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Quantum Theory of Energy Loss by Test Ions to Plasma

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Abstract

A fully quantum theoretical calculation is carried out for the average energy loss rate by test ions released in a plasma. The calculation is based on the divergence-free kinetic theory of Kihara and Honda. The rate is given in terms of several auxiliary functions which may be evaluated numerically for a given test ion once the Maxwellian temperatures and densities of each plasma species (electrons and ions) are specified. The examples considered earlier by Sigmar and Joyce are re-evaluated and the accuracies of their approximate calculations are found to be generally quite good. Several new quantitative results are presented.

1. Introduction

In a recent paper George *et al.* (1979; hereinafter referred to as Paper I) presented a fully quantum theoretical calculation of the energy loss by a test ion to plasma electrons. The calculation was based on the divergence-free kinetic equation of Kihara (1964) and Honda (1964*a*, 1964*b*) and contained all known energy loss formulae as appropriate limiting cases. As far as the energy loss by a nonrelativistic ion to quiescent plasma electrons with a Maxwellian velocity distribution is concerned, we thus believe that Paper I constitutes the most general solution of the problem.

The question we wish to take up in the present work is concerned with the energy loss to plasma ions. The loss to ions and the contribution of ions to the dynamical dielectric response function are both certainly negligible as long as the test ions are much faster than the thermal electrons. As the ion slows down and approaches the thermal energy, however, the loss to plasma ions becomes appreciable. The energy loss by fusion-produced ions to plasma ions is thus important in a hot plasma; the speed of a $3\cdot5$ MeV α particle is approximately $1/5$ of the electron thermal speed at a temperature T_e (in energy units) of 10 keV. We also note that, for slower ions, the terms proportional to the electron–ion mass ratio m_e/M in the loss rate are not quite negligible (Butler and Buckingham 1962). These terms were neglected in Paper I but will be kept in the present calculation.

A calculation of the energy loss rate to both electrons and ion species within a plasma was presented some time ago by Sigmar and Joyce (1971). Their calculation was based on the Balescu–Lenard kinetic equation, and thus the screening and collective effects were correctly taken into account but a cutoff had to be introduced in order to make finite the contribution from collisions accompanying large momentum transfers. Quantum diffraction effects (George and Hamada 1978) were only approximately taken into consideration through an appropriate choice of the cutoff. From a theoretical point of view, such a treatment is not completely satisfactory.

The calculation presented here is based on the fully quantum theoretical and divergence-free kinetic equation of Kihara (1964) and Honda (1964*a*, 1964*b*). Thus no cutoff is required and quantum diffraction effects are correctly taken into account. The final energy loss rate is given in terms of the quantum-theoretical Coulomb logarithm and several functions defined by one-dimensional integrals which can be evaluated numerically once the plasma parameters are specified. The accuracy of the calculation critically depends on the ratio $b_{\min}/b_{\mathrm{D}}$, where b_{D} is the effective Debye length and $b_{\min}$ the effective minimum impact parameter, either classical or quantum theoretical (George and Hamada 1978). The error involved in the final result is at most of the order of $(b_{\min}/b_{\mathrm{D}})\ln(b_{\mathrm{D}}/b_{\min})$ for $b_{\mathrm{D}} \gg b_{\min}$. For a typical magnetically confined fusion plasma of density $\sim 10^{14}\,\mathrm{cm}^{-3}$ and temperature $\sim 10\,\mathrm{keV}$, the ratio $b_{\min}/b_{\mathrm{D}}$ is of the order of 10^{-8}. For an inertially confined high density plasma of density $\sim 10^{22}\,\mathrm{cm}^{-3}$ and temperature $\sim 10\,\mathrm{keV}$, the ratio is of the order of 10^{-4}. The accuracy of our results is therefore expected to be sufficiently high.

2. Energy Loss Rate

We consider Coulomb collisions of a test ion of mass M, charge Ze, and velocity V with plasma particles, i.e. electrons and ions. The jth species of plasma particles is characterized by mass m_j, charge e_j and number density n_j. We assume that none of the species are degenerate and that all species are represented by a Maxwellian velocity distribution

$$f_j(\boldsymbol{v}) = n_j(m_j/2\pi T_j)^{3/2}\exp(-m_j\boldsymbol{v}^2/2T_j)\,, \tag{1}$$

where T_j is the temperature of species j in energy units.

According to Honda (1964*a*, 1964*b*), the average time variation of energy of the test ion can be evaluated in two parts:

$$\mathrm{d}E/\mathrm{d}t = (\mathrm{d}E/\mathrm{d}t)_{\mathrm{I}} + (\mathrm{d}E/\mathrm{d}t)_{\mathrm{II}}\,, \tag{2}$$

where the first term is the result obtained in the Born approximation and the second term represents the correction to it. In the quantum limit where all close collisions are dominated by quantum diffraction effects, the second term vanishes so that the Born approximation alone gives the correct loss rate.

The Born approximation leads to (Honda 1964*b*)

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{I}} = \frac{4(Ze)^2}{\hbar}\sum_j e_j^2\int d\boldsymbol{g}\, f_j(\boldsymbol{g}-\boldsymbol{V})\int \mathrm{d}\boldsymbol{k}\,\frac{\omega}{|k^2\,\varepsilon(\boldsymbol{k},\omega)|^2}\,\delta(\boldsymbol{k}\,.\,\boldsymbol{g}+\hbar k^2/2\mu_j)\,, \tag{3}$$

where

$$\omega = \boldsymbol{k}\,.\,\boldsymbol{V} + \hbar k^2/2M\,, \tag{4}$$

$\boldsymbol{g} = \boldsymbol{V}-\boldsymbol{v}$ is the relative velocity and $\mu_j = m_j M/(m_j+M)$ is the reduced mass. The dielectric response function $\varepsilon(\boldsymbol{k},\omega)$ is given by (Kihara and Aono 1963)

$$\varepsilon(\boldsymbol{k},\omega) = 1 + \sum_j (k_{\mathrm{D}j}/k)^2\{X(\rho_j) + \mathrm{i}\,Y(\rho_j)\}\,, \tag{5}$$

where

$$k_{\mathrm{D}j} = (4\pi n_j e_j^2/T_j)^{\frac{1}{2}} \tag{6}$$

is the Debye wave number of species j and

$$X(\rho) = 1 - 2\rho\exp(-\rho^2)\int_0^\rho \mathrm{d}t\,\exp(t^2), \qquad Y(\rho) = \pi^{\frac{1}{2}}\rho\exp(-\rho^2), \tag{7}$$

with

$$\rho_j = (\omega/k)(m_j/2T_j)^{\frac{1}{2}}. \tag{8}$$

The correction to the Born approximation is given by (Honda 1964*b*)

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{II}} = (Ze)^2 \sum_j \frac{\omega_{pj}^2}{n_j}\int \mathrm{d}\boldsymbol{g}\, f_j(\boldsymbol{g}-\boldsymbol{V})\left\{\left(1+\frac{m_j}{M}\right)\frac{\boldsymbol{g}\,.\,\boldsymbol{V}}{g^3} - \frac{m_j}{Mg}\right\}\left\{\gamma + \mathrm{Re}\,\psi\left(\frac{\mathrm{i}Zee_j}{\hbar g}\right)\right\}, \tag{9}$$

where

$$\omega_{pj} = (4\pi n_j e_j^2/m_j)^{\frac{1}{2}} \tag{10}$$

is the plasma frequency of species j, $\gamma = 0\cdot57721\ldots$ is Euler's constant and ψ is the digamma function, Re denoting the real part (Abramowitz and Stegun 1972). We wish to evaluate the rates (3) and (9) in the present section.

(*a*) *Born Approximation*

In order to evaluate the rate (3) we first assume that

$$k_{\mathrm{Q}j} = (\mu_j/\hbar)(2T_j/m_j)^{\frac{1}{2}} \gg k_{\mathrm{D}j}, \tag{11}$$

for all j. This condition is satisfied when we have $T_j \gg \hbar\omega_{pj}$. Now let k_1 be a wave number such that

$$k_{\mathrm{Q}j} \gg k_1 \gg k_{\mathrm{D}j}. \tag{12}$$

We then divide the integration over k into two regions: (i) $k_1 \geqslant k \geqslant 0$ and (ii) $\infty > k \geqslant k_1$. It will be seen that the final result is independent of k_1 to a very good approximation as long as the conditions (12) are satisfied.

Region (i) above corresponds to distant collisions. The dielectric response function which describes the Debye shielding and collective excitation is important here but we expect that the quantum diffraction effect is negligible. In order to see this explicitly we first note that the term $\hbar k^2/2\mu_j$ within the δ function in the integrand of equation (3) gives only a small contribution as long as $g \gg \hbar k_1/2\mu_j$. Since in general we have $g \gtrsim (2T_j/m_j)^{\frac{1}{2}} = \hbar k_{\mathrm{Q}j}/\mu_j$ this is indeed the case according to the conditions (12). Hence we have

$$\int \mathrm{d}\boldsymbol{g}\, f_j(\boldsymbol{g}-\boldsymbol{V})\,\delta(\boldsymbol{k}\,.\,\boldsymbol{g} + \hbar k^2/2\mu_j)$$

$$= (n_j x_j/kV\sqrt{\pi})\exp\{-(\hat{\boldsymbol{k}}\,.\,\boldsymbol{x}_j)^2\}(1 - m_j\hbar\boldsymbol{k}\,.\,\boldsymbol{V}/2T_j\mu_j) + O(k_1^2/k_{\mathrm{Q}j}^2), \tag{13}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ and

$$\boldsymbol{x}_j = (m_j/2T_j)^{\frac{1}{2}}\,\boldsymbol{V} \tag{14}$$

is the test ion velocity measured in terms of the thermal speed of species j.

Let us next consider the dielectric response function $\varepsilon(\boldsymbol{k},\omega)$. Defining the effective Debye wave number by

$$k_{\mathrm{D}} = \left(\sum_j k_{\mathrm{D}j}^2\right)^{\frac{1}{2}}, \tag{15}$$

we see from equations (5) and (7) that $\varepsilon(\boldsymbol{k},\omega)$ is appreciably different from unity only for $k \lesssim k_{\mathrm{D}}$. Since we are only interested in a test ion speed such that $V \gg \hbar k_{\mathrm{D}}/M$, we may replace ρ_j defined by equation (8) by

$$\rho_j = (\boldsymbol{x}_j . \hat{\boldsymbol{k}}), \tag{16}$$

which makes $|\varepsilon(\boldsymbol{k},\omega)|^2$ an even function of $\boldsymbol{k}$.

Substituting the result (13) into equation (3) we then obtain in region (i)

$$\left(\frac{\mathrm{d}E(\mathrm{i})}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{(Ze)^2}{\pi^{3/2}V}\sum_j \omega_{pj}^2 m_j x_j \int \mathrm{d}\Omega_k \left(\frac{(\hat{\boldsymbol{k}}.\boldsymbol{x}_j)^2}{\mu_j} - \frac{1}{2M}\right)\exp\{-(\hat{\boldsymbol{k}}.\boldsymbol{x}_j)^2\}$$

$$\times \int_0^{k_1} \mathrm{d}k \frac{k^3}{|k^2\varepsilon|^2}, \tag{17}$$

which indeed shows that there is no quantum diffraction effect in region (i). Writing

$$A = \sum_j (k_{\mathrm{D}j}/k_{\mathrm{D}})^2 X(\hat{\boldsymbol{k}}.\boldsymbol{x}_j), \qquad B = \sum_j (k_{\mathrm{D}j}/k_{\mathrm{D}})^2 Y(\hat{\boldsymbol{k}}.\boldsymbol{x}_j), \tag{18}$$

and noting the definition (5), we find

$$\int_0^{k_1} \mathrm{d}k \frac{k^3}{|k^2\varepsilon|^2} = \ln\left(\frac{k_1}{k_{\mathrm{D}}}\right) - \tfrac{1}{4}\ln\left(A^2+B^2\right) - \frac{A}{2B}\arctan\left(\frac{B}{A}\right) + O\left(\frac{k_{\mathrm{D}}^2}{k_1^2}\right). \tag{19}$$

It is appropriate to introduce two ubiquitous auxiliary functions at this point:

$$\Psi(x) = \operatorname{erf}(x) - \Phi(x), \qquad \Phi(x) = (2/\sqrt{\pi})x\exp(-x^2). \tag{20}$$

Substituting the result (19) into equation (17) and carrying out the remaining integrations we then find

$$\left(\frac{\mathrm{d}E(\mathrm{i})}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{(Ze)^2}{V}\sum_j \omega_{pj}^2\left\{\left(\Psi(x_j) - \frac{m_j}{M}\Phi(x_j)\right)\ln\left(\frac{k_1}{k_{\mathrm{D}}}\right) + G_j(x_j)\right\}, \tag{21}$$

where

$$G_j(x_j) = -\frac{1}{\sqrt{\pi}}\int_0^{x_j} \mathrm{d}s \exp(-s^2)\left\{\left(1+\frac{m_j}{M}\right)s^2 - \frac{m_j}{2M}\right\}\left\{\ln\left(A_j^2+B_j^2\right) + \frac{2A_j}{B_j}\arctan\left(\frac{B_j}{A_j}\right)\right\}, \tag{22}$$

with

$$A_j = \sum_i (k_{\mathrm{D}i}/k_{\mathrm{D}})^2 X(sx_i/x_j), \qquad B_j = \sum_i (k_{\mathrm{D}i}/k_{\mathrm{D}})^2 Y(sx_i/x_j). \tag{23}$$

The arctan function in equation (22) should be evaluated in the range 0–π since $Y > 0$.

As equations (13) and (19) indicate, the errors involved in (21) are at most of the order k_D/k_{Qj} for the choice

$$k_1 \approx (k_D k_{Qj})^{\frac{1}{2}}.$$

If we neglect the contributions from plasma ions altogether and also neglect terms of order m_e/M, we find that G defined by equation (22) is related to May's (1969) quantity $\Delta_1(x)$ through the relation

$$G(x) = \Psi(x)\Delta_1(x). \tag{24}$$

The loss rate (21) then reduces to the one obtained previously (equation 26 of Hamada 1978).

Let us next turn to region (ii), $\infty > k \geqslant k_1 \gg k_{Dj}$. We are now concerned with close collisions. In view of equations (5) and (7) it is now a good approximation to replace $\varepsilon(\boldsymbol{k},\omega)$ by unity. The error caused by this approximation is at most of the order $(k_{Dj}/k_1)^2$. The integration over $\boldsymbol{k}$ in equation (3) can then be easily carried out:

$$\left(\frac{\mathrm{d}E(\mathrm{ii})}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{(Ze)^2}{\pi^{3/2}V}\sum_j m_j\omega_{pj}^2 x_j \int \mathrm{d}\boldsymbol{s}\exp\{-(\boldsymbol{x}_j-\boldsymbol{s})^2\}\left(\frac{\boldsymbol{x}_j\cdot\boldsymbol{s}}{\mu_j s^3}-\frac{1}{Ms}\right)\left\{\ln s+\ln\left(\frac{2k_{Qj}}{k_1}\right)\right\}, \tag{25}$$

where we have set

$$\boldsymbol{s} = (m_j/2T_j)^{\frac{1}{2}}\boldsymbol{g}. \tag{26}$$

Using the relations

$$\int \mathrm{d}\boldsymbol{s}\, s^{-1}\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\} = \pi^{3/2}\operatorname{erf}(x)/x, \tag{27a}$$

$$\int \mathrm{d}\boldsymbol{s}\, s^{-3}(\boldsymbol{x}\cdot\boldsymbol{s})\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\} = \pi^{3/2}\,\Psi(x)/x, \tag{27b}$$

we arrive at

$$\left(\frac{\mathrm{d}E(\mathrm{ii})}{\mathrm{d}t}\right)_{\mathrm{I}} = -\frac{(Ze)^2}{V}\sum_j \omega_{pj}^2\left\{\left(\Psi(x_j)-\frac{m_j}{M}\Phi(x_j)\right)\ln\left(\frac{2k_{Qj}}{k_1}\right)+F_1(x_j)+\frac{m_j}{M}F_2(x_j)\right\}, \tag{28}$$

where

$$F_1(x) = \pi^{-3/2}x\int \mathrm{d}\boldsymbol{s}\, s^{-3}\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\}(\boldsymbol{x}\cdot\boldsymbol{s})\ln s$$

$$= \tfrac{1}{2}\pi^{-\frac{1}{2}}\int_0^\infty \mathrm{d}s\, s^{-2}[(2sx+1)\exp\{-(x+s)^2\}+(2sx-1)\exp\{-(x-s)^2\}]\ln s, \tag{29}$$

$$F_2(x) = F_1(x)-\pi^{-3/2}x\int \mathrm{d}\boldsymbol{s}\, s^{-1}\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\}\ln s$$

$$= F_1(x)-\pi^{-\frac{1}{2}}\int_0^\infty \mathrm{d}s\,[\exp\{-(x-s)^2\}-\exp\{-(x+s)^2\}]\ln s. \tag{30}$$

For the choice $k_1 \approx (k_D k_{Qj})^{\frac{1}{2}}$, the errors involved in equation (28) are at most of the order of $(k_D/k_{Qj})\ln(k_{Qj}/k_D)$.

The functions F_1 and F_2 are universal functions with the following limiting values:

$$F_1(x \gg 1) = \ln x - \tfrac{3}{4}x^{-2} + O(x^{-4}), \qquad F_1(x \ll 1) = -\tfrac{2}{3}\pi^{-\frac{1}{2}}\gamma x^3 + O(x^5),$$

$$F_2(x \gg 1) = -\tfrac{1}{2}x^{-2} + O(x^{-4}), \qquad F_2(x \ll 1) = \pi^{-\frac{1}{2}}\{\gamma x - (\gamma + \tfrac{2}{3})x^3\} + O(x^5).$$

We note that $F_1(x)$ is related to May's (1969) quantity $\Delta_2(x)$ through the relation

$$F_1(x) = \tfrac{1}{2}\Psi(x)\{\Delta_2(x) + \ln 2 + 1\}. \tag{31}$$

If the energy loss to plasma electrons alone is considered and the terms of order m_e/M are neglected, the rate (28) reduces to the corresponding result obtained by Hamada (1978, equation 28 thereof; note that k_Q in the present paper is $\sqrt{2}$ times the one used there).

Adding the results (21) and (28) we find that the energy loss rate in the Born approximation is given by

$$-\left(\frac{dE}{dt}\right)_{\mathrm{I}} = \frac{(Ze)^2}{V}\sum_j \omega_{pj}^2\left\{\left(\Psi(x_j) - \frac{m_j}{M}\Phi(x_j)\right)\ln\left(\frac{2k_{Qj}}{k_D}\right) + F_1(x_j) + \frac{m_j}{M}F_2(x_j) + G_j(x_j)\right\}. \tag{32}$$

The functions F_1 and F_2 are universal functions but G_j has to be evaluated numerically for each of the given plasma parameters and test ions.

(*b*) *Correction to Born Approximation*

Our next task is to evaluate the correction (9) to the Born approximation (32). The integral proportional to Euler's constant can be easily evaluated by using the relations (27). In order to evaluate the remaining term we first define a nondimensional parameter ν_j by

$$\nu_j = (Ze|e_j|/\hbar)(m_j/2T_j)^{\frac{1}{2}}. \tag{33}$$

Then we have $Ze|e_j|/g\hbar = \nu_j/s$ with s defined by equation (26). After some manipulations we obtain

$$\left(\frac{dE}{dt}\right)_{\mathrm{II}} = \frac{(Ze)^2}{V}\sum_j \omega_{pj}^2\left\{\left(\Psi(x_j) - \frac{m_j}{M}\Phi(x_j)\right)\gamma - H_1(x_j,\nu_j) - \frac{m_j}{M}H_2(x_j,\nu_j)\right\}, \tag{34}$$

where

$$\begin{aligned} H_1(x,\nu) &= -\pi^{-3/2}x\int d\boldsymbol{s}\,\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\}(\boldsymbol{s}.\boldsymbol{x}/s^3)\,\mathrm{Re}\,\psi(\mathrm{i}\nu/s) \\ &= -\tfrac{1}{2}\pi^{-\frac{1}{2}}\int_0^\infty ds\, s^{-2}[(2sx+1)\exp\{-(x+s)^2\} \\ &\qquad + (2sx-1)\exp\{-(x-s)^2\}]\,\mathrm{Re}\,\psi(\mathrm{i}\nu/s), \end{aligned} \tag{35}$$

$$\begin{aligned} H_2(x,\nu) &= H_1(x,\nu) + \pi^{-3/2}x\int d\boldsymbol{s}\, s^{-1}\exp\{-(\boldsymbol{x}-\boldsymbol{s})^2\}\,\mathrm{Re}\,\psi(\mathrm{i}\nu/s) \\ &= H_1(x,\nu) + \pi^{-\frac{1}{2}}\int_0^\infty ds\,[\exp\{-(x-s)^2\} - \exp\{-(x+s)^2\}]\,\mathrm{Re}\,\psi(\mathrm{i}\nu/s), \end{aligned} \tag{36}$$

which are further universal functions. We note that

$$H_1(x,v) = \Psi(x)\{\Delta_3(x,v)+\gamma\}, \tag{37}$$

where $\Delta_3(x,v)$ is defined and tabulated in Paper I. Thus the rate (34) reduces to the corresponding expression obtained in Paper I when the loss to electrons only is considered and m_e/M is negligible compared with unity.

As functions of x, the quantities H_1 and H_2 have the following limiting forms:

$$H_1(x \ll 1, v) = -4x^3 \Lambda_0(v)/3\sqrt{\pi} + O(x^5),$$

$$H_2(x \ll 1, v) = 2x\Lambda_0(v)/\sqrt{\pi} - 2x^3\{5\Lambda_0(v) - 2\Lambda_1(v)\}/3\sqrt{\pi} + O(x^5),$$

$$H_1(x \gg 1, v) = -\operatorname{Re}\psi(z) - \tfrac{1}{2}x^{-2}\operatorname{Re}\{2z\psi'(z) + \tfrac{1}{2}z^2\psi''(z)\} + O(x^{-4}),$$

$$H_2(x \gg 1, v) = -\tfrac{1}{2}x^{-2}\operatorname{Re}\{z\psi'(z)\} + O(x^{-4}),$$

where

$$\Lambda_n(v) = \int_0^\infty \mathrm{d}t\, t^n \exp(-t)\operatorname{Re}\psi(\mathrm{i}vt^{-\frac{1}{2}})$$

and $\psi'(z) = \mathrm{d}\psi(z)/\mathrm{d}z$ etc., with $z = \mathrm{i}v/x$. The function $\Lambda_0(v)$ has been tabulated in Paper I. Derivatives of digamma functions have simple series expansions (Abramowitz and Stegun 1972) which may be readily evaluated numerically. As functions of v, on the other hand, the limiting forms of H_1 and H_2 are

$$H_1(x, v \gg \bar{s}) = -\Psi(x)\ln v + F_1(x) + O(\bar{s}^2/v^2),$$

$$H_2(x, v \gg \bar{s}) = \Phi(x)\ln v + F_2(x) + O(\bar{s}^2/v^2),$$

$$H_1(x, v \ll \bar{s}) = \gamma\Psi(x) + O(v^2/\bar{s}^2),$$

$$H_2(x, v \ll \bar{s}) = -\gamma\Phi(x) + O(v^2/\bar{s}^2),$$

where $\bar{s}$ is an appropriate average of the variable s defined by equation (26), namely $\bar{s} \approx 1$ if $x \lesssim 1$ and $\bar{s} \approx x$ if $x \gtrsim 1$. The functions F_1 and F_2 are defined by equations (29) and (30) respectively. It is clear that $(\mathrm{d}E/\mathrm{d}t)_{\mathrm{II}}$ given by equation (34) vanishes in the quantum limit $v_j \ll \bar{s}_j$ (all j), as it should since then the Born approximation is valid.

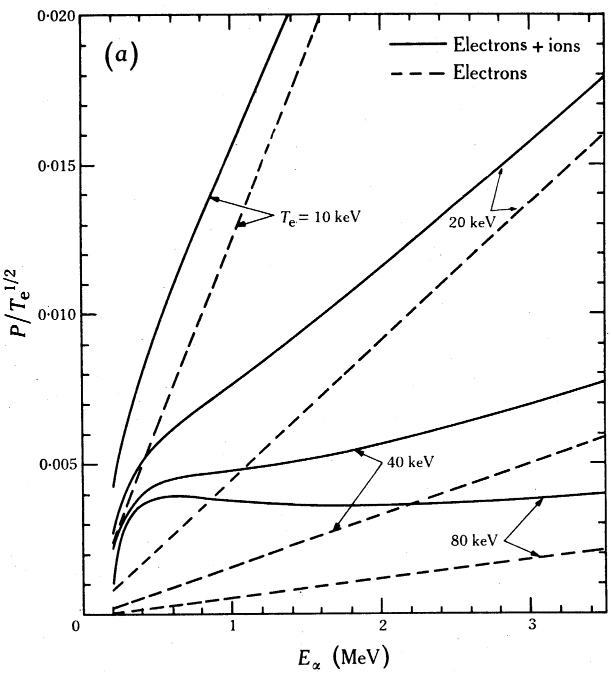
(*c*) *Final Formula*

Combining equations (32) and (34) according to equation (2), we finally arrive at the complete formula for the energy loss rate:

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{(Ze)^2}{V}\sum_j \omega_{pj}^2\left\{\left(\Psi(x_j) - \frac{m_j}{M}\Phi(x_j)\right)\ln\left(\frac{2k_{Qj}}{\Gamma k_D}\right) + F_1(x_j) + \frac{m_j}{M}F_2(x_j)\right.$$
$$\left. + G_j(x_j) + H_1(x_j, v_j) + \frac{m_j}{M}H_2(x_j, v_j)\right\}, \tag{38}$$

where $\Gamma = \exp\gamma = 1{\cdot}78107\ldots$. Since equations (32) and (34) reduce to corresponding formulae given in Paper I, our result (38) clearly reduces to the loss rate obtained in Paper I when plasma ions are altogether discarded and terms of order m_e/M are neglected.

The energy loss rate (38) is formally written as the sum of contributions from individual plasma species. The contribution from electrons, however, is not equal to the loss rate in a one-component electron plasma since k_D in the Coulomb logarithm and $G_e(x_e)$ both contain effects of ambient ion species. Similarly, the contribution of ions of species j is affected by electrons and other ion species. We also note that all terms proportional to m_e/M in equation (38) vary as $x_e = (m_e/2T_e)^{\frac{1}{2}}V$ while those independent of m_e/M vary as x_e^3 for sufficiently small values of x_e. Corrections of order m_e/M are thus important only for $x_e \lesssim (m_e/M)^{\frac{1}{3}}$. These points will be discussed more quantitatively in Section 4 below.



Figs 1*a* and 1*b*. Showing for a 3·5 MeV α particle injected into a DT plasma characterized by the parameters (41) of the text: (*a*) the energy loss rate of the α particle as a function of its energy E_α, for the four indicated plasma electron temperatures T_e, given in terms of the quantity $P/T_e^{\frac{1}{2}}$ defined by equation (40); and (*b*) the relative fractions (left scale) and absolute values (right scale) of the energy deposited onto electrons and ions of the plasma, as a function of T_e. In (*a*) the full and dashed curves for $T_e = 10$ keV reach 0·0449 and 0·0431 keV$^{-\frac{1}{2}}$ respectively at $E_\alpha = 3 \cdot 5$ MeV.

3. Comparison with Calculation by Sigmar and Joyce

Sigmar and Joyce (1971; subsequently referred to as SJ in this section) based their calculation on the Balescu–Lenard kinetic equation which contains a divergent term arising from close Coulomb collisions. They avoided the divergence by assigning appropriate values to the Coulomb logarithms (either classical or quantum theoretical) depending on the plasma parameters and test ions. Our formula (38) requires no such procedures. In the present section we examine the accuracy of the approximate SJ calculation.

Following SJ we define a nondimensional quantity P by

$$P = -(\mathrm{d}E/\mathrm{d}t)(n_e k_{De}^{-3}/T_e \omega_{pe}), \quad (39)$$

so that, from the definitions (6) and (10) of k_{De} and ω_{pe}, with an electron temperature T_e (keV) and density n_e (cm^{-3}),

$$-\mathrm{d}E/\mathrm{d}t = (P/T_e^{\frac{1}{2}}) \times n_e \times 4\cdot342 \times 10^{-12} \quad \mathrm{MeV\,s^{-1}}. \quad (40)$$

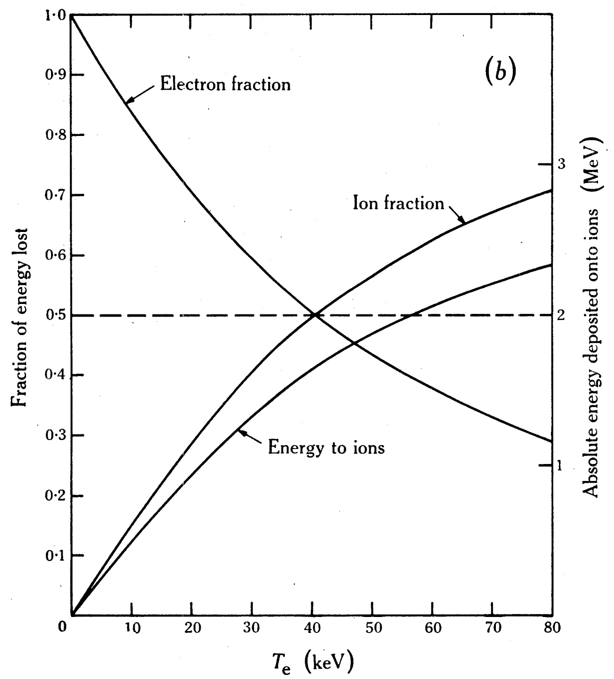


Fig. 1*b* [see caption on facing page]

We have evaluated the quantity $P/T_e^{\frac{1}{2}}$ for an α particle released in a deuterium–tritium (DT) plasma characterized by

$$n_e = 2n_D = 2n_T = 10^{14}\,\mathrm{cm}^{-3}, \qquad T_D = T_T = 100\,\mathrm{keV}, \quad (41)$$

for four electron temperatures T_e = 10, 20, 40 and 80 keV. The results are shown in Fig. 1*a*. On comparison with the results obtained by SJ, we see that their overall accuracy is remarkably good. For $E_\alpha > 0\cdot5$ MeV, SJ underestimate the loss rate very slightly (by $<5\%$). For $E_\alpha < 0\cdot5$ MeV, however, there are significant discrepancies. At $E_\alpha = 0\cdot2$ MeV and $T_e = 80$ keV, for example, the loss rate calculated from equation (38) exceeds the corresponding SJ value by a factor of 2. The simplified version of the SJ calculation made by Kammash and Galbraith (1973) definitely overestimates the loss rate.

Again following SJ we call an injected ion thermalized when it is slowed down to the energy $2T_{th}$ where T_{th} is the highest of the temperatures of all species. Suppose that a $3\cdot5$ MeV α particle is released into a DT plasma characterized by the parameters

(41). When it becomes thermalized at $E_\alpha = 2T_D = 2T_T = 200$ keV, an energy of 3·3 MeV has been absorbed by electrons and ions. In Fig. 1*b* we show the fractions of the energy lost to electrons and to ions as functions of electron temperature. Absolute values of energy deposited onto ions are also shown. Comparing with corresponding values of SJ, we see that they overestimate the electron fraction and hence underestimate the ion fraction slightly for $T_e > 30$ keV. The difference is less than 0·02. The break-even point between electron and ion fractions is at $T_e = 41$ keV, as compared with 44 keV found by SJ.

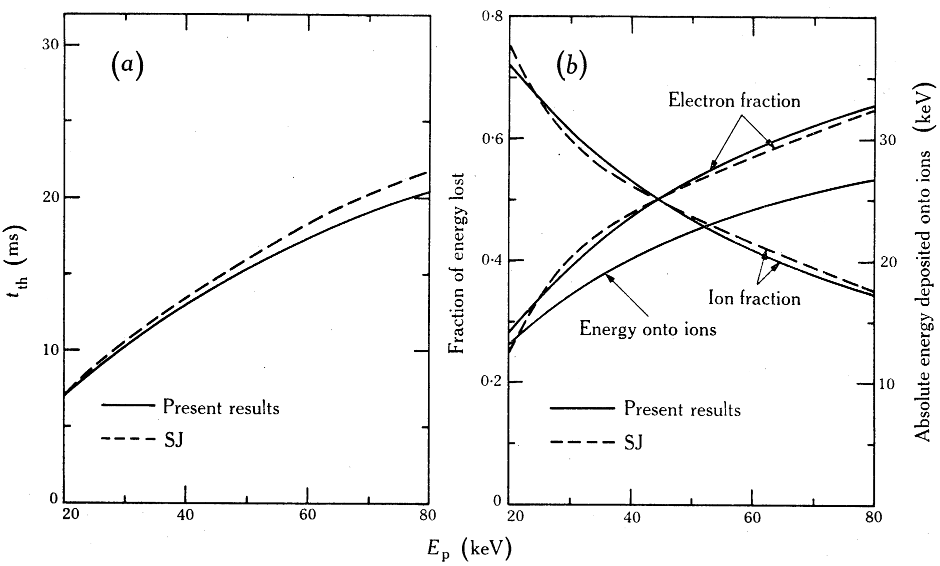


Fig. 2. Comparison of the present calculated results with those of SJ (Sigmar and Joyce 1971) for a proton injected into a hydrogen plasma characterized by the parameters (42) of the text: (*a*) the thermalization time t_{th} of the proton as a function of the proton energy E_p, where t_{th} is defined as the time taken for the proton to slow down to an energy of $2T_e = 2$ keV; and (*b*) the relative fractions (left scale) and absolute values (right scale) of the energy deposited onto electrons and ions of the plasma.

We have next calculated the energy loss of a proton injected into a hydrogen plasma characterized by the parameters

$$n_e = n_H = 5\times 10^{13}\ \mathrm{cm}^{-3}, \qquad T_e = 2T_H = 1\ \mathrm{keV}. \tag{42}$$

Fig. 2*a* shows the thermalization time t_{th} as a function of proton energy E_p. As indicated, SJ slightly overestimate t_{th} for $E_p > 20$ keV. In Fig. 2*b* we show the energy absorbed by plasma ions as well as the relative fractions of absorbed energy by electrons and ions. There are small discrepancies between the present results and those of SJ except at the break-even point at $E_p = 44$ keV.

Finally, we have found that for a 3·5 MeV α particle released in a DT plasma such that

$$n_e = 2n_D = 2n_T = 10^{14}\ \mathrm{cm}^{-3}, \qquad T_e = 6\ \mathrm{keV}, \qquad T_D = T_T = 4\ \mathrm{keV}, \tag{43}$$

it takes 238 ms to thermalize, i.e. to slow down to $2T_e = 12$ keV. SJ find a value of 250 ms for the thermalization time. We have also found that during the thermalization process 86·4% of the energy goes into the electrons and 13·6% into the ions. These fractions are in good agreement with the corresponding SJ values of 87% and 13%.

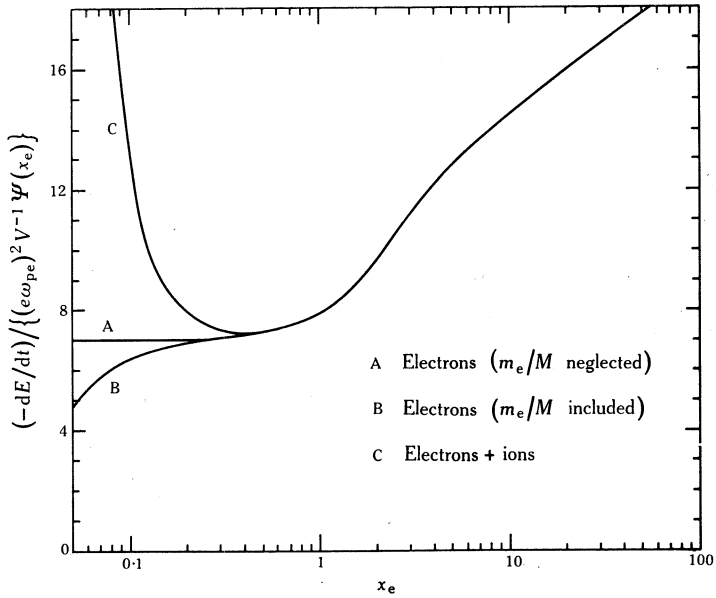


Fig. 3. Energy loss rate dE/dt of a proton injected into a lithium plasma characterized by the parameters (44) of the text, as a function of the proton speed $x_e = (m_e/2T_e)^{\frac{1}{2}} V$. Curve A gives the loss to electrons with m_e/M neglected (as calculated in Paper I), curve B the loss to electrons with the m_e/M correction included and curve C the loss to electrons plus lithium ions. For $x_e \gtrsim 0 \cdot 5$ all three curves are practically identical.

4. Further Quantitative Results

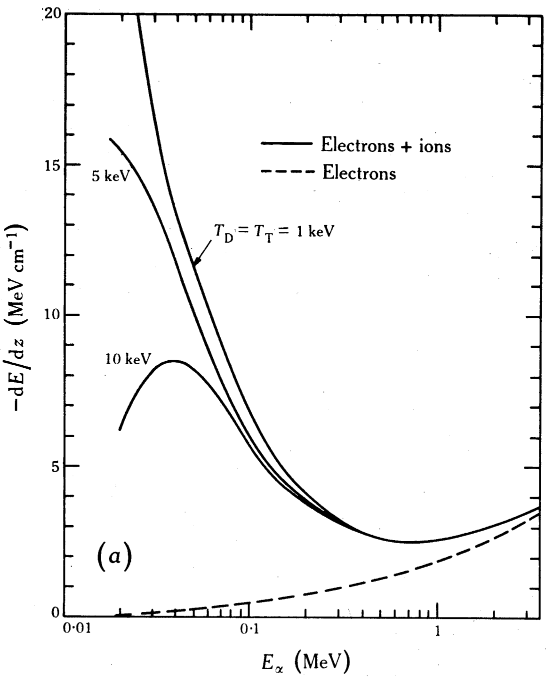
Let us first consider the example studied in Paper I: the energy loss rate of an injected proton in a lithium plasma such that

$$n_e = 3n_{Li} = 10^{14}\,\mathrm{cm}^{-3}, \qquad T_e = T_{Li} = 2\,\mathrm{eV}. \tag{44}$$

Burke and Post (1974) measured the energy loss by protons of several keV ($x_e \approx 1$) in such a plasma. The results of our calculations are shown in Fig. 3. As far as energy loss to electrons is concerned, the correction of order m_e/M is entirely negligible for $x_e \gtrsim 0 \cdot 3$ ($E_p \gtrsim 300$ eV).

The calculation presented in Paper I refers to the one-component electron plasma where ambient plasma ions are replaced by a smeared-out neutralizing background. In the present calculation, on the other hand, plasma ions do affect the energy loss to electrons through their contributions to the dielectric response function. Comparing the present results with those of Paper I, we have confirmed that the one-component approximation used in the calculation of energy loss to electrons is generally excellent. The error caused by this approximation is, in the example considered, of the order of -1% at $x_e = 0 \cdot 05$ and decreases rapidly in magnitude as x_e increases, reaching $-0 \cdot 01\%$ already at $x_e = 0 \cdot 5$.

Turning now to the question of the energy loss to ions, we see from Fig. 3 that this loss becomes appreciable at $x_e \lesssim 0 \cdot 5$. Butler and Buckingham (1962) showed that the energy losses to electrons and ions should be equal at $x_e = 0 \cdot 1$ in the present



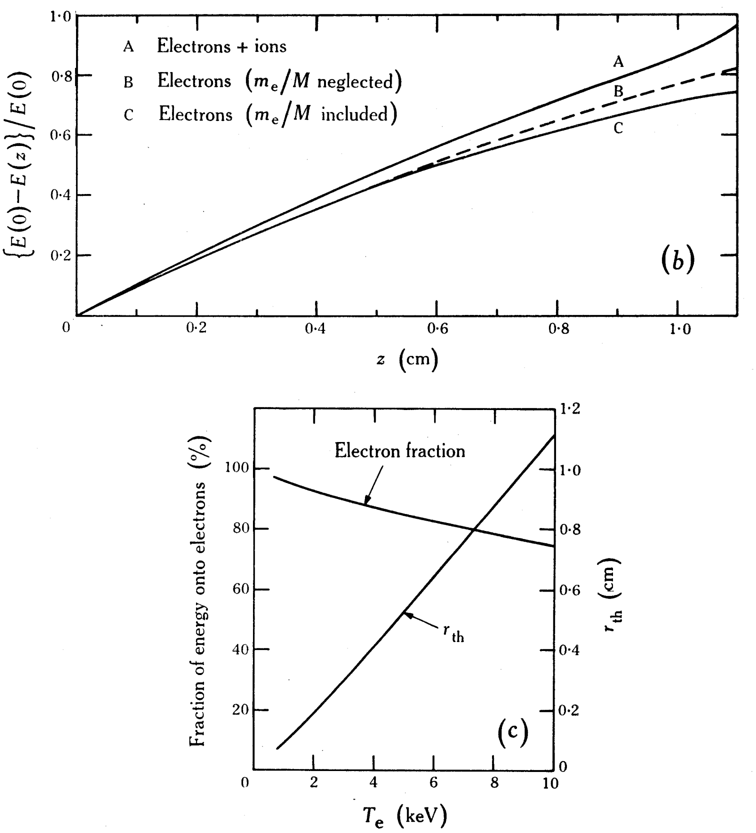
Figs 4a–4c. Showing for a 3·5 MeV α particle in a DT plasma characterized by the density (45) of the text: (*a*) the energy loss rate per traversed distance dE/dz of the α particle as a function of its energy E_α, for $T_e = 10$ keV and $T_D = T_T = 1$, 5 and 10 keV as indicated; (*b*) the fractional energy loss of the α particle as a function of the traversed distance z, for $T_e = 10$ keV and $T_D = T_T = 1$ keV; and (*c*) the fractional energy loss to electrons (left scale) and the thermalization length r_{th} (right scale) of the α particle as a function of T_e, for $T_D = T_T = 1$ keV. The curves in (*b*) represent the calculated energy loss (A) to electrons and ions, (B) in a one-component electron plasma, neglecting terms of order m_e/M (taken from George and Hamada 1978), and (C) to electrons when the m_e/M correction is included.

example. This feature is clearly seen in Fig. 3. The energy loss to ions rapidly decreases as x_e increases and at $x_e \approx 1$ it is already only of the order of 0·1% of the loss to electrons. Analysis of the experimental results of Burke and Post (1974) in terms of the one-component plasma approximation, neglecting terms of order m_e/M, is therefore amply justified.

Next we turn to the energy loss of an α particle in a plasma of high density characteristic of an inertially confined plasma. Specifically we consider a 3·5 MeV α particle released in a DT plasma of solid state density:

$$n_e = 2n_D = 2n_T = 5{\cdot}8 \times 10^{22}\ \mathrm{cm}^{-3}. \tag{45}$$

In Fig. 4*a* we show the energy loss rate per unit distance as function of energy E_α. For $E_\alpha < 0{\cdot}2$ MeV the energy loss to ions dominates. At $E_\alpha = 0{\cdot}2$ MeV, however, 95% of the initial energy is already lost so that this dominance is really immaterial. In fact, we have confirmed that for $T_e = 10$ keV the fraction of energy deposited onto ions during the thermalization of a 3·5 MeV α particle is 25%, essentially independent of ion temperature T_i at least for 1 keV $\leqslant T_i \leqslant$ 10 keV.



Figs 4*b* and 4*c* [see caption on facing page]

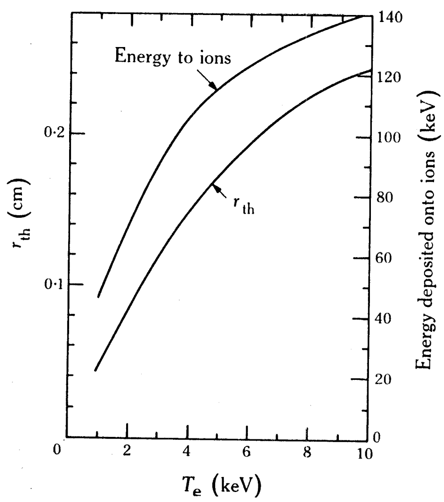


Fig. 5. Energy lost to ions (right scale) and thermalization length r_{th} (left scale) as a function of T_e, for a 200 keV deuterium ion injected into a tritium plasma characterized by the parameters (46) of the text.

Fig. 4*b* gives the fractional energy loss of a $3 \cdot 5$ MeV α particle released in a DT plasma as a function of the distance traversed. Here $T_e = 10$ keV, $T_D = T_T = 1$ keV, and the density is given by equation (45). Curve B refers to the one-component electron plasma as evaluated by George and Hamada (1978), neglecting terms of order m_e/M. The m_e/M correction reduces the loss down to the curve C in Fig. 4*b*.

Fig. 4*c* shows the thermalization length r_{th} of a fusion-produced $3 \cdot 5$ MeV α particle in a plasma as a function of electron temperature. Here the densities are those of equation (45) and $T_D = T_T = 1$ keV. The thermalization length is defined as the distance traversed before the α particle slows down to $2T_e$. Over the covered range of T_e, the thermalization length increases more or less linearly with T_e, in contrast to the often quoted $T^{3/2}$ rule (Chou 1972). The fraction of the energy deposited onto the electrons is also shown.

In connection with plasma heating by beam injection, we finally study the energy loss of deuterium ions injected at 200 keV into a tritium plasma of solid state density:

$$n_e = n_T = 4 \cdot 5 \times 10^{22}\ \mathrm{cm}^{-3}, \qquad T_T = 1\ \mathrm{keV}. \tag{46}$$

For $T_e = 1$–10 keV, the thermalization length r_{th} and the energy deposited onto the ions are shown in Fig. 5. The break-even point, at which equal fractions of energy go to electrons and ions, is at $T_e = 3 \cdot 2$ keV.

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