Aust. J. Phys., 1980, **33**, 771–87

Flow Dynamics in Pulsar Magnetosphere Models with Particle Inertia

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Abstract

Steadily rotating neutron star magnetospheres, with the Lorentz force balanced by inertia, are studied. It is assumed that charged particles leave the star with nonrelativistic speeds and that any returning are decelerated so as to be nonrelativistic on impact. No assumptions are made as to where in the magnetosphere substantial acceleration occurs or as to the degree of charge separation. It is found that the main qualitative features of the flow dynamics for the general oblique rotator can be understood by regarding the poloidal speed v_p as a parameter for the azimuthal speed v_ϕ and Lorentz factor γ of a species, and then considering flow curves that represent the variation of v_p^2/c^2 along each poloidal streamline. There are two possible flow branches—solutions for v_ϕ and γ—for each curve. One is applicable where the dimensionless non-corotational electric potential $\tilde{\Phi} \equiv e\Phi/mc^2$ of the species is <1, which includes all the magnetosphere inside and on the light cylinder, and also where $\tilde{\Phi} \geqslant 1$ provided $v_p^2/c^2 < 1 - 1/x^2$ there; x is the dimensionless cylindrical radial coordinate. The other is applicable inside and on the light cylinder and also where $\tilde{\Phi} < 1$ outside $x = 1$ provided $v_p^2/c^2 > 1 - 1/x^2$ there, but is never valid where $\tilde{\Phi} > 1$. The flow branches can meet, with infinite gradients of v_ϕ and γ, corresponding to failure of the dissipation-free flow.

1. Introduction

The canonical pulsar model consists of a rotating neutron star with the magnetic axis inclined to the rotation axis. Let $\tilde{\omega}$, ϕ and z be cylindrical polar coordinates with the z axis as the rotation axis of the pulsar. The system under consideration is steadily rotating at angular frequency Ω. Hence, it follows from Faraday's law and $\nabla . \boldsymbol{B}=0$ that the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are connected by (Mestel 1971)

$$\boldsymbol{E} + c^{-1}\Omega\tilde{\omega}\boldsymbol{t} \times \boldsymbol{B} = -\nabla\Phi, \tag{1}$$

where c is the speed of light in free space, $\boldsymbol{t}$ is the unit toroidal vector and the gauge-invariant quantity Φ is related to the familiar scalar and vector potentials ϕ and $\boldsymbol{A}$ by $\Phi = \phi - (\Omega\tilde{\omega}/c)A_\phi$ (Endean 1972*a*).

Within the star, the approximation of perfect conductivity is adequate for the present purposes, so that Φ can be put equal to zero there. In the simplest model, the star is embedded in free space; it emits an electromagnetic wave of frequency Ω, except for the axisymmetric case in which the magnetic and rotation axes are parallel or antiparallel. But in this *vacuum* model, there exists a powerful component $E_{\|}$ of $\boldsymbol{E}$ along $\boldsymbol{B}$ near the star. As pointed out by Goldreich (1969), Goldreich and Julian (1969) and Michel (1969) for the axisymmetric case, any charges available tend to flow so as to cancel $E_{\|}$: they suggested that charges will be pulled out of the star and that it may be a better approximation to suppose that $\Phi = 0$ in the magnetosphere. The particle density required is least in the strictly charge-separated case, and can correspond to a very small mass density (Mestel 1971). This suggested the investigation of magnetospheres in the *zero-inertia limit*: the plasma particles are tied to the magnetic field lines and do not carry energy or angular momentum, but they do constitute sources of the electromagnetic field. The electric current density consists of a part parallel to $\boldsymbol{B}$ plus the current of corotation of the net charge density: the condition $E_{\parallel} = 0$ or $\Phi = 0$, together with equation (1), is satisfied by any velocity of the form $\kappa\boldsymbol{B}+\Omega\tilde{\omega}\boldsymbol{t}$, with κ a scalar.

In the axisymmetric Goldreich–Julian (1969) model, there is a corotating region threaded by magnetic field lines that close within the light cylinder, but lines anchored in the 'polar caps' cross the light cylinder. Since inertial terms were neglected, the equations do not automatically keep particle speeds below c: Goldreich and Julian had to introduce a negative contribution $\kappa\boldsymbol{B}_{\mathrm{t}}$ to the toroidal velocity sufficient to keep the toroidal speed below c. There is necessarily associated with this a poloidal velocity $\kappa\boldsymbol{B}_{\mathrm{p}}$. This electron flow, together with an outward flow of positive ions introduced to prevent the star from charging up, constitute an electrically driven stellar wind. The magnetic field lines are constrained to follow the wind and so cannot cross the equator and close, but extend to infinity. Thus, as pointed out by Mestel *et al.* (1979, hereafter referred to as MPW), in the Goldreich–Julian model it is the kinematic need to keep particle speeds below c that leads to the existence of a stellar wind.

There are certain physical difficulties with the Goldreich–Julian model. One, which was pointed out by Okamoto (1974), is an unacceptable magnetic field structure in the charge-separated case: the critical magnetic field line separating the regions of positive and negative corotating charge must be a straight line parallel to the equator. Another, which was pointed out in the original paper, is that the outflowing ions pass through the corotating electron zone—instability and dissipation would be expected to intervene here. Jackson (1976*a*) pointed out that this difficulty is avoided if the positive outgoing current consists of inflowing electrons. These electrons might come from the surrounding nebula, or it might be that electrons leaving the star manage to drift across poloidal magnetic field lines somewhere near or beyond the light cylinder $\Omega\tilde{\omega} = c$ and return to the star. The latter case requires a major extension of the Goldreich–Julian treatment. The most obvious procedure is to include relativistic inertia which, as well as providing drift across poloidal magnetic field lines, has the added advantage of automatically ensuring that particle speeds remain below c; this, in turn, means that a stellar wind is no longer necessarily present (MPW).

In this way, MPW argued, one is led to consider the possibility of constructing a self-consistent model with circulating electrons, together with corotating electrons and ions. Models involving circulating electrons have been proposed by Jackson (1976*a*, 1976*b*), Rylov (1977) and MPW.

The purpose of this paper is not to propose any particular model but to reach an understanding of the main features of the flow dynamics for the general oblique rotator, using the exact relativistic dynamical equations expressing balance between the Lorentz force and inertia.

2. Equations of Motion

For simplicity, we take the plasma to be cold and non-dissipative, with inertia the only nonelectromagnetic term in the equation of motion. It has been pointed out by Endean (1972*a*, 1972*b*) that, under the steady-rotation constraint (1), there exists a constant of the motion Ψ_k for particles of species k:

$$\Psi_k \equiv \Phi + \frac{\gamma_k m_k c^2}{e_k}\left(1 - \frac{\Omega\tilde{\omega}}{c}\frac{v_{k\phi}}{c}\right), \tag{2}$$

where e_k, m_k, γ_k and $v_{k\phi}$ represent the charge, rest mass, Lorentz factor and ϕ component of velocity of the particles of the species concerned.

For multispecies, cold, relativistic plasmas, the flux conservation theorem of magnetohydrodynamics can be generalized to a 'fluxoid' conservation theorem in order to incorporate the effects of particle inertia (Buckingham *et al.* 1972, 1973): the quantity $\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)$, where $\boldsymbol{p}_k \equiv \gamma_k m_k \boldsymbol{v}_k$, is 'frozen-in' to the motion of species k. The steady-rotation condition $\partial/\partial t = -\Omega\,\partial/\partial\phi$ (Mestel 1971; Endean 1972*a*) is valid for, in particular, cylindrical polar components of vectors. Using these results, Burman and Mestel (1978) showed how to simplify considerably the equations of motion. The integral Ψ_k is constant on lines of the reduced flow velocity $\boldsymbol{u}_k$, defined as $\boldsymbol{v}_k - \Omega\tilde{\omega}\boldsymbol{t}$. In this paper it will be assumed that all particles are nonrelativistic when immediately outside the star. In this case, the constant value $m_k c^2/e_k$ taken by Ψ_k on the stellar surface is propagated along the lines of $\boldsymbol{u}_k$ throughout whatever portion of the magnetosphere contains particles of that species. Thus equation (2) becomes

$$1 - \frac{e_k \Phi}{m_k c^2} = \gamma_k\left(1 - \frac{\Omega\tilde{\omega}}{c}\frac{v_{k\phi}}{c}\right), \tag{3}$$

and the equation of motion of species k reduces to the very simple form (Burman and Mestel 1978)

$$\boldsymbol{u}_k \times \{\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)\} = \boldsymbol{0}. \tag{4}$$

It is worth emphasizing that in deriving equations (3) and (4), Burman and Mestel (1978) imposed the physical boundary condition that particles leaving the star are emitted with nonrelativistic speeds and that any particles returning to the star, or accreted by it, are decelerated so as to be nonrelativistic on impact. This does not prejudge the question of whether emitted particles become relativistic only near the light cylinder, or whether $E_{\parallel}$ accelerates them to high Lorentz factors near the star. This question can be settled theoretically only by a full treatment of the magnetospheric structure.

The equation of motion (4) implies that $\boldsymbol{u}_k$ is parallel to $\nabla \times (\boldsymbol{A} + c\boldsymbol{p}_k/e_k)$; that is (Burman and Mestel 1979),

$$\boldsymbol{u}_k = \kappa_k \boldsymbol{B}_k, \tag{5}$$

where κ_k is a scalar and $\boldsymbol{B}_k \equiv \boldsymbol{B} + (c/e_k)\nabla \times \boldsymbol{p}_k$. The vorticity term in the expression for $\boldsymbol{u}_k$ represents an 'inertial drift'. When inertial effects are negligible, equation (5) reduces to the isorotation law $\boldsymbol{u}_k = \kappa_k \boldsymbol{B}$, which means that the lines of the reduced flow velocity coincide with those of the magnetic field; the vorticity or inertial term introduces a departure of the reduced flow lines from the magnetic field lines, implying a departure of the poloidal flow from the poloidal magnetic field lines. Equation (5) is a differential equation relating the flow velocity of the species to the magnetic field. It can also be regarded as a generalized isorotation law, expressing the coincidence of the lines of the reduced flow velocity with those of the 'magneto-inertial' or 'magnetoidal' field $\boldsymbol{B}_k$.

Use of $\partial/\partial t = -\Omega\,\partial/\partial\phi$ and $\boldsymbol{u}_k \equiv \boldsymbol{v}_k - \Omega\tilde{\omega}\boldsymbol{t}$ reduces the continuity equation of species k to $\nabla.(n_k \boldsymbol{u}_k) = 0$, where n_k is the number density. It hence follows from equation (5) and $\nabla.\boldsymbol{B}_k=0$ that $\boldsymbol{u}_k.\nabla(n_k \kappa_k) = 0$, which means that $n_k \kappa_k$ is constant on the lines of $\boldsymbol{u}_k$ which, by equation (5), are identical with the lines of the magnetoidal field (Burman and Mestel 1979).

It should be noted that, since its right-hand side is positive inside the light cylinder, the integral (3) shows immediately that the inequality

$$e_k \Phi/m_k c^2 < 1 \qquad \text{for } \Omega\tilde{\omega}/c < 1 \tag{6}$$

must hold.

The above dynamical equations do not depend on taking any particular geometry for the pulsar magnetosphere. In the next section they will be applied to the special case of axisymmetry.

3. Axisymmetric Model with Inertia

Although it is likely that charge separation occurs in much of the magnetosphere, separation will not be imposed on the equations: the possible occurrence of mixed plasma in some domains is allowed for, but any interaction between the species, other than through the total electromagnetic field, is ignored.

The model is axisymmetric, meaning that all quantities are independent of ϕ. A stream function χ_k is introduced for each species so that the continuity equations are automatically satisfied:

$$e_k n_k u_{k\tilde{\omega}} = -\tilde{\omega}^{-1}\,\partial\chi_k/\partial z\,, \qquad e_k n_k u_{kz} = \tilde{\omega}^{-1}\,\partial\chi_k/\partial\tilde{\omega}\,. \tag{7a, b}$$

Using equations (7), we can integrate the ϕ component of the equation of motion (4) to

$$\tilde{\omega}(p_{k\phi} + e_k A_\phi/c) = Q_k(\chi_k)\,, \tag{8}$$

where Q_k is an arbitrary function of a single variable. The presence of the inertial term $p_{k\phi}$ in equation (8) relates to the detachment of the poloidal flow from the poloidal magnetic field lines. That Ψ_k is constant on lines of $\boldsymbol{u}_k$ can be expressed here as the statement that Ψ_k is a function of χ_k only: $\Psi_k = \Psi_k(\chi_k)$. As already noted, particles are assumed in this paper to be nonrelativistic *at* the surface of the star, so each Ψ_k is constant not merely on lines of $\boldsymbol{u}_k$, but throughout all space occupied by that species. From equations (7) and (8), the $\tilde{\omega}$ and z components of the equation of motion (4) can both be expressed as

$$e_k n_k u_{k\phi}\, Q_k'(\chi_k) = \{\nabla \times (\boldsymbol{p}_k + e_k \boldsymbol{A}/c)\}_\phi\,, \tag{9}$$

where the prime denotes differentiation with respect to the argument. Note that equation (9) is just the ϕ component of $\boldsymbol{u}_k = \kappa_k \boldsymbol{B}_k$, and thus $cn_k Q_k'(\chi_k) = 1/\kappa_k$; that $n_k \kappa_k$ is a function of χ_k only also follows from its constancy on lines of $\boldsymbol{u}_k$.

For convenience, the dimensionless cylindrical polar radial coordinate $x \equiv \Omega\tilde{\omega}/c$ will be used, as will the quantity

$$\varepsilon_k \equiv (\Omega/m_k c^2)\{Q_k(\chi_k) - \tilde{\omega} e_k A_\phi/c\}\,. \tag{10}$$

The integral (8) can now be expressed as

$$\gamma_k x v_{k\phi}/c = \varepsilon_k\,. \tag{11}$$

Note from this that ε_k and $v_{k\phi}$ have the same sign: ε_k is positive or negative according to whether the particles are forward moving or backward moving in the inertial rest frame of the star.

Eliminating γ_k between the integrals (3) and (11) yields

$$xv_{k\phi}/c = \varepsilon_k/(1 - e_k \Phi/m_k c^2 + \varepsilon_k), \tag{12}$$

and hence the integral (3) gives

$$\gamma_k = 1 - e_k \Phi/m_k c^2 + \varepsilon_k. \tag{13}$$

Using the definition of γ_k to eliminate it from the integrals (3) and (11) results in two equations, each relating the three components of $\boldsymbol{v}_k$. Using equation (12) to eliminate $v_{k\phi}$ from either of these gives

$$\frac{v_{k\mathrm{p}}^2}{c^2} = 1 - \frac{1 + \varepsilon_k^2/x^2}{(1 - e_k \Phi/m_k c^2 + \varepsilon_k)^2}, \tag{14}$$

where $v_{k\mathrm{p}}$ denotes the poloidal speed $(v_{k\varpi}^2 + v_{kz}^2)^{\frac{1}{2}}$ of species k. From equations (7),

$$(n_k e_k c\varpi)^2 = \frac{c^2}{v_{k\mathrm{p}}^2}\left\{\left(\frac{\partial \chi_k}{\partial \varpi}\right)^2 + \left(\frac{\partial \chi_k}{\partial z}\right)^2\right\}. \tag{15}$$

Equation (9), which expresses both the ϖ and z components of the equation of motion of the species or the ϕ component of $\boldsymbol{u}_k = \kappa_k \boldsymbol{B}_k$, can be written, using equations (7) and $cn_k Q_k' = 1/\kappa_k$, as

$$\frac{u_{k\phi}}{\kappa_k} = B_\phi - \frac{m_k c^2}{e_k}\left\{\frac{\partial}{\partial \varpi}\left(\frac{\gamma_k}{n_k e_k c\varpi}\frac{\partial \chi_k}{\partial \varpi}\right) + \frac{\partial}{\partial z}\left(\frac{\gamma_k}{n_k e_k c\varpi}\frac{\partial \chi_k}{\partial z}\right)\right\}. \tag{16}$$

Equations (7) and (12)–(15) provide forms for the velocity components, the Lorentz factor, the poloidal speed and the number density of any species, based on the equation of continuity, the two integrals (3) and (8) of the motion and the boundary conditions on the stellar surface. Equation (16) is the corresponding form of the remaining dynamical equation.

Equations (12), (13) and (14) for $v_{k\phi}$, γ_k and $v_{k\mathrm{p}}$ involve the parameter ε_k, but (14) can be regarded instead as giving ε_k in terms of $v_{k\mathrm{p}}$ (together with Φ). This suggests an alternative procedure: development of equations for $v_{k\phi}$ and γ_k without ε_k, but with $v_{k\mathrm{p}}$ appearing as a parameter.

Using the definition $\gamma_k^{-2} \equiv 1 - v_{k\mathrm{p}}^2/c^2 - v_{k\phi}^2/c^2$ to eliminate γ_k from the integral (3) of the motion and then rearranging leads to a result that can be regarded as a quadratic equation for $v_{k\phi}$ involving Φ and having $v_{k\mathrm{p}}$ as a parameter; thus

$$\frac{v_{k\phi}}{c} = \frac{x \pm (1 - e_k \Phi/m_k c^2) d_k}{(1 - e_k \Phi/m_k c^2)^2 + x^2}, \tag{17}$$

where

$$d_k \equiv [\{(1 - e_k \Phi/m_k c^2)^2 + x^2\}(1 - v_{k\mathrm{p}}^2/c^2) - 1]^{\frac{1}{2}}. \tag{18}$$

The integral (3) becomes, using equation (17) for $v_{k\phi}$,

$$\gamma_k = \frac{(1 - e_k \Phi/m_k c^2)^2 + x^2}{1 - e_k \Phi/m_k c^2 \mp x d_k}. \tag{19}$$

The results of this paragraph depend only on the definition of γ_k and the integral (3) of the motion, and so are not restricted to the axisymmetric case but are valid for the oblique rotator.

For the axisymmetric case, the other integral of the motion, namely (11), now becomes using equations (17) and (19)

$$\frac{x^2 \pm (1 - e_k \Phi/m_k c^2) x d_k}{1 - e_k \Phi/m_k c^2 \mp x d_k} = \varepsilon_k. \tag{20}$$

Equations (17), (19) and (15) provide convenient forms for $v_{k\phi}$, γ_k and n_k in which $v_{k\mathrm{p}}$ is treated as a parameter, ε_k having been eliminated; ε_k is given in terms of $v_{k\mathrm{p}}$ and Φ by equation (20).

4. Some Implications of the Equations

In this section, the main features of the flow dynamics will be extracted from the equations developed above. For convenience, the suffix k labelling the species will be dropped.

It has already been pointed out in Section 2 that $e\Phi/mc^2 < 1$ for $x < 1$. Note further that when $e\Phi/mc^2 = 1$, equation (17) becomes $v_\phi/c = 1/x$, which cannot occur inside *or on* the light cylinder: the inequality (6) can be replaced by

$$e\Phi/mc^2 < 1 \qquad \text{for } x \leqslant 1, \tag{21}$$

a result which is valid for the oblique rotator, not just for the axisymmetric case. Since $\gamma > 1$, equation (13) shows that in the axisymmetric case

$$e\Phi/mc^2 < \varepsilon. \tag{22}$$

For $e\Phi < 0$, inequality (21) is automatically satisfied; inequality (22) is automatically satisfied for particles that are forward moving in the inertial frame and states that $|\varepsilon| < |e\Phi|/mc^2$ for backward-moving particles. For $e\Phi > 0$, inequality (22) shows that the particles cannot be backward moving in the inertial frame; for forward-moving particles, (22) must hold everywhere, and together with (21), implies that $e\Phi/mc^2 < \min(1, \varepsilon)$ for $x \leqslant 1$.

In a region of outflow, $e\Phi$ must decrease from zero on the stellar surface to negative values nearby, so that the non-corotational electric force $-e\nabla\Phi$ acts to accelerate the particles away from the star. Further away, in the model proposed by MPW, $e\Phi$ increases, reaching the value mc^2 beyond the light cylinder: with $e\Phi$ increasing, the non-corotational electric force $-e\nabla\Phi$ acts inwards moderating the centrifugal 'slingshot' effect.

Imagine a region, inside the light cylinder, in which the azimuthal velocity component is approximately equal to the corotational speed: $v_\phi \simeq \Omega\tilde{\omega}$. Equations (12) and (13) show that here

$$\varepsilon \simeq x^2(1 - e\Phi/mc^2)/(1-x^2), \qquad \gamma \simeq (1 - e\Phi/mc^2)/(1-x^2). \tag{23a, b}$$

Suppose that $v_\phi \simeq \Omega\tilde{\omega}$, $v_p^2/c^2 \ll x^2$ and $v_p^2/c^2 \ll 1-x^2$, so that $\gamma \simeq (1-x^2)^{-\frac{1}{2}}$) Since $v_\phi \simeq \Omega\tilde{\omega}$, the first inequality means that the species is essentially corotating with the star. As $\gamma \simeq (1-x^2)^{-\frac{1}{2}}$, the approximations (23) show that

$$e\Phi/mc^2 \simeq 1-(1-x^2)^{\frac{1}{2}}, \qquad \varepsilon \simeq x^2(1-x^2)^{-\frac{1}{2}} \tag{24a, b.}$$

in a zone of corotation. Note that the inequalities (6) and (22) are satisfied, as they must be. In particular, the approximations $e\Phi/mc^2 \simeq \frac{1}{2}x^2$ and $\varepsilon \simeq x^2$, both for $x^2 \ll 1$, describe the behaviour of Φ and ε not too far from the star in a corotation zone.

Consider now the case in which $v_\phi \simeq \Omega\tilde{\omega}$ but v_p is at least of similar order to v_ϕ, so that the region is not one of corotation but contains a substantial poloidal flow. Thus, we have $\gamma \simeq (1-x^2-v_p^2/c^2)^{-\frac{1}{2}}$, and equations (23) give

$$\frac{e\Phi}{mc^2} \simeq 1-\frac{1-x^2}{(1-x^2-v_p^2/c^2)^{\frac{1}{2}}}, \qquad \varepsilon \simeq \frac{x^2}{(1-x^2-v_p^2/c^2)^{\frac{1}{2}}} \tag{25a, b}$$

in a zone of poloidal flow. Again, the inequalities (6) and (22) are satisfied, as they must be. In particular, the approximations

$$e\Phi/mc^2 \simeq \tfrac{1}{2}x^2 - v_p^2/2c^2, \qquad \varepsilon \simeq x^2(1+\tfrac{1}{2}x^2+v_p^2/2c^2) \tag{26a, b}$$

for $x^2 \ll 1$ and $v_p^2/c^2 \ll 1$, represent the behaviour of Φ and ε not too far from the star in a poloidal flow zone, so long as the poloidal flow remains nonrelativistic. Note that, provided v_p^2 varies more rapidly near the star than the first power of x, equations (1) and (26a) show that $E_{\parallel} \to 0$ as $x \to 0$ meaning that the Goldreich–Julian conditions are satisfied near the star. (The equations could readily be extended to incorporate the finite size of the star, which has been neglected in obtaining the integral (3) of the motion.) In reality, some residual nonzero value of $E_{\parallel}$ will be required to overcome the surface work function and accelerate the particles away.

Now turn to the equations based on the quadratic equation for v_ϕ, from which an ambiguity of sign has entered equation (17) for v_ϕ and (19) for γ. It is necessary to examine the extent to which the two branches are physically acceptable. Although the relativistic nature of the calculations has ensured that $\gamma^2 > 1$, the algebraic manipulations on which equations (17) and (19) are based have introduced the possibility that formally negative values of γ can arise: conditions to eliminate these must be introduced.

First note that

$$x^2d^2-(1-e\Phi/mc^2)^2 = \{(1-e\Phi/mc^2)^2+x^2\}\{x^2(1-v_p^2/c^2)-1\}. \tag{27}$$

Since $1-e\Phi/mc^2 > 0$ inside and on the light cylinder, it follows from equation (27) that

$$xd < 1-e\Phi/mc^2 \qquad \text{for } x \leqslant 1, \tag{28a}$$

and further that, for $x > 1$,

$$xd \left.\begin{matrix} < \\ = \\ > \end{matrix}\right\} |1-e\Phi/mc^2|, \qquad \text{for } v_p^2/c^2 \left.\begin{matrix} > \\ = \\ < \end{matrix}\right\} 1-1/x^2. \tag{28b}$$

Since γ must be positive, equation (19) shows that

$$\pm xd < 1 - e\Phi/mc^2 \tag{29}$$

must be satisfied everywhere. Inside and on the light cylinder $1 - e\Phi/mc^2 > 0$: inequality (29) with the lower (minus) sign is satisfied, while inequality (28a) shows that (29) with the upper (plus) sign is satisfied. Thus, no spurious solution has been introduced in the region inside the light cylinder: both branches are physically acceptable there. Outside the light cylinder, first consider a region in which $e\Phi/mc^2 < 1$: inequality (29) with the lower sign is satisfied, while inequality (28b) shows that (29) with the upper sign is satisfied provided $v_p^2/c^2 > 1 - 1/x^2$. When $e\Phi/mc^2 = 1$, inequality (29) with the lower sign is satisfied, but the condition $v_p^2/c^2 < 1 - 1/x^2$ must hold for d, and hence γ, to be real with γ finite; inequality (29) with the upper sign cannot be satisfied. In a region with $e\Phi/mc^2 > 1$, inequality (28b) shows that (29) with the lower sign is satisfied provided $v_p^2/c^2 < 1 - 1/x^2$; inequality (29) with the upper sign cannot be satisfied.

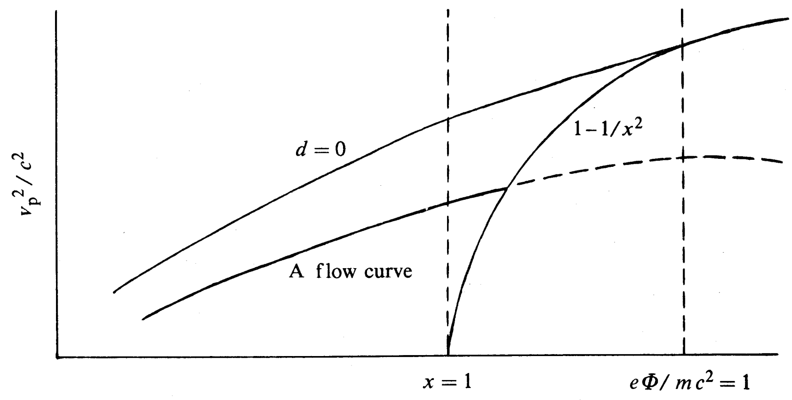

Fig. 1. Sketch of a flow diagram showing a typical flow curve and the flow curve $d = 0$, together with the function $1 - 1/x^2$ for $x \geqslant 1$. A flow curve expresses the variation of v_p^2/c^2 with distance along a poloidal streamline. It represents both flow branches where it is solid; where it is dashed only one branch can exist. The light cylinder $x = 1$ and the surface $e\Phi/mc^2 = 1$ are indicated. In the region on or below $d = 0$ but above $1 - 1/x^2$, both flow branches are valid. In the region below $1 - 1/x^2$, or on $1 - 1/x^2$ except where it meets $d = 0$ (which occurs on $e\Phi/mc^2 = 1$), one branch is valid. Elsewhere no solutions exist. The two solution branches coincide on $d = 0$, where particles can transfer from one branch to the other with dissipation of energy.

In summary, the lower sign branch is applicable for $e\Phi/mc^2 < 1$, which includes all the magnetosphere inside and on the light cylinder, and also where $e\Phi/mc^2 \geqslant 1$ provided $v_p^2/c^2 < 1 - 1/x^2$ there; the upper sign branch is applicable inside and on the light cylinder and also where $e\Phi/mc^2 < 1$ outside $x = 1$ provided $v_p^2/c^2 > 1 - 1/x^2$ there, but it is never valid where $e\Phi/mc^2 > 1$. (See Fig. 1.)

Equation (19) gives an infinite value of γ when $\pm xd = 1 - e\Phi/mc^2$, yielding $v_p^2/c^2 = 1 - 1/x^2$ and $v_\phi/c = 1/x$ which cannot be satisfied inside or on the light cylinder. At a pole of γ, the condition (29) is minimally violated: it follows from the discussion of (29) that such poles can occur, for the lower sign branch, only where

$e\Phi/mc^2 \geqslant 1$ and, for the upper sign branch, only where $e\Phi/mc^2 \leqslant 1$ outside $x = 1$. Formally, such poles separate the allowed positive ranges of γ from the unphysical negative ranges.

If there exists a surface on which $e\Phi/mc^2 = 1$ then, since the upper sign branch of the flow cannot extend into regions in which $e\Phi/mc^2 > 1$, it is clear that particles on this branch penetrating beyond the light cylinder cannot reach $e\Phi/mc^2 = 1$: their outward motion is restricted by the γ pole. This failure of the description of the flow based on balance of the Lorentz force by inertia was discovered by MPW, whose electrodynamic analysis showed that a surface on which $e\Phi/mc^2 = 1$ does occur in their model. As they pointed out, in the vicinity of the γ pole the particles will lose energy by dissipative processes that have been neglected here; the inertial and dissipative effects will cause the poloidal flow to cross poloidal magnetic field lines and return to the star. Thus, provided a surface on which $e\Phi/mc^2 = 1$ exists, the upper sign branch may describe the outflow of poloidally circulating particles which, in a more complete model, would be reflected back to the star without penetrating this surface.

If a surface $e\Phi/mc^2 = 1$ exists, then the behaviour of particles on the lower sign branch of the flow depends on the way in which v_p^2/c^2 varies with x where $e\Phi/mc^2 \geqslant 1$. If $v_p^2/c^2 < 1 - 1/x^2$ is maintained there, then outflowing particles can move off to infinity without encountering a pole of γ, thus forming a stellar wind; but if the flow reaches a point where $v_p^2/c^2 = 1 - 1/x^2$, corresponding to a γ pole, another poloidally circulating flow occurs, distinguished from the former in that the particles can penetrate the surface $e\Phi/mc^2 = 1$. The lower sign branch can instead represent an accretion flow.

Near the star, equations (17) for v_ϕ and (19) for γ show that

$$\gamma \simeq 1 - e\Phi/mc^2 + x^2 \pm xd, \qquad u_\phi/c \simeq \pm d. \tag{30a, b}$$

Hence the lower and upper sign branches can be characterized as representing flows that, near the star, are backward moving and forward moving respectively in the corotating frame.

It should be noted that for d, and hence γ and v_ϕ, to be real

$$\frac{v_p^2}{c^2} \leqslant 1 - \frac{1}{(1 - e\Phi/mc^2)^2 + x^2} \tag{31a}$$

must hold everywhere. It follows, in order for the right-hand side to be non-negative, that

$$(1 - e\Phi/mc^2)^2 + x^2 \geqslant 1 \qquad \text{for } x < 1 \tag{31b}$$

must hold. In particular, inequality (31b) implies that

$$e\Phi/mc^2 \leqslant \tfrac{1}{2}x^2 \qquad \text{where } |e\Phi|/mc^2 \ll 1, \tag{32}$$

placing a limit on $e\Phi/mc^2$ near the star that is consistent with the approximation (26a).

Consider the behaviour of the equations in the vicinity of the light cylinder. In order for d, and so v_ϕ and γ, to be real

$$\frac{v_p^2}{c^2} \leqslant \frac{(1 - e\Phi/mc^2)^2}{(1 - e\Phi/mc^2)^2 + 1} \tag{33}$$

must hold sufficiently close to $x = 1$. If it happens that the poloidal flow is non-relativistic there, with v_p^2/c^2 much less than the right-hand side of inequality (33), then (18) gives (for $e\Phi/mc^2 < 1$)

$$d \simeq 1 - \frac{e\Phi}{mc^2} - \left(1 - \frac{e\Phi}{mc^2}\right)^{-1}\left[\frac{v_p^2}{2c^2}\left\{\left(1 - \frac{e\Phi}{mc^2}\right)^2 + 1\right\} + 1 - x\right].$$

Hence, equations (17) and (19) show that

$$\frac{v_\phi}{c} \simeq 1 - \frac{v_p^2}{2c^2} \quad \text{or} \quad \frac{v_\phi}{c} \simeq \frac{1-(1-e\Phi/mc^2)^2}{1+(1-e\Phi/mc^2)^2}, \tag{34a, b}$$

$$\gamma \simeq \frac{1-e\Phi/mc^2}{v_p^2/2c^2+1-x} \quad \text{or} \quad \gamma \simeq \frac{1+(1-e\Phi/mc^2)^2}{2(1-e\Phi/mc^2)}. \tag{35a, b}$$

If it happens that the poloidal flow is as relativistic as possible here, so that v_p^2/c^2 is close to the right-hand side of inequality (33), then $d \simeq 0$; hence equations (17) and (19) give

$$\frac{v_\phi}{c} \simeq \frac{1}{(1-e\Phi/mc^2)^2+1} \simeq \frac{v_p^2/c^2}{(1-e\Phi/mc^2)^2}, \tag{36a, b}$$

$$\gamma \simeq \frac{(1-e\Phi/mc^2)^2+1}{1-e\Phi/mc^2} \simeq \frac{1-e\Phi/mc^2}{v_p^2/c^2} \tag{37a, b}$$

for both branches. Equations (34)–(37) illustrate the fact that v_ϕ and γ are well behaved at the light cylinder: particles on either branch of the flow encounter no difficulty, at least in principle, in crossing the light cylinder.

Suppose that a surface $e\Phi/mc^2 = 1$ exists and consider the behaviour of the equations in the vicinity of the γ pole of the upper sign branch. Sufficiently close to the pole, the condition

$$x^2\left(\frac{v_p^2}{c^2} - 1 + \frac{1}{x^2}\right) \ll \frac{(1-e\Phi/mc^2)^2}{(1-e\Phi/mc^2)^2+x^2} \tag{38}$$

must be valid so long as the pole occurs before $e\Phi/mc^2 = 1$ is reached. With this condition, equation (18) gives

$$d \simeq \frac{1-e\Phi/mc^2}{x} - \frac{x}{2}\left(\frac{v_p^2}{c^2} - 1 + \frac{1}{x^2}\right)\frac{(1-e\Phi/mc^2)^2+x^2}{1-e\Phi/mc^2}.$$

Hence, for the upper sign branch, equations (17) and (19) give

$$\frac{v_\phi}{c} \simeq \frac{1}{x} - \frac{x}{2}\left(\frac{v_p^2}{c^2} - 1 + \frac{1}{x^2}\right), \tag{39}$$

$$\gamma \simeq \frac{1-e\Phi/mc^2}{x^2 v_p^2/2c^2 + \frac{1}{2}(1-x^2)}. \tag{40}$$

If it happens that the condition (38) holds in the region extending from somewhat inside the light cylinder to the γ pole, so that the flow is nonrelativistic near the light cylinder with

$$\frac{v_p^2}{c^2} \ll \frac{(1-e\Phi/mc^2)^2}{(1-e\Phi/mc^2)^2+1}, \tag{41}$$

then equations (39) and (40) are valid over this region: they reduce to equations (34a) and (35a) near $x = 1$ and also describe a particle's approach to the γ pole.

Consider the behaviour of the equations on a surface where $e\Phi/mc^2 = 1$. Equation (18) shows that for d and hence v_ϕ and γ to be real with γ finite, the condition $v_p^2/c^2 < 1-1/x^2$ must be satisfied. Equation (17) gives $v_\phi/c = 1/x$ and, for the lower sign branch (the only allowed one here with finite γ), equation (19) gives $\gamma = x\{x^2(1-v_p^2/c^2)-1\}^{-\frac{1}{2}}$. If it happens that $v_p^2/c^2 = 1-1/x^2$ where $e\Phi/mc^2 = 1$, then equations (18) and (19) show that d vanishes there with γ infinite for both branches: the two branches coincide in a common pole of γ on the surface $e\Phi/mc^2 = 1$ and no particles can penetrate this surface.

The condition for the two flow branches to coincide is $d = 0$, which is

$$\frac{v_p^2}{c^2} = 1 - \frac{1}{(1-e\Phi/mc^2)^2+x^2}; \tag{42}$$

thus v_p^2/c^2 must at least equal $1-1/x^2$. When equation (42) is satisfied, (17) and (19) give

$$\frac{v_\phi}{c} = \frac{x}{(1-e\Phi/mc^2)^2+x^2} = x\left(1-\frac{v_p^2}{c^2}\right), \tag{43a, b}$$

$$\gamma = \frac{(1-e\Phi/mc^2)^2+x^2}{1-e\Phi/mc^2} = \frac{1}{(1-v_p^2/c^2)(1-e\Phi/mc^2)}. \tag{44a, b}$$

Thus v_ϕ and γ are finite at a coincidence of the flow branches, except that γ diverges where $e\Phi/mc^2 = 1$ if a coincidence occurs there.

It follows from the definition (18) of d that

$$2d\nabla d = \nabla[\{(1-e\Phi/mc^2)^2+x^2\}(1-v_p^2/c^2)]. \tag{45}$$

If the flow branches coincide at an isolated point in space, so that the condition (42) holds at that point but not at neighbouring points, then the right-hand side of equation (45) must be nonzero at the point, and so ∇d diverges there. Hence v_ϕ and γ will have infinite gradients at such a point, corresponding to failure of the dissipation-free flow. If the flow branches coincide over a continuous domain of points in space, so that condition (42) holds over that domain, then the right-hand side of equation (45) vanishes within that domain, but is nonzero on its boundary: v_ϕ and γ will have infinite gradients at least on the boundary. If the flow branches coincide at all points in space where they can both exist—a domain which cannot extend outside the surface $e\Phi/mc^2 = 1$ if that exists—then v_ϕ and γ are given by equations (43) and (44) everywhere in the flow, and they are both finite with finite derivatives, except for the γ pole where $e\Phi/mc^2 = 1$.

The condition (42) for coincidence of the flow branches can be rearranged into the form

$$\frac{e\Phi}{mc^2} = 1 - \left(\frac{1}{1 - v_p^2/c^2} - x^2\right)^{\frac{1}{2}}. \tag{46}$$

For an essentially corotating zone, this reduces on neglecting v_p^2/c^2 to equation (24a) for Φ, as is to be expected since the two branches are essentially in coincidence there. Taking $v_p^2/c^2 \ll 1$ and $x^2 \ll 1$, equation (46) reduces to (26a) for Φ: in a zone of nonrelativistic poloidal flow near the star the two flow branches are close to coincidence, but become more distinct further from the star.

In the axisymmetric model, each poloidal streamline can be labelled by the colatitude θ_s at which it intersects the stellar surface. Thus, v_p^2/c^2 can be regarded as a family of functions of x, parametrized by θ_s, each representing a poloidal streamline. Consider the behaviour of these flow curves, and suppose that, as in the model developed by MPW, there exists a surface on which $e\Phi/mc^2 = 1$—which can only occur outside the light cylinder. The curve $d = 0$, on which the two flow branches are coincident, must always lie above a curve representing the function $1 - 1/x^2$ for $x > 1$, except at $e\Phi/mc^2 = 1$ where they meet; as a possible flow curve, $d = 0$ terminates at $e\Phi/mc^2 = 1$ in a γ pole. All other possible flow curves must lie below $d = 0$ (see Fig. 1). As x increases from the stellar surface, these curves each represent both flow branches until they intersect the curve $1 - 1/x^2$, somewhere between the light cylinder and the surface $e\Phi/mc^2 = 1$, where the upper sign branch terminates in a γ pole. The flow curve continues on, representing the lower sign branch only, which remains valid as x increases unless the flow curve again meets the curve $1 - 1/x^2$, in which case this branch meets a γ pole at that point and the flow curve terminates.

5. Comparison with Other Work

In this section, some of the results derived above will be contrasted with those obtained by MPW and by Ardavan (1976*a–d*).

From the definition of u_ϕ (see Section 2), the definition of γ can be rearranged to give

$$x^{-2}(\gamma x u_\phi/c)^2 + 2\gamma(\gamma x u_\phi/c) + \{1 - (1 - x^2 - v_p^2/c^2)\gamma^2\} = 0. \tag{47}$$

The integral (3) of the motion can be expressed, using the definition of u_ϕ, in the form

$$\gamma = \frac{1 - e\Phi/mc^2 + \gamma x u_\phi/c}{1 - x^2}. \tag{48}$$

Substituting the right-hand side of equation (48) for γ into (47), except where γ appears in the combination $\gamma x u_\phi/c$, gives a quadratic equation for $\gamma x u_\phi/c$ that will be seen below to be equivalent to equation (4.34) of MPW, except that their small gravitational correction factor $\Gamma_0 - 1$ has been omitted here. This quadratic equation has the solutions

$$-\gamma x u_\phi/c = x\frac{(1 - e\Phi/mc^2)x v_p^2/c^2 \pm (1 - x^2)d}{1 - x^2(1 - v_p^2/c^2)}, \tag{49}$$

and the implications could be deduced as in Section 4 above. The approach of MPW (for which they acknowledge a contribution by their referee R. Buckley) is quite different, as will now be discussed.

On introducing f, where $f \equiv v_p^2/x^2u_\phi^2$, in order to facilitate comparison with the analysis of MPW, and substituting the right-hand side of equation (48) for γ, except where γ appears in the combination $\gamma x u_\phi/c$, equation (47) becomes

$$(\gamma x u_\phi/c)^2 = \frac{(1-e\Phi/mc^2)^2+x^2-1}{x^{-2}+(1-x^2)f}, \tag{50}$$

and this is just equation (4.34) of MPW, with the gravitational correction factor omitted. (In their notation $-\gamma x u_\phi/c$ is $1/V$ and f is R_2/R_1^2.) Their analysis is based on treating f as a parameter rather than as a function of the dependent variable in the quadratic equation. In effect, the contribution of the poloidal speed, which actually enters the quadratic equation (47) through the term that does not contain the dependent variable, has been transferred by them to the term in $(\gamma x u_\phi/c)^2$; it then follows, on elimination of those γ factors which do not appear in the combination $\gamma x u_\phi/c$, that the linear term in the quadratic equation appears to drop out, leaving equation (50).

As MPW noted, the conditions (31b) above (which followed from the general condition for v_ϕ to be real) and $x^2(x^2-1)f < 1$ for $x > 1$ must be satisfied in order for equation (50) to have real 'solutions'. They argued that *for fixed* R_1 *and* R_2 (*fixed* f in the present notation), the latter condition imposes an upper limit x_m on x, given by $x_m^2(x_m^2-1) = 1/f$, yielding $x_m^2 = \frac{1}{2}\{1+(1+4/f)^{\frac{1}{2}}\}$, so that $x_m - 1 \simeq 1/2f$ for $f \gg 1$. At $x = x_m$, equation (50) shows that γ is infinite. Hence, from the integral (3), $-u_\phi/c = (x_m^2-1)/x_m$ or $v_\phi/c = 1/x_m$ at $x = x_m$. These results show that $v_p^2/c^2 = 1-1/x_m^2$ at $x = x_m$. The two flow branches of MPW correspond to the two 'solutions' of equation (50) with f treated as a parameter that is independent of the 'solution':

$$-\gamma x u_\phi/c = \pm\left(\frac{(1-e\Phi/mc^2)^2+x^2-1}{x^{-2}+(1-x^2)f}\right)^{\frac{1}{2}}. \tag{51}$$

In their model, because of dissipation, only one of these branches is used outside the light cylinder.

These expressions for v_ϕ and v_p at a pole of γ agree with those obtained in Section 4 above. But the analysis of Section 4 clarifies the distinction between the γ poles of the two flow branches and, in particular, shows that they coincide only in the special case in which the two poles occur on $e\Phi/mc^2 = 1$. MPW allowed x_m to occur either between $x = 1$ and $e\Phi/mc^2 = 1$ or outside $e\Phi/mc^2 = 1$; in my analysis, these two cases correspond to the poles of the two separate flow branches. The analysis of MPW has correctly predicted the fact that γ becomes infinite, but has obscured the behaviour of the dissipation-free flow curves, including the ranges of existence of the flow branches and details of the occurrence of the γ poles.

In their model, MPW regard the poloidal flow as following the poloidal magnetic field lines, except in a dissipation region just outside the light cylinder: in the dissipation-free part of the flow they neglect inertial drifts so far as the flow geometry is concerned, but take some account of inertia through the variable Φ. They identify their two branches of the non-dissipative flow—the two 'solutions' (51) with $-u_\phi$ *apparently* positive and negative—with outflow and inflow respectively. They argue that dissipation-free outflow crosses the light cylinder but terminates at x_m, and that dissipation-free inflow cannot start beyond the light cylinder. However, the analysis of Section 4, particularly equations (33)–(37), has demonstrated that both flow branches are well behaved at the light cylinder, so that particles on either branch have no difficulty *in principle* in crossing the light cylinder. Nevertheless, in practice, the Lorentz factor for the upper sign branch of Section 4, given in a special case by equation (35a) near $x = 1$, might well be too large there for the dissipation-free flow theory to be valid. Presumably the dissipative forces near the light cylinder in the MPW model cause the actual flow branches to cross, so that outflowing particles lose energy, become inflowing particles and return to the star.

In some work on axisymmetric pulsar magnetospheres, Ardavan (1976*a–c*) included the inertial term in the equation of motion of the plasma as a whole, but invoked the magnetohydrodynamic approximation in which not only inertial drifts are neglected but so is $E_{\parallel}$ meaning that the crucially important variable Φ is taken to be constant. The severity of the latter restriction was subsequently recognized by Ardavan (1976*d*). He incorporated the variable Φ, but neglected inertial drifts and thus obtained an approximate integral of the motion. He then derived an expression for γ that contains a denominator which vanishes on the 'Alfvén cylinder', where he applied a 'critical condition' to ensure that γ remains finite.

In contrast to Ardavan's work, the analysis of Section 4 above has shown that the expression obtained for γ which accounts fully for inertial effects has a denominator that cannot vanish within the light cylinder: the expression (19) for γ is well behaved inside the light cylinder without requiring any critical condition; in fact the numerator can never vanish, so no critical condition is possible and poles of γ must occur when $v_p^2/c^2 = 1 - 1/x^2$. It is noteworthy that MPW, who neglected inertial drifts, did not require any critical condition. The discrepancy between Ardavan's results and those of both MPW and the present paper is attributable to his neglecting inertial drifts in obtaining his claimed constant of the motion. As MPW pointed out, the inertial drifts are vital for the energy and angular momentum integrals, and hence for the Endean integral involving the variable Φ, even when they hardly affect the flow geometry: in much of the flow it may be a good approximation to neglect inertial drifts in describing the flow geometry, but neglecting inertial drifts to obtain an integral of the motion is inadmissable.

6. Oblique Rotator

Because of its symmetry and consequent stationary nature, the axisymmetric model cannot represent an actual pulsar, though it could represent possible nonpulsed gamma-ray sources (MPW). Magnetospheres of neutron stars with magnetic and rotation axes that need be neither parallel nor antiparallel will now be considered.

The flow dynamics can be investigated by using the definition of γ_k together with the integral (3) of the motion and treating the poloidal speed v_{kp} of the species as a parameter. Eliminating the Lorentz factor gives a quadratic equation for the toroidal speed, with solutions (17), exactly as in Section 3. If, alternatively, $v_{k\phi}$ is eliminated, then the solutions (19) for γ_k result. Thus, almost all of the analysis of the dynamics given in Section 4 carries over immediately to the problem of the oblique rotator.

Mestel (1980) has given an analysis of the flow dynamics for the oblique rotator which, like the work of MPW, is based on the approximation in which inertial drifts are neglected in treating the geometry of the flow while some account of inertial effects is taken through the variable Φ. This approach is directed towards a model of the type proposed by MPW, in which the poloidal flow is tied to the poloidal magnetic field lines almost until dissipative forces take over. Mestel's analysis will now be compared with that developed in this paper. For simplicity, the suffix k labelling the species will again be dropped.

On using η, where $\eta \equiv v_{\mathrm{p}}^2/u_\phi^2$, the definition of γ can be expressed in the form

$$\gamma^{-2} = 1 - x^2 - 2xu_\phi/c - (1+\eta)u_\phi^2/c^2\,. \tag{52}$$

In terms of u_ϕ instead of v_ϕ, the integral (3) is

$$\gamma^{-1} = \frac{1 - x^2 - xu_\phi/c}{1 - e\Phi/mc^2}\,. \tag{53}$$

So long as inertial drifts are negligible, $\eta \simeq B_{\mathrm{p}}^2/B_\phi^2$ so that η can be treated as a parameter that is independent of u_ϕ. (Mestel used $B_{\mathrm{p}}^2/B_\phi^2$ as his definition of η.) Eliminating u_ϕ where it appears explicitly, or eliminating γ, between equations (52) and (53) leads to the following expressions for u_ϕ/c and γ (Mestel 1980), based on treating η as an independent parameter:

$$\frac{u_\phi}{c} = \frac{x\{1-x^2-(1-e\Phi/mc^2)^2\} \pm (1-e\Phi/mc^2)\Delta}{(1+\eta)(1-e\Phi/mc^2)^2+x^2}\,, \tag{54}$$

$$\gamma = \frac{(1+\eta)(1-e\Phi/mc^2)^2+x^2}{(1-e\Phi/mc^2)\{1+\eta(1-x^2)\} \mp x\Delta}\,, \tag{55}$$

where

$$\Delta \equiv \{(1-e\Phi/mc^2)^2+x^2-1\}^{\frac{1}{2}}\{1+\eta(1-x^2)\}^{\frac{1}{2}}\,. \tag{56}$$

The procedure of treating η as an independent parameter has transferred the contribution of the poloidal speed, which actually enters the quadratic equation for u_ϕ through the term that does not contain the dependent variable, to the term in u_ϕ^2.

Mestel (1980) pointed out that, for the upper sign branch, equation (55) implies that γ is infinite on the light cylinder and that, for the lower sign branch, γ will be infinite if the magnetic field line on which the particles under consideration are travelling passes through a point, outside the light cylinder, where

$$x^2 = 1 + 1/\eta \simeq 1 + B_\phi^2/B_{\mathrm{p}}^2\,. \tag{57a, b}$$

At such a point, Δ vanishes and the two dissipation-free flow branches described by equations (54) and (55) coincide; but in the MPW-type model, because of dissipation related to the apparent pole on the light cylinder, only one of these branches is used outside the light cylinder. At a point given by (57a), equation (54) shows that $v_\phi/c = 1/x$ and, since $\eta \equiv v_{\mathrm{p}}^2/u_\phi^2$, that $v_{\mathrm{p}}^2/c^2 = 1 - 1/x^2$. These expressions for v_ϕ and v_{p} at a pole of γ agree with those obtained in Section 4 above. But the apparent occurrence of a pole on the light cylinder for the upper sign branch is an artefact of the neglect of inertial drifts in this vicinity: equation (54) implies that, for this branch, $u_\phi = 0$ on the light cylinder so that η is infinite there and its treatment as an independent parameter fails. As discussed in Section 4, both dissipation-free flow branches are well behaved at the light cylinder, so that particles on either branch have no difficulty *in principle* in crossing the light cylinder, although the Lorentz factor for the upper sign branch might well be too large there for the dissipation-free flow theory to be valid. The qualitative features of the dissipation-free flow curves are clearer when v_p is used as a parameter rather than η.

The singular behaviour at points given by equations (57) in fact follows directly from consideration of the isorotation approximation (Goldreich and Julian 1969; Scharlemann and Wagoner 1973; Scharlemann 1974). Expressing the constraint $v < c$ in the form $(u_\phi/c + x)^2 + (v_p/c)^2 < 1$, and using the isorotation law $\boldsymbol{u} \simeq \kappa \boldsymbol{B}$ to substitute for u_ϕ and v_p, gives

$$\boldsymbol{B}^2(\kappa/c)^2 + 2xB_\phi(\kappa/c) + x^2 - 1 < 0\,. \tag{58}$$

Thus, so long as inertial drifts remain negligible outside the light cylinder, the condition

$$B_\phi^2/B_p^2 \geqslant x^2 - 1 \qquad \text{for } x > 1\,, \tag{59}$$

obtained by Goldreich and Julian (1969), must be satisfied in order for κ to be real. When the strict inequality in (59) is violated, large accelerations occur (Scharlemann 1974) and inertial effects cannot be neglected; as pointed out by Scharlemann, these accelerations are centrifugal rather than electromagnetic in origin. Equality in (59) implies that the isorotation approximation has led to $v = c$ and γ infinite. Equation (57b) represents the same result: taking the variable Φ into account while still neglecting inertial drifts has no effect on the occurrence of the γ poles, though it misleadingly *appears* to place the γ pole of one branch on the light cylinder. It is now clear that taking inertia fully and exactly into account does not remove the γ poles: they occur when $v_p^2/c^2 = 1 - 1/x^2$, corresponding exactly to equation (57a) with η *defined as* v_p^2/u_ϕ^2.

7. Concluding Remarks

The purpose of this paper has been to elucidate the essential features of the flow dynamics in steadily rotating neutron star magnetospheres using the exact dynamical equations expressing balance between the Lorentz force and relativistic inertia, neglecting dissipative forces. Inaccuracies and obscurities in the work of other authors, who used approximate methods, have been avoided by using the exact dynamical equations with inertial effects fully incorporated.

MPW neglected inertial drifts in discussing the flow geometry, but pointed out that they are vital for the energy and angular momentum integrals, and hence for the Endean integral containing Φ, even if they hardly affect the flow geometry. Their work, which was on the axisymmetric case, and Mestel's (1980) subsequent work on the oblique case, were directed towards models in which inertial drifts have a negligible effect on the flow geometry almost until dissipative forces take over near the light cylinder. This approach may well be a valid one for model building, and has correctly predicted the fact that γ becomes infinite, but has tended to obscure the qualitative features of the dissipation-free flow curves, including details of the occurrence of the γ poles and the ranges of existence of the flow branches.

The work of this paper has been based on the equations of continuity and motion of pressure-free charged fluids, together with the source-free members of Maxwell's set of equations, namely Faraday's law and $\nabla . \boldsymbol{B} = 0$, under the steady-rotation constraint. The electrodynamic equations containing source terms, namely the Ampère–Maxwell and Gauss laws, remain to be invoked in order to study the effects of the flows on the electrodynamic field. I intend to discuss this topic later.

Acknowledgments

I thank Professor L. Mestel, F. R. S., for preprints of the papers by Mestel *et al.* (1979) and Mestel (1980), and for some very informative discussions. I thank a referee for his comments which have led to a considerable improvement in the presentation of this paper.

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Manuscript received 12 October 1979, accepted 15 April 1980