formulation of the random fields is in Coulomb gauge although their physical manifestation is Lorentz invariant so that we will want to work in Lorentz gauge whenever possible. Whichever gauge is in use, the potentials are generated by the sources which are obtained from solutions of the Dirac equation in the usual way as

$$\rho = e\psi^\dagger \psi, \quad \mathbf{j} = e\psi^\dagger \boldsymbol{\alpha} \psi. \quad (4)$$

A covariant form of these equations is obtained by writing

$$x_4 = it, \quad A_4 = i\phi, \quad j_4 = i\rho, \quad (5a)$$

$$\gamma_4 = \beta, \quad \gamma_k = -i\beta\alpha_k, \quad (5b)$$

which yields

$$\{\gamma_\mu (\partial / \partial x_\mu - ieA_\mu) + m\} \psi = 0, \quad (6a)$$

$$j_\mu = -ie\bar{\psi} \gamma_\mu \psi. \quad (6b)$$

The coupled Maxwell–Dirac equations are nonlinear and we must obviously resort to approximation for their treatment. However, there are some general properties which we wish to point out explicitly since it is certainly invalid to attempt an approximation scheme which violates a general property of the exact solutions.

Energy Conservation (Glassey and Strauss 1979)

First note that the Dirac Hamiltonian is time-dependent and therefore not energy conservative. However, this is simply due to the fact that energy may be exchanged with the EM field which is not a closed system whenever we allow incoming and outgoing radiation. The EM fields are not free fields, so the appropriate energy density for them is (S, p. 302)

$$\frac{1}{2}(E^2 + B^2) - \rho\phi + \text{div}(\phi\mathbf{E}). \quad (7)$$

It is well known that

$$\partial \left\{ \frac{1}{2}(E^2 + B^2) \right\} / \partial t = -\text{div} \mathbf{P}_{\text{em}} + \mathbf{j} \cdot \mathbf{E}, \quad (8)$$

where \mathbf{P}_{em} is the Poynting vector. The rate of change of the expectation value of the Dirac Hamiltonian is easily calculated as

$$-e \int d\mathbf{r} \psi^\dagger \left(\boldsymbol{\alpha} \cdot \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \phi}{\partial t} \right) \psi = - \int d\mathbf{r} \left(\mathbf{j} \cdot \frac{\partial \mathbf{A}}{\partial t} - \rho \frac{\partial \phi}{\partial t} \right). \quad (9)$$

Thus, if we take for our integration region a large volume V which completely contains the particles, we find that

$$\begin{aligned} & \frac{d}{dt} \int_V d\mathbf{r} \left\{ \frac{1}{2}(E^2 + B^2) - \rho\phi + \psi^\dagger H\psi \right\} \\ &= - \int_S \mathbf{P}_{em} \cdot d\mathbf{S} - \int_V d\mathbf{r} \left(\mathbf{j} \cdot \text{grad } \phi - \phi \frac{\partial \rho}{\partial t} \right) = - \int_S \mathbf{P}_{em} \cdot d\mathbf{S}, \quad (10) \end{aligned}$$

which tells us that the energy of the coupled system is only altered through the influence of incoming and outgoing radiation. We think it important to point this out explicitly because of the difficulties which are commonly ascribed to semiclassical theories when these facts are forgotten. For example, the excitation of an atomic system from one energy level to a higher one must be accompanied by the donation of energy from incoming radiation, and this radiation is modified by the transition currents which act as sources in d'Alembert's equation.

Time Reversal

Under time reversal, we expect that charge densities are unaltered while current densities should change sign. Again, the vector potential changes sign but not the scalar potential. These changes will be accommodated into our full set of equations provided we make a change of representation in the Dirac wavefunction. Consequently, we define time reversed quantities as follows:

$$\rho'(\mathbf{r}, t) = \rho(\mathbf{r}, -t), \quad \mathbf{j}'(\mathbf{r}, t) = -\mathbf{j}(\mathbf{r}, -t), \quad \phi'(\mathbf{r}, t) = \phi(\mathbf{r}, -t), \quad (11a)$$

$$A'(\mathbf{r}, t) = -A(\mathbf{r}, -t), \quad \psi'(\mathbf{r}, t) = \alpha_1 \alpha_3 \psi^*(\mathbf{r}, -t). \quad (11b)$$

Looking first at the Dirac equation (1), if we take the complex conjugate, replace t by $-t$ and premultiply by $\alpha_1 \alpha_3$, we find that ψ' satisfies the same equation as ψ , except that the potentials are replaced by the time reversed ones. The essential point, however, is that when we substitute the time reversed wavefunction into the expressions (4) for the charge and current densities, we have

$$\psi'^\dagger \psi' = \psi^\dagger \psi, \quad \psi'^\dagger \boldsymbol{\alpha} \psi' = -\psi^\dagger \boldsymbol{\alpha} \psi. \quad (12)$$

Thus the change in representation which is brought about by time reversal has no effect on the observable source densities apart from the desired change of sign of the current.

S-matrix Expansion

We will also use the familiar S-matrix expansion (S, p. 184),

$$S = 1 - i \int_{-\infty}^{\infty} dt_1 H_1(t_1) - \int_{-\infty}^{\infty} dt_1 H_1(t_1) \int_{-\infty}^{t_1} dt_2 H_1(t_2) + \dots, \quad (13)$$

where the Dirac equation is written in the interaction representation and the wavefunction ψ is replaced by $\exp(iH_0 t)\psi$:

$$i \partial\psi/\partial t = H_1 \psi. \tag{14}$$

For an electromagnetic interaction, H_1 is given by

$$H_1 = -ie \int \bar{\psi} \gamma_\mu \psi A_\mu d^3r, \tag{15}$$

in the usual way (S, p. 185).

The free-particle solutions to the Dirac equation are (S, p. 91ff)

$$\psi_p^{(s)}(\mathbf{r}) = (m/EV)^{\frac{1}{2}} u^{(s)}(\mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{x}), \tag{16}$$

where $\mathbf{p} \cdot \mathbf{x} = p_\mu x_\mu$, $p_\mu = (\mathbf{p}, iE)$, the $x_\mu = (\mathbf{r}, it)$ are four-vectors and E can be both positive and negative.

3. Random Fields

The transverse EM potentials for the random fields are expanded in plane waves following the procedure of Boyer (1975), except that we use the notation of Akhiezer and Berestetskii (AB, p. 6). We write

$$A(\mathbf{r}, t) = \int d\mathbf{k} (16\pi^3 \omega)^{-\frac{1}{2}} \exp(i\mathbf{k} \cdot \mathbf{r}) \{ -i\mathbf{f}(\mathbf{k}, t) + i\mathbf{f}^*(-\mathbf{k}, t) \}, \tag{17}$$

where $\omega = |\mathbf{k}|$. Also we note that the energy of the fields is

$$\int d\mathbf{k} \omega \mathbf{f}^*(\mathbf{k}, t) \cdot \mathbf{f}(\mathbf{k}, t), \tag{18}$$

where we have used the usual conversion factor to replace the integral as a sum over modes. We require the energy to be $\frac{1}{2}\omega$ per mode, and consequently we must write

$$\mathbf{f}(\mathbf{k}) = i(V/16\pi^3)^{\frac{1}{2}} \sum_\lambda \mathbf{e}_{k\lambda} \exp(-i\omega t + i\theta_{k\lambda}), \tag{19}$$

where $\mathbf{e}_{k\lambda}$ is the polarization vector labelled by λ . The phases $\theta_{k\lambda}$ are random variables uniformly distributed over the interval $[0, 2\pi]$ with no mutual correlation between them. We will need the following relations:

$$\langle \exp(\pm i\theta_{k\lambda}) \exp(\mp i\theta_{k'\lambda'}) \rangle = \begin{cases} \delta_{\lambda\lambda'} \delta_{kk'} \\ (2\pi)^3 V^{-1} \delta_{\lambda\lambda'} \delta(\mathbf{k} - \mathbf{k}') \end{cases} \tag{20a}$$

$$\langle \exp(\pm i\theta_{k\lambda}) \exp(\pm i\theta_{k'\lambda'}) \rangle = 0, \tag{20b}$$

$$\sum_{\lambda=1,2} (\mathbf{e}_{k\lambda})_i (\mathbf{e}_{k\lambda})_j = \delta_{ij} - k_i k_j / \omega^2, \tag{20c}$$

which differ from those given by Boyer only in our choice of normalization. Note that if we introduce the coefficients $a_{k\lambda} = \sqrt{\frac{1}{2}} \exp(i\theta_{k\lambda})$ and write $kx = \mathbf{k} \cdot \mathbf{r} - \omega t$, then the random field has the expansion

$$A(\mathbf{r}, t) = (V)^{-\frac{1}{2}} \sum_{k\lambda} (2\omega)^{-\frac{1}{2}} \mathbf{e}_{k\lambda} \{ \exp(i\mathbf{k} \cdot \mathbf{x}) a_{k\lambda} + \exp(-i\mathbf{k} \cdot \mathbf{x}) a_{k\lambda}^* \}, \tag{21}$$

which is identical with that used in QED (AB, p. 159). The essential difference, however, is that the vacuum expectation values of aa^* and a^*a are $\frac{1}{2}$ in the present case while they are 1 and 0 respectively in QED.

The average of any component of the random field is

$$\langle A_i(\mathbf{r}, t) \rangle = 0, \quad (22)$$

so that they will not contribute to any process in which they occur to only first order of perturbation theory. For terms of second order in the random fields, we will need the correlation function

$$\begin{aligned} &\langle A_i(\mathbf{r}, t) A_j(\mathbf{r}', t') \rangle \\ &= \frac{\pi}{2(2\pi)^4} \int \frac{d\mathbf{k}}{\omega} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right) \left(\exp\{i\omega(t-t') - i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')\} + \text{c.c.} \right), \end{aligned} \quad (23)$$

which is readily expressed in four-integral form as

$$\frac{1}{2i(2\pi)^4} \int_{C_1} d^4k \frac{\exp\{ik(x-x')\}}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right) = \frac{1}{2} D_{ij}^{(1)}(x-x'), \quad (24)$$

where the contour C_1 is shown in Fig. 1a. We will see later that this correlation function is precisely what we will need to convert the transverse Green functions which account for semiclassical interactions into the usual photon propagators of QED. These Green functions are defined by

$$D_{ij}^{A,R}(x) = \frac{1}{(2\pi)^4} \int_{C_A, C_R} d^4k \frac{\exp(ikx)}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right), \quad (25)$$

where the contours C_A, C_R for k_0 are shown in Fig. 1b. It is obvious from the definition of the various functions in equations (24) and (25) that the combination

$$D_{ik}^C = \frac{1}{2} \{ D_{ik}^{(1)} - iD_{ik}^A - iD_{ik}^R \} \quad (26)$$

has the same integral representation with a different contour. For future reference we give the explicit form,

$$D_{ij}^C(x) = \frac{1}{i(2\pi)^4} \int_C d^4k \frac{\exp(ikx)}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right), \quad (27)$$

where the contour C is shown in Fig. 1c.

It is interesting to note that the correlation function (24) is, in QED, the vacuum expectation value

$$\langle \bar{N}(A_i^\dagger(x) A_j^\dagger(x')) - N(A_i^\dagger(x) A_j^\dagger(x')) \rangle_0 \quad (28)$$

(see AB, p. 181ff), where N stands for normal ordering and \bar{N} for antinormal ordering. Ackerhalt *et al.* (1973), considering the physical basis for the radiative shifts of atomic energy levels, found that changing from a symmetric ordering scheme (half normal and half antinormal) to a normal ordering scheme was equivalent to changing from radiation reaction to random fields as the physical basis. The difference between the two kinds of ordering is precisely $\bar{N} - N$ as given by the expectation value (28).

We will also need the Fourier transform of the correlation function which must be read from equation (23) rather than (24) if we want to express it in terms of δ functions. Rewriting equation (23) as

$$\langle A_i(0) A_j(x) \rangle = \frac{1}{(2\pi)^4} \int d^4 k \exp(ikx) \frac{\pi}{2\omega} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right) \left(\delta(k_0 + \omega) + \delta(k_0 - \omega) \right), \quad (29)$$

we find that the required Fourier transform is

$$\frac{\pi}{2\omega} \left(\delta_{ij} - \frac{k_i k_j}{\omega^2} \right) \left(\delta(k_0 + \omega) + \delta(k_0 - \omega) \right). \quad (30)$$

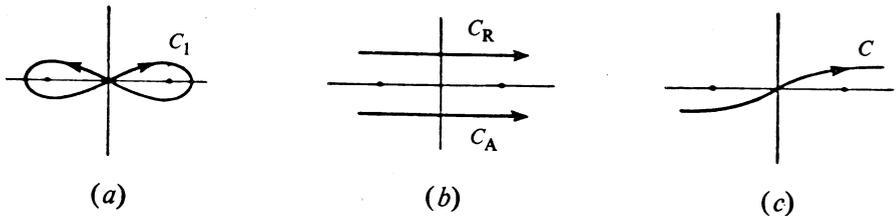


Fig. 1. Contours of integration in the complex k_0 plane.

4. Semiclassical Theory

There is no difficulty in using the Dirac equation in unquantized form, as is shown in detail by Sakurai (S, pp. 231–41). We simply split the Hamiltonian into a soluble part H_0 and a perturbation. For the present we assume that H_0 is the free-particle Hamiltonian, although it makes little difference in principle if it is an atomic Hamiltonian. Following Sakurai, we split the Dirac equation as

$$(\gamma_\mu \partial/\partial x_\mu + m)\psi = ie\gamma_\mu A_\mu \psi, \quad (31)$$

and then introduce the Green function which satisfies

$$(\gamma_\mu \partial/\partial x_\mu + m)K(x - x') = -i\delta^{(4)}(x - x'). \quad (32)$$

The only boundary conditions which are appropriate for a proper physical description of particle–antiparticle states lead to the electron propagator, namely

$$K(x - x') = \frac{-i}{(2\pi)^4} \int d^4 p \exp\{ip(x - x')\} \frac{-i\gamma \cdot p + m}{p^2 + m^2 - i\epsilon}, \quad (33)$$

and they amount to assuming that we know the initial positive energy states and the final negative energy ones. Thus the application of reasonable physical rules of interpretation leads to the correct form for the Green function, and it is immaterial whether or not second quantization is invoked as the method of reaching the final expressions. The solution of equation (31) by a perturbation expansion now appears in the standard form (S, p. 233)

$$\begin{aligned} \psi(x) = & \psi_i(x) + \int d^4 x' K(x, x') \{-e\gamma_\mu A_\mu(x')\} \psi_i(x') \\ & + \int d^4 x' \int d^4 x'' K(x, x') \{-e\gamma_\mu A_\mu(x')\} K(x', x'') \{-e\gamma_\nu A_\nu(x'')\} \psi_i(x'') + \dots, \end{aligned} \quad (34)$$

where ψ_i is the initial plane wave state. By taking the matrix element with a final plane wave state, we obtain the S -matrix expansion

$$S_{fi} = -ie \int d^4x' \bar{\psi}_f(x') \gamma_\mu A_\mu(x') \psi_i(x') \\ + (-ie)^2 \int d^4x' \int d^4x'' \bar{\psi}_f(x') \gamma_\mu A_\mu(x') K(x', x'') \gamma_\nu A_\nu(x'') \psi_i(x'') + \dots \quad (35)$$

For processes which do not involve internal interactions or the random fields, all of the necessary calculations leading to the S -matrix elements for such phenomena as Mott scattering, Compton scattering etc. may now be completed even though we have not quantized the EM fields. All of this is discussed in detail in S (pp. 210–40).

Turning now to the interactions, we see that a typical term will involve

$$i \int d^4x' j_\mu(x') A_\mu(x') \\ = i \int d^4x' \int d\mathbf{r}'' \int_{-\infty}^0 ds j_\mu(\mathbf{r}', t') D_{\mu\nu}^R(\mathbf{r}' - \mathbf{r}'', -s) j_\nu(\mathbf{r}', t' + s), \quad (36)$$

where we have used the retarded Green function to evaluate the potentials. Using the time reversal symmetry of the situation we may also employ the advanced Green function to write the same contribution as

$$i \int d^4x' \int d\mathbf{r}'' \int_0^\infty ds j_\mu(\mathbf{r}', t') D_{\mu\nu}^A(\mathbf{r}' - \mathbf{r}'', -s) j_\nu(\mathbf{r}', t' + s), \quad (37)$$

and on taking one-half of each term we will again recover the full perturbation expansion of QED, except that this time D^C will be replaced by $-\frac{1}{2}i(D_{ij}^A + D_{ij}^R)$. For a process such as Möller scattering we see that a semiclassical treatment suffices. After adding in the Coulomb interaction, using the manipulations from S (pp. 252–3), we find that the transition matrix element of semiclassical theory differs from QED only in the prescription of how we avoid the pole in the photon propagator. But in this case we do not have to integrate over the momentum four-vector of the photon, so this is irrelevant.

Transition Currents

One of the practical problems with a semiclassical theory is that it is difficult to calculate the effect of the wavefunction transitions on the EM fields. The sources are obtained, at least in principle, by using the energy density (7), but this information is of little use since it presupposes a complete solution for the wavefunction as a function of time. What we need to know is the contribution made by each first-order transition and this may be read from the form of the perturbation expansion. First, a transition which is caused by incoming radiation must absorb that radiation, and for this to be so the transition current density has to be

$$-ie\bar{\psi}_i \gamma_\mu \psi_f. \quad (38)$$

Thus for an initial and final state of momentum $p_\mu^{(i)}$ and $p_\mu^{(f)}$ we have a source whose space-time dependence is $\exp\{i(p^{(f)} - p^{(i)}) \cdot x\}$, so that radiation whose propagation vector is $p_\mu^{(f)} - p_\mu^{(i)}$ is produced by the transition. Of course we must choose the

advanced solution to d'Alembert's equation in this case, since the sources absorb the radiation. For a transition which emits radiation, the transition current density has to be

$$-ie\bar{\psi}_f \gamma_\mu \psi_i, \tag{39}$$

where the retarded solution is now appropriate. This is the current density which appears as the source in equation (36), and is paired with a similar term by the retarded Green function; similarly for the contribution (37). Barwick (1978) uses similar expressions for transition currents in his demonstration that the outgoing radiation of Compton scattering may be calculated correctly using semiclassical theory. However, the concrete physical interpretation of superposition states which he gives is not necessary in the above analysis.

5. Equivalence of Random and Quantum Electrodynamics

We commence by working in Coulomb gauge, although we will quickly make the transition to a covariant formulation. Consider the S -matrix expansion (35) in which the potentials may now involve random fields in addition to the external and interaction fields. Each term in the expansion will have to be averaged over the random phases which will involve averaging products $A_i A_j \dots A_k$ of the potentials of the random fields. If there are an odd number of factors in this product, then equation (22) assures us of a zero result; if there are an even number of factors then each pair averages to the correlation function (23). These pairings must be taken in every possible combination as they are in QED, and as they are in the semiclassical theory. A moment's thought will lead to the conclusion that the effect of the random fields, when they are added to the semiclassical theory, is to replace the (transverse) Green function by the (transverse) photon propagator of QED, namely

$$D_{ik}^C = \frac{1}{2}(D_{ik}^{(1)} - iD_{ik}^R - iD_{ik}^A). \tag{40}$$

Of course, the change will only affect the final result when the corresponding line of the Feynman diagram has an energy momentum four-vector k_μ which is free, so that it is relevant to keep track of which way the k_0 integral goes around the pole. Such diagrams only occur when we begin to consider the so-called 'radiative corrections'.

Coulomb Interaction

The Coulomb interaction must also be added to each occurrence of the transverse propagator, exactly as in a formulation of QED using Coulomb gauge. The essential point in making this addition is that the transverse propagator always appears in conjunction with Dirac matrices and spinors in combinations of the form

$$\begin{aligned} (\bar{u}'_1 \gamma_i u_1) \dots D_{ik} \dots (\bar{u}'_2 \gamma_k u_2) &= (\bar{u}'_1 \gamma_i u_1) \dots \{i(k^2 - i\epsilon)\}^{-1} \dots (\bar{u}'_2 \gamma_i u_2) \\ &\quad - (\bar{u}'_1 \gamma \cdot \hat{k} u_1) \dots \{i(k^2 - i\epsilon)\}^{-1} \dots (\bar{u}'_2 \gamma \cdot \hat{k} u_2). \end{aligned} \tag{41}$$

Similarly, the Coulomb interaction leads to terms of the form

$$(\bar{u}'_1 \gamma_4 u_1) \dots (i|k|^2)^{-1} \dots (\bar{u}'_2 \gamma_4 u_2). \tag{42}$$

Using the properties of the Dirac spinors (S, p. 252), we may show that the term (42) is equivalent to

$$(\bar{u}'_1 \gamma_4 u_1) \dots \{i(k^2 - i\epsilon)\}^{-1} \dots (\bar{u}'_2 \gamma_4 u_2) + (\bar{u}'_1 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} u_1) \dots \{i(k^2 - i\epsilon)\}^{-1} \dots (\bar{u}'_2 \boldsymbol{\gamma} \cdot \hat{\mathbf{k}} u_2), \quad (43)$$

and when this is combined with the transverse propagator, we obtain the covariant photon propagator

$$D_{\mu\nu}^C(k) = \{i(k^2 - i\epsilon)\}^{-1}. \quad (44)$$

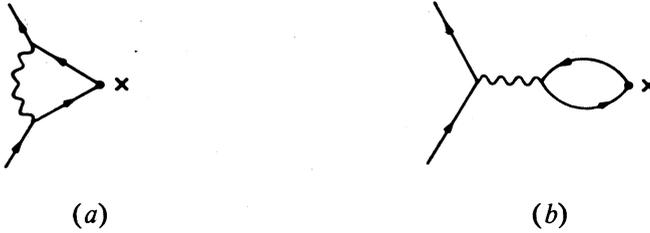


Fig. 2. Feynman diagrams which contribute to the Lamb shift to lowest order in (renormalized) perturbation theory: (a) the interaction of the electron with its own EM field; (b) the vacuum polarization contribution.

Covariant Random Field

The Coulomb interaction (42) is not singular at $k_0 = \pm\omega$ so that, if we split the covariant photon propagator into covariant parts $D^{(1)}$ and $D^A + D^R$, then the Coulomb interaction makes no contribution to $D^{(1)}$, either when the four-momentum of the corresponding photon line is fixed (it is irrelevant which way we skirt the singularity) or free (the Coulomb part has no singularity). This prompts us to consider the generalized four-vector random field

$$A_\mu = \int d\mathbf{r} \frac{1}{2}(V/\omega)^{\frac{1}{2}} \exp(i\mathbf{k} \cdot \mathbf{x}) \sum_{\lambda=1}^4 \epsilon_{\mu\lambda} \exp(i\theta_{k,\lambda}), \quad (45)$$

where the polarization vectors are defined exactly as in QED (S, p. 254), namely

$$\epsilon_{\mu\nu} = (\mathbf{e}_{k,\lambda}, 0) \quad \lambda = 1, 2, \quad (46a)$$

$$= (\hat{\mathbf{k}}, 0) \quad \lambda = 3, \quad (46b)$$

$$= (0, 0, 0, i) \quad \lambda = 4. \quad (46c)$$

The correlation function of the random field then becomes

$$\langle A_\mu(x) A_\nu(x') \rangle = \frac{1}{2} D_{\mu\nu}^{(1)}(x - x'). \quad (47)$$

Using this correlation function from the start, together with a covariant choice of gauge (for example, the Lorentz gauge) would lead directly to the conclusions at which we have already arrived, but without the necessity of treating the Coulomb interaction separately.

6. Lamb Shift

The explanation of the Lamb shift is one of the outstanding success stories of QED. To lowest order in (renormalized) perturbation theory, there are two processes which contribute, represented by the Feynman diagrams of Fig. 2. Fig. 2a corresponds to the interaction of the electron with its own EM field and was first considered by Bethe (1947) who used mass renormalization and an energy cutoff to obtain a finite result in agreement with experiment. Soon after Bethe's paper, Welton (1948) published a heuristic explanation by calculating the mean square amplitude of motion of a classical point particle in a random field, and then arguing that in the Schrödinger equation we must replace the Coulomb potential $V(x)$ by a smeared-out average. Reference to Fig. 2a will show that the internal photon line has its four-momentum vector free, so that random fields will indeed play an important role in evaluating the contribution semiclassically. Fig. 2b is generally referred to as the vacuum polarization term, and in it the photon momentum is fixed by energy-momentum conservation. Consequently the random fields make no contribution and this is the reason why Uehling (1935) was able to give a semiclassical explanation of this term.

Random Field Calculation

We will sketch out the derivation of Bethe's (1947) formula using the random fields as the perturbation in a nonrelativistic treatment so the connection with the original work is made transparent. The perturbing Hamiltonian is

$$H' = -(e/m)\mathbf{p} \cdot \mathbf{A} \tag{48}$$

or, on substituting the expansion (21) for the random fields,

$$H' = \frac{-e}{mV^{\frac{1}{2}}} \sum_{\mathbf{k}\lambda} \frac{\mathbf{p} \cdot \mathbf{e}_{\mathbf{k}\lambda}}{(2\omega)^{\frac{1}{2}}} \left(\exp(i\mathbf{k} \cdot \mathbf{x}) a_{\mathbf{k}\lambda} + \exp(-i\mathbf{k} \cdot \mathbf{x}) a_{\mathbf{k}\lambda}^* \right). \tag{49}$$

We will use second-order perturbation theory, which expresses the energy shift in terms of products like $a_{\mathbf{k}\lambda} a_{\mathbf{k}'\lambda'}$. When we average over the random phases, we use

$$\langle a_{\mathbf{k}\lambda} a_{\mathbf{k}'\lambda'}^* \rangle = \langle a_{\mathbf{k}\lambda}^* a_{\mathbf{k}'\lambda'} \rangle = \frac{1}{2} \delta_{\lambda\lambda'} \delta_{\mathbf{k}\mathbf{k}'}, \tag{50a}$$

$$\langle a_{\mathbf{k}\lambda} a_{\mathbf{k}'\lambda'} \rangle = \langle a_{\mathbf{k}\lambda}^* a_{\mathbf{k}'\lambda'}^* \rangle = 0, \tag{50b}$$

and this gives the result

$$\begin{aligned} \Delta E_A = \frac{e^2}{2m^2V} \sum_I \sum_{\mathbf{k}\lambda} \frac{1}{2\omega} & \left(\frac{\{e \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{e \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I + \omega} \right. \\ & \left. + \frac{\{e \cdot \mathbf{p} \exp(i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{e \cdot \mathbf{p} \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I - \omega} \right). \tag{51} \end{aligned}$$

Bethe's renormalized formula may now be recovered by using the dipole approximation and replacing the sum over \mathbf{k} by an integral. The reason why we do not have to renormalize is that the relations (50), which differ from the parallel relations of QED, cause the terms of equation (51) to occur in pairs in such a way that the term in the sum with $I = A$ cancels. In Bethe's treatment this term is consciously omitted on the ground that it is part of the EM mass of the electron.

Spontaneous Absorption Problem

As is well known, by adding the infinitesimal quantity $i\delta$ to the denominators of equation (51) and using the fact that

$$\frac{1}{E_A - E_I \pm \omega + i\delta} = \frac{1}{E_A - E_I \pm \omega} - i\pi\delta(E_A - E_I \pm \omega), \quad (52)$$

we obtain a complex energy level shift whose imaginary part is the decay rate. The problem is that when we do this, the first term on the right-hand side corresponds to spontaneous emission and the second to spontaneous absorption. We solved this problem (in principle) when we showed in Section 3 that the semiclassical interactions and the random fields together give the quantum propagator (26). However, we would like to briefly sketch how this works in the context of the present calculation. To this end we first write down the expression for ΔE_A analogous to equation (51), but obtained from the Dirac equation with random fields:

$$\Delta E_A = \frac{1}{4V} \sum_I \sum_{k,\lambda} \frac{1}{\omega} \left(\frac{\{-ie\gamma_\mu \varepsilon_{\mu\lambda} \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{-ie\gamma_\nu \varepsilon_{\nu\lambda} \exp(i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I + \omega + i\delta} + \frac{\{-ie\gamma_\mu \varepsilon_{\mu\lambda} \exp(i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{-ie\gamma_\nu \varepsilon_{\nu\lambda} \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I - \omega + i\delta} \right). \quad (53)$$

Now if we used quantized EM fields in the same calculation, we would obtain for our result twice the first term in the summation (spontaneous emission) while the second term (spontaneous absorption) would be absent. The difference between these two expressions is our present concern, namely

$$\frac{1}{32\pi^3} \sum_I \int d\mathbf{k} \frac{1}{\omega} \left(\frac{\{-ie\gamma_\mu \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{-ie\gamma_\mu \exp(i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I + \omega + i\delta} - \frac{\{-ie\gamma_\mu \exp(i\mathbf{k} \cdot \mathbf{r})\}_{AI} \{-ie\gamma_\mu \exp(-i\mathbf{k} \cdot \mathbf{r})\}_{IA}}{E_A - E_I - \omega + i\delta} \right), \quad (54)$$

where we have summed over polarization states and replaced the sum over \mathbf{k} by an integral. We now write $\mathbf{k} = \omega\hat{\mathbf{k}}$, then the integral over the directions of \mathbf{k} is trivial, leading to

$$\frac{1}{8\pi^2} \sum_I \int_0^\infty d\omega \frac{\sin \omega R}{R} \left(\frac{(-ie\gamma_\mu)_{AI} (-ie\gamma_\mu)_{IA}}{E_A - E_I + \omega + i\delta} - \frac{(-ie\gamma_\mu)_{AI} (-ie\gamma_\mu)_{IA}}{E_A - E_I - \omega + i\delta} \right). \quad (55)$$

The substitution $\omega \rightarrow -\omega$ in the second term results in the collapse of the two integrals into one integral along a contour from $-\infty$ to $+\infty$, which passes over the poles at $\omega = E_I - E_A$. Evaluating this by standard methods of complex variable theory, we finally reduce our expression to

$$\sum_I \{\exp(i\omega_{AI} R)/8\pi R\} (-ie\gamma_\mu)_{AI} (-ie\gamma_\mu)_{IA}, \quad (56)$$

where $\omega_{AI} = E_A - E_I$. The meaning of this formula is made plain by writing each of the matrix elements as a three-dimensional integral to give

$$\frac{1}{2} \iint d\mathbf{r} d\mathbf{r}' \{ -ie\bar{u}_A(\mathbf{r})\gamma_\mu u_I(\mathbf{r}) \} \frac{\exp(i\omega_{AI}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \{ -ie\bar{u}_I(\mathbf{r}')\gamma_\mu u_A(\mathbf{r}') \}, \quad (57)$$

which is simply the interaction of two semiclassical transition current densities (see equations 38 and 39). The exponential factor comes about because of the time dependence of these current densities.

7. Conclusions

We have shown that the inclusion of the random vacuum EM field, as proposed by Boyer and others, in a semiclassical theory reproduces the results of QED. Suggestions along these lines have already been made (Senitzky 1973; Milonni and Smith 1975) on the basis of results obtained from simple nonrelativistic atomic model calculations. By considering the semiclassical radiation reaction, as is done in Jaynes's 'neoclassical theory' (Crisp and Jaynes 1969; Stroud and Jaynes 1970) together with the vacuum fields, we have shown that a perturbation expansion for the S matrix involves the photon propagator of QED. This enables one to foresee the circumstances in which a semiclassical calculation (without random fields) will be adequate, and when it is necessary either to second quantize or to add random fields, simply by inspecting the Feynman diagrams for the calculation in hand. Thus we see immediately that Möller and Compton scattering can be treated semiclassically, while the Lamb shift needs QED or random fields. We have reproduced Bethe's (1947) nonrelativistic expression and discussed the relativistic generalization. The fact that the contribution from the random fields needs no renormalization is intuitively pleasing, since we would expect that the infinities of QED come from self-interactions rather than from any random nature of the fields influencing the particles. Indeed, we are currently looking into a scheme whereby the classical solution of the self-interaction problem through a Lorentz invariant separation of the Coulomb fields from the radiation fields (Rohrlich 1965) is used to eliminate the divergences from the outset. Such a theory would still of course be semiclassical, and we will need to understand its precise connection with QED. Certainly in considering emission and absorption we have seen the need to consider carefully the boundary conditions, and to keep in mind conservation principles, as is done in the Lorentz-Dirac theory.

In the light of our results it is clear that the failure of some semiclassical theories to give an adequate explanation of radiative corrections has been due to the neglect of the vacuum field which plays an important role in QED. Thus, in the same paper Barwick (1978) finds that a semiclassical explanation of the outgoing radiation from Compton scattering is possible, and then suggests that the Lamb shift may be calculated as a part of the EM self-energy alone. It is hoped that our work will both broaden the understanding of a number of the difficulties facing the present form of QED and elucidate the validity of the various semiclassical arguments so common in practical calculational problems.

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