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Nonlinear Magnetic Convection at Small Prandtl Numbers

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Abstract

Nonlinear magnetic convection is investigated using the modal equations for cellular convection. The boundary layer method is used assuming large Rayleigh number R and small Prandtl number σ for different ranges of the Chandrasekhar number Q. The heat flux F is determined for the value of the wave number which maximizes F. For a weak field, F is independent of Q but increases with $R\sigma$; for a moderate field, F decreases with Q but increases with R and σ; for a strong field, F decreases with Q, increases with R and is independent of σ. F eventually becomes $O(1)$ as $Q \to O(R)$, and the layer becomes stable.

Introduction

This paper studies nonlinear magnetic convection at small Prandtl numbers under the so-called modal forms of the equations for momentum, heat and magnetic field. Briefly, these equations are constructed by expanding the fluctuating quantities in a complete set of functions of the horizontal coordinate, and then truncating the expansion. The single mode equations are derived simply by retaining only the first term in the expansion. A more detailed discussion of these equations and their derivations are given by Gough *et al.* (1975); earlier, the same system of equations was derived in a different way by Roberts (1966) using a procedure proposed by Glansdorff and Prigogine (1964). Numerical computations of the single mode equations for thermal convection have recently been carried out by Toomre *et al.* (1977), and good agreement with the asymptotic results obtained by Gough *et al.* (1975) was found.

The treatment in this paper is for the steady state. Numerical studies by Weiss (1975) indicate that a steady state can be reached by a hydromagnetic convective flow of finite amplitude. Of course, sufficiently strong convective flows are time dependent, but the present study aims at exploring the properties of nonlinear magnetic convection in the simpler case of a steady state, as an approximation to the physical situation. The particular interest and importance of magnetic convection in geophysical and astrophysical situations, where the Prandtl number is not large and nonlinearities are strong, have motivated the present study.

Before we start formulating the problem and considering the technical details, it is useful to note briefly the physical effects on the flow conditions for various strengths of the external field. When the imposed vertical field is weak, the convection and the heat transport are unaffected and the fluctuating field is produced kinematically. For a moderate or strong field, the Lorentz force is significant, the convective cells are reduced in horizontal size and the heat transport is decreased. The rigidity imparted by the external field tends to suppress the convection and thus seriously reduce the heat transport. For a sufficiently strong field, the convection is suppressed entirely and the layer becomes stable.

Governing Equations

We consider an infinite horizontal layer of fluid of depth d permeated by a magnetic field. The upper and lower surfaces are assumed to be stress free and are maintained at temperatures T_0 and $T_0+\Delta T$ $(\Delta T > 0)$ respectively. The magnetic field $\boldsymbol{H}^* = (H_1^*, H_2^*, H_3^*)$ can be written as $\boldsymbol{H}^* = [\![\boldsymbol{H}^*]\!]+\boldsymbol{h}^*$, where the open brackets denote a horizontal average. Since $[\![\boldsymbol{H}^*]\!]$ is only a function of the vertical variable z and we have $\nabla . [\![\boldsymbol{H}^*]\!] = 0$, the component $[\![H_3^*]\!]$ must be a constant, taking the value of an assumed impressed field with vanishing horizontal components. If the field strength is measured in units of $[\![H_3^*]\!]$ then we have $\boldsymbol{H}^* = \boldsymbol{K}+\boldsymbol{h}^*$, where $\boldsymbol{K}$ is a unit vector in the vertical direction.

The modal equations of the hydromagnetic convection are derived from the Boussinesq equations for momentum, magnetic field and heat by expanding the fluctuating variables in the planform functions $f_n(x,y)$ of linear theory (Gough *et al.* 1975). The nondimensional steady state forms of these equations, after truncating the expansion by retaining only the first term, are

$$(\mathrm{D}^2-a^2)^2 W = Ra^2 T + C\sigma^{-1}\{2\mathrm{D}W(\mathrm{D}^2-a^2)W + W(\mathrm{D}^2-a^2)\mathrm{D}W\}$$

$$-\tau Q(\mathrm{D}^2-a^2)\mathrm{D}h_3, \tag{1a}$$

$$\tau(\mathrm{D}^2-a^2)h_3 + \mathrm{D}W = 0, \tag{1b}$$

$$(\mathrm{D}^2-a^2)T + (1-WT+F)W = C(2W\mathrm{D}T + T\mathrm{D}W). \tag{1c}$$

The advection of the magnetic field in equation (1b) and nonlinear interactions of $\boldsymbol{h}^*$ with $\boldsymbol{h}^*$ and $\boldsymbol{u}$ in equations (1a) and (1b) are neglected, since the present analysis is restricted to the parameter range $\tau \gg 1$, where $\tau = \eta/K$ is the ratio of magnetic diffusivity η to thermal diffusivity K. In equations (1), $h_3 f_1$ and Wf_1 are the vertical components of the magnetic vector $\boldsymbol{h}^*$ and velocity vector $\boldsymbol{W}$ respectively, Tf_1 is the deviation of the temperature from its horizontal average, $R = \alpha g \Delta T d^3/K\nu$ is the Rayleigh number, $Q = [\![H_3^*]\!]^2 d^2/\mu\rho_0 \nu\eta$ is the Chandrasekhar number and $\sigma = \nu/K$ is the Prandtl number, μ being the magnetic permeability, ρ_0 the reference density (constant), ν the kinematic viscosity, α the coefficient of thermal expansion and g the acceleration due to gravity. Also, a is the horizontal wave number, D denotes d/dz, $F = \langle WT \rangle$ is the heat flux, the angle brackets indicating a further vertical average over the whole layer, and $C = [\![\tfrac{1}{2}f_1^3(x,y)]\!]$ is the parameter derived from the planform function $f_1(x,y)$. The constant C vanishes for rolls and rectangles and has a value of $6^{-\frac{1}{2}}$ for a hexagonal planform. We shall assume here that C is nonzero and takes the value of $6^{-\frac{1}{2}}$ as a representative magnitude. For $C = 0$, the system of equations (1) reduce to the so-called mean field equations for magnetic convection; these have been solved for this problem recently (Riahi 1980).

We shall rescale our dependent variables such that

$$\omega = (FR)^{-\frac{1}{2}}W, \qquad \theta = (R/F)^{\frac{1}{2}}T, \qquad H = (FR)^{-\frac{1}{2}}\tau h_3. \tag{2}$$

The governing equations then take the forms

$$(\mathrm{D}^2-a^2)^2\omega = a^2\theta + C\sigma^{-1}(FR)^{\frac{1}{2}}\{2\mathrm{D}W(\mathrm{D}^2-a^2)\omega+\omega(\mathrm{D}^2-a^2)\mathrm{D}\omega\}$$
$$-Q(\mathrm{D}^2-a^2)\mathrm{D}H, \tag{3a}$$

$$(\mathrm{D}^2-a^2)H+\mathrm{D}\omega = 0, \tag{3b}$$

$$(FR)^{-1}(\mathrm{D}^2-a^2)\theta+(1-\omega\theta+F^{-1})\omega = C(FR)^{-\frac{1}{2}}(2\omega\,\mathrm{D}\theta+\theta\,\mathrm{D}\omega). \tag{3c}$$

We shall use the following constraint to evaluate the heat transport:

$$F = (1-R^{-1}\langle|\nabla\theta|^2\rangle)/\langle(1-\omega\theta)^2\rangle, \tag{4}$$

which is obtained by multiplying equation (3c) by θ and taking the vertical average over the layer.

The boundary conditions to be considered for the free surfaces at $z = 0$ and 1 are

$$\omega = \mathrm{D}^2\omega = \theta = H = 0. \tag{5}$$

The subsequent analysis and solution of equations (3)–(5) supposes throughout that both the Rayleigh number and the heat flux are large. The magnitude of the Chandrasekhar number Q is allowed to vary, and different classes of solutions are found for different orders of its magnitude. In each case, the principal focus is on the unique solution that maximizes F.

Solutions by Boundary Layer Method

(*a*) *Weak Field*

The wave number a is supposed to be large (which can be justified *a posteriori*), so that the convection cells are narrow. The solutions can be obtained by matching asymptotic approximations in the interior with two distinct regions near each boundary. In the interior of the layer, the inertial, buoyancy and convection terms are important, and equations (3) are satisfied by

$$\omega = (\sigma C^{-1}z)^{1/3}(FR)^{-1/6}, \qquad \theta = \omega^{-1}, \tag{6a, b}$$

$$H = \tfrac{1}{3}(\sigma C^{-1}z^{-2})^{1/3}(FR)^{-1/6}a^{-2}. \tag{6c}$$

Near each surface and adjacent to the interior are intermediate layers of thickness $O(a^{-1})$, in which vertical derivatives are important in the inertial term. Defining appropriate boundary layer coordinates $\zeta_\mathrm{t} = a(1-z)$ and $\zeta_\mathrm{b} = az$ for the upper and lower of these layers respectively, we find that equations (3) yield: as $\zeta_\mathrm{t}\to 0$,

$$\omega = (\sigma/C)^{1/3}(FR)^{-1/6}(\tfrac{1}{2})^{1/3}(3\zeta_\mathrm{t})^{2/3}, \qquad \theta = \omega^{-1}, \tag{7a, b}$$

$$H = \tfrac{6}{5}(\sigma/C)^{1/3}(FR)^{-1/6}a^{-1}(\tfrac{3}{4})^{2/3}\zeta_\mathrm{t}^{5/3}; \tag{7c}$$

and, as $\zeta_\mathrm{b}\to 0$,

$$\omega = (\sigma/Ca)^{1/3}(FR)^{-1/6}\zeta_\mathrm{b}(3\log\zeta_\mathrm{b}^{-1})^{1/3}, \qquad \theta = \omega^{-1}, \tag{7d, e}$$

$$H = -\tfrac{1}{2}a^{-1}(\sigma/Ca)^{1/3}(FR)^{-1/6}\zeta_\mathrm{b}^2(3\log\zeta_\mathrm{b}^{-1})^{1/3}. \tag{7f}$$

Closer to each surface and adjacent to the intermediate layers are thermal layers, in which thermal conduction is significant and θ is brought to its zero boundary value. We define δ_t and δ_b as the thicknesses of the top and bottom thermal layers respectively. Also, $\eta_t = (1-z)/\delta_t$ and $\eta_b = z/\delta_b$ are defined to be the corresponding variables in these layers. We then find from the governing equations (3), after applying matching conditions (matching the solutions with the corresponding ones in the intermediate layers), that the solutions in the lower thermal layer are

$$\omega = A_b \eta_b, \qquad \theta = \frac{C^{\frac{1}{2}}\eta_b}{2A_b}\int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\{\tfrac{1}{2}C\eta_b^2(1-t)\}\, dt, \tag{8a, b}$$

$$H = -\tfrac{1}{2}\delta_b A_b \eta_b^2, \tag{8c}$$

where

$$A_b = (\sigma/Ca)^{1/3}(FR)^{-1/6} a\delta_b \{3\log(a\delta_b)^{-1}\}^{1/3}, \qquad \mu^2 = 1 + C^{-2}. \tag{9}$$

Similarly, we find the following solutions in the upper thermal layer:

$$\omega = A_t \eta_t, \qquad \theta = -\frac{C^{\frac{1}{2}}\eta_t}{2A_t}\int_1^{\mu^2} (\mu^2 - t^2)^{-\frac{1}{4}} \exp\{\tfrac{1}{2}C\eta_t^2(t-1)\}\, dt, \tag{10a, b}$$

$$H = \tfrac{1}{2}\delta_t A_t \eta_t^2, \tag{10c}$$

where

$$A_t = (\sigma/2C)^{1/3}(FR)^{-1/6}(3a\delta_t)^{2/3}. \tag{11}$$

To determine F, we must evaluate the expressions $\langle|\nabla\theta|^2\rangle$ and $\langle(1-\omega\theta)^2\rangle$ in equation (4). After use of a formal procedure to maximize the heat flux (Chan 1971), we find

$$a = (\tfrac{1}{3})^{1/2}(32)^{1/32}(\tfrac{1}{7})^{15/32}(\tfrac{7}{6}I)^{3/16}(R\sigma/C)^{9/32}(\log R\sigma)^{-1/32}, \tag{12a}$$

$$\delta_b = (\tfrac{7}{6}I)^{1/8}(224)^{3/16}(R\sigma/C)^{-5/16}(\log R\sigma)^{-3/16}, \tag{12b}$$

$$\delta_t = (\tfrac{1}{48})^{1/5}(\tfrac{7}{6}I)^{3/20}(224)^{9/40}(R\sigma/C)^{-3/8}(\log R\sigma)^{-1/40}, \tag{12c}$$

$$F = (\tfrac{1}{70})^{3/16}(\tfrac{6}{7}I^{-1})^{9/8}(\tfrac{5}{16})^{3/16}(R\sigma/C)^{5/16}(\log R\sigma)^{3/16}, \tag{12d}$$

where

$$I = 1{\cdot}062(1 + C^2)^{1/4}. \tag{13}$$

The detailed analysis (not included here) shows that the conditions for the validity of the solutions are

$$R^{-1} \ll \sigma \ll (R^{-1}\log R)^{1/9}, \qquad Q \ll R^{5/8}\sigma^{-3/8}(\log R\sigma)^{1/24}. \tag{14}$$

(b) Moderate Field

The solutions for this case can be obtained by matching asymptotic approximations in the interior with three distinct regions near the lower boundary and two distinct regions near the upper boundary. The solutions (6) hold again in the interior of the layer. Near the lower and upper boundary and adjacent to the interior are an inner layer of thickness ε and an intermediate layer of thickness $O(a^{-1})$ respectively. In the inner layer the inertial buoyancy and Lorentz forces are all of comparable magnitudes, and equations (3) yield, as $\zeta \to 0$,

$$\omega = (\sigma\varepsilon/C)^{1/3}(FR)^{-1/6}(\tfrac{2}{3})^{1/2}\zeta(\log \zeta^{-1})^{1/2}, \qquad \theta = \omega^{-1}, \tag{15a, b}$$

$$H = (\sigma/C)^{1/3}(FR)^{-1/6}(\tfrac{2}{3})^{1/2}a^{-2}\varepsilon^{-2/3}(\log \zeta^{-1})^{1/2}, \tag{15c}$$

where $\zeta = z/\varepsilon$ is the layer variable. The detailed analysis indicates that

$$\varepsilon = (Q/3a^2)^{3/4}(\sigma/C)^{1/2}(FR)^{-1/4} \tag{16a}$$

and that ε must satisfy the condition

$$1 \gg \varepsilon \gg a^{-1}. \tag{16b}$$

The intermediate layer near the upper boundary has essentially the same structure as the corresponding one for the case of a weak field and so equations (7a)–(7c) hold.

Closer to the lower boundary is another intermediate layer of thickness $O(a^{-1})$. The solutions in this layer obtained after applying matching conditions are

$$\omega = (\sigma\varepsilon/C)^{1/3}(FR)^{-1/6}(\tfrac{2}{3})^{1/2}(a\varepsilon)^{-1}(\log a\varepsilon)^{1/2}\zeta_b, \qquad \theta = \omega^{-1}, \tag{17a, b}$$

$$H = (\tfrac{2}{3})^{1/2}(\sigma/C)^{1/3}(FR)^{-1/6}a^{-2}\varepsilon^{-2/3}(\log a\varepsilon)^{1/2}\{1-\exp(-\zeta_b)\}, \tag{17c}$$

with $\zeta_b = az$ as before.

There are further layers closer to each boundary and adjacent to the intermediate layers, in which thermal conduction is significant and θ is brought to its zero boundary value. Using the same notation as in subsection (*a*) above, the governing equations (3) yield the expressions (10) and (11) and the following results:

$$\omega = A_b\eta_b, \qquad \theta = \frac{C^{\frac{1}{2}}\eta_b}{2A_b}\int_1^{\mu^2}(\mu^2-t^2)^{-\frac{1}{4}}\exp\{\tfrac{1}{2}C\eta_b^2(1-t)\}\,\mathrm{d}t, \tag{18a, b}$$

$$H = a^{-1}A_b\eta_b, \tag{18c}$$

where

$$A_b = (\sigma\varepsilon/C)^{1/3}(FR)^{-1/6}(\tfrac{2}{3})^{1/2}(\delta_b/\varepsilon)(\log a\varepsilon)^{1/2}. \tag{19}$$

Using equation (4) to evaluate F, we find

$$\varepsilon = (5QR^{-1})^{3/4}, \tag{20a}$$

$$a = 3^{-9/20}.2^{-7/20}.5^{-1/4}I^{1/5}(\sigma/C)^{3/10}R^{1/4}Q^{1/20}\{\log(Q^{16/5}\sigma^{6/5}R^{-2})\}^{-1/20}, \tag{20b}$$

$$\delta_b = 3^{3/10}.2^{-1/10}.5^{1/2}I^{1/5}(\sigma/C)^{-1/5}Q^{3/10}R^{-1/2}\{\log(Q^{16/5}\sigma^{6/5}R^{-2})\}^{-3/10}, \tag{20c}$$

$$\delta_t = 3^{-4/25}.2^{3/25}.5^{3/10}I^{4/25}(\sigma/C)^{-9/25}R^{-2/5}Q^{1/25}\{\log(Q^{16/5}\sigma^{6/5}R^{-2})\}^{-1/25}, \tag{20d}$$

$$F = (\tfrac{4}{5})^{3/2}(\tfrac{1}{24})^{3/10}I^{-6/5}R^{1/2}Q^{-3/10}(\sigma/C)^{1/5}\{\log(Q^{16/5}\sigma^{6/5}R^{-2})\}^{3/10}. \tag{20e}$$

The conditions obtained for the validity of these solutions are

$$R^{5/8}\sigma^{-3/8} \ll Q \ll R^{5/7}\sigma^{-2/7}\log R\sigma, \qquad \text{for} \qquad R^{-1} \ll \sigma \ll R^{-1/8}; \tag{21a}$$

$$R^{5/8}\sigma^{-3/8} \ll Q \ll \sigma^{-6}\log(\sigma^{-1}R^{1/9}) \qquad \text{for} \qquad R^{-1/8} \ll \sigma \ll R^{-1/9}. \tag{21b}$$

(c) Moderately Strong Field

The preceding boundary layer solution in the ranges (21) was based essentially on the condition $a \ll \delta_b^{-1}$. Now as Q further increases beyond the ranges (21), we will obtain a new condition $a = \delta_b^{-1}$. Thus near the lower boundary, the intermediate layer will coincide with the thermal layer. The solutions in this new thermal layer are then found to be

$$\omega = A_b\eta_b, \tag{22a}$$

$$\theta = \tfrac{1}{2}C^{\frac{1}{2}}\left(\frac{\mu+1}{\mu-1}\right)^{1/4C\mu}\eta_b\int_1^{\mu}\left(\frac{\mu-t}{\mu+t}\right)^{1/4C\mu}(\mu^2-t^2)^{-\frac{1}{4}}\exp\{\tfrac{1}{2}C\eta_b^2(1-t)\}\,dt, \tag{22b}$$

$$H = a^{-1}A_b\{1-\exp(-\eta_b)\}. \tag{22c}$$

The maximization of F can proceed as before, and we find:

$$\varepsilon = 3^{-5/2}.2^{7/4}(\sigma/C)^{1/2}J^2Q^{5/2}R^{-2}\{\log(Q^2R^{-4/3}\sigma^{2/3})\}^{-7/4}, \tag{23a}$$

$$a = (3R/2JQ)\{\log(Q^2R^{-4/3}\sigma^{2/3})\}, \tag{23b}$$

$$\delta_b = a^{-1}, \tag{23c}$$

$$\delta_t = 2^{11/5}.3^{-12/5}J^{8/3}(\sigma/C)^{-1/5}R^{-11/5}Q^2\{\log(Q^2R^{-4/3}\sigma^{2/3})\}^{-2}, \tag{23d}$$

$$F = (3R/2Q)J^{-2}\{\log(Q^2R^{-4/3}\sigma^{2/3})\}, \tag{23e}$$

where

$$\begin{aligned} J \approx \frac{(2\pi)^{3/2}}{3C}\Bigg\{&\left(\frac{\mu-1}{2}\frac{\sqrt{3}}{2}+\frac{\mu+1}{2}-\frac{1}{C}\right)\left(1-\frac{\sqrt{3}}{2}\right)^{\frac{1}{4}(1/C\mu-3)} \\ &\times\left(\frac{(\mu-1)\sqrt{3}}{2}+(3\mu+1)\right)^{-\frac{1}{4}(1/C\mu+5)}+\left(\frac{\mu+1}{2}-\frac{1}{C}\right)\left(3\mu+1\right)^{-\frac{1}{4}(1/C\mu+5)} \\ &+\left(-\frac{\mu-1}{2}\frac{\sqrt{3}}{2}+\frac{\mu+1}{2}-\frac{1}{C}\right)\left(1+\frac{\sqrt{3}}{2}\right)^{\frac{1}{4}(1/C\mu-3)} \\ &\times\left(-\frac{(\mu-1)\sqrt{3}}{2}+(3\mu+1)\right)^{-\frac{1}{4}(1/C\mu+5)}\Bigg\}(\mu-1)^{-7/4}(\mu+1)^{1/4C\mu}. \end{aligned} \tag{24}$$

The following conditions are found to be necessary for the validity of these solutions:

$$R^{2/3}\sigma^{-1/3} \ll Q \ll R^{4/5}\sigma^{-1/5}(\log\sigma R)^{7/10}, \qquad \text{for} \qquad R^{-1} \ll \sigma \ll R^{-2/11}; \tag{25a}$$

$$R^{4/5}\sigma^{2/5}(\log \sigma R^{2/11}) \ll Q \ll R^{4/5}\sigma^{-1/5}(\log \sigma R)^{7/10},$$

$$\text{for} \quad R^{-2/11} \ll \sigma \ll (\log \sigma R)^{7/6}(\log \sigma R^{2/11})^{-5/3}. \quad (25\text{b})$$

(*d*) *Strong Field*

For Q larger and beyond the ranges (25), the inner layer thickness ε no longer satisfies the condition (16b) and we must then have the new condition

$$a^{-1} \ll \delta_b \ll 1. \quad (26)$$

A detailed analysis for this condition indicates that the value of a which maximizes F satisfies the relation

$$a \sim Q^{2/3}\sigma^{2/3}R^{-1/3}(\log RQ^{-1})^{-1/6}. \quad (27)$$

The solutions can be obtained by matching asymptotic approximations in the interior with two distinct regions (thermal and magnetic layers) near each boundary. In the interior of the layer, the buoyancy and Lorentz forces balance, although the inertial force could become significant. The governing equations (3) then yield the results, as $z \to 0$,

$$\omega = a(2Q^{-1})^{\frac{1}{2}}z(\log z^{-1})^{\frac{1}{2}}, \qquad \theta = \omega^{-1}, \quad (28\text{a, b})$$

$$H = a^{-1}(2Q^{-1})^{\frac{1}{2}}(\log z^{-1})^{\frac{1}{2}}. \quad (28\text{c})$$

Similar asymptotic results near $z = 1$ can be obtained by replacing z with $1-z$ in equations (28). Near each surface and adjacent to the interior are thermal layers in which thermal conduction is significant and θ is brought to its zero boundary value. We define $\eta_b = z/\delta_b$ to be the variable in the lower layer, δ_b being the thickness of the lower thermal layer. After matching the solutions to the corresponding solutions in the interior, we obtain the results

$$\omega = A_b\eta_b, \qquad \theta = \eta_b/A_b(1+\eta_b^2), \qquad H = A_b a^{-2}\delta_b^{-1}, \quad (29)$$

where

$$A_b = (2Q^{-1})^{\frac{1}{2}}a\delta_b(\log \delta_b^{-1})^{\frac{1}{2}}. \quad (30)$$

The analogous solutions and analysis in the upper thermal layer (which has the same structure as the lower one) are straightforward and shall not be repeated here.

The solutions for ω and θ satisfy the required boundary conditions at $z = 0$ and 1. A further thinner layer near each boundary is then needed to adjust the solution to satisfy the correct boundary condition on H. This is a magnetic layer of thickness a^{-1} with $\zeta_b = za$ as its variable (for the lower layer). We then find from the governing equations that

$$H = A_b a^{-2}\delta_b^{-1}\{1-\exp(-\zeta_b)\}. \quad (31)$$

A similar analysis and solution in the upper magnetic layer is straightforward.

To determine F, we evaluate as before the expressions for $\langle|\nabla\theta|^2\rangle$ and $\langle(1-[\![\omega\theta]\!])^2\rangle$ in equation (4) and find

$$\delta_b = \tfrac{1}{2}\pi Q(R\log RQ^{-1})^{-1}, \quad (32\text{a})$$

$$\delta_t = \delta_b, \tag{32b}$$

$$F = 2\pi^{-2}Q^{-1}R(\log RQ^{-1}), \tag{32c}$$

δ_t denoting the thickness of the upper thermal layer. The conditions for the validity of these solutions are

$$R^{4/5}\sigma^{-2/5}(\log\sigma R^{1/2})^{7/10} \ll Q \leqslant R^{4/5}\sigma^{-8/5}(\log\sigma R^{1/8})^{-2/5},$$
$$\text{for} \quad R^{-1/8} \ll \sigma \ll (\log\sigma R^{1/8})^{-1/3}(\log\sigma R^{1/2})^{-7/12}; \tag{33a}$$

$$R^{4/5}\sigma^{-2/5}(\log\sigma R^{1/2})^{7/10} \ll Q \ll R$$
$$\text{for} \quad R^{-1/2} \ll \sigma \leqslant R^{-1/8}. \tag{33b}$$

Discussion

The boundary layer analysis here has shown that it is appropriate to divide the parameter space into four different regions: (*a*) For a weak field, F is independent of Q; the stabilizing effect of the field is so small that the maximizing flow behaves as if there were no field. The horizontal scale of convection cells is also independent of Q. (*b*) For a moderate field, the convection cells are reduced in size perpendicular to the imposed field. The heat transport decreases with Q. A new inner layer is developed, in which the Lorentz force is comparable in magnitude with either the inertial or buoyancy force. (*c*) For a moderately strong field, the behaviour of the flow falls in between that for a moderate field and a strong field. The dependence of F on σ is quite weak, while its dependence on Q and R is essentially the same as in the case of a strong field. (*d*) For a sufficiently strong field, the inner layer mentioned in (*b*) disappears. However, a new thin magnetic layer is developed which is responsible for bringing the fluctuating field to its correct boundary values.

It is interesting to note that in each of the cases discussed above, F approaches a value $O(1)$ as σ approaches its smallest possible value and Q attains its largest possible value. Thus, convection in a fluid with larger σ seems to be less affected by the stabilizing effects of an imposed field than in a fluid with a smaller σ. It is also seen from the boundary layer solutions that for sufficiently large Q the flow properties depend weakly on σ.

Recently Riahi (1980) investigated the problem of magnetic convection under the so-called mean field approximation ($C = 0$). It was clear that in this case the dependence of the flow on σ disappears, and the results are believed to describe correctly the average properties of the flow at large σ. Thus some similarities may be expected between the results of the present study and those for the case $C = 0$ at sufficiently large Q, and in fact, for a moderately strong or strong field, the dependence of F on Q and R is found to be unchanged qualitatively in both cases $C = 0$ and $C \neq 0$. But the conditions on the validity of the solutions, the size of the convection cells and other results are different.

Busse (1975) investigated the effect of a weak vertical field on two-dimensional steady convection. He found, in particular, that the influence of the field decreased with increasing amplitude of convection so that a finite amplitude onset of steady convection became possible at values of R considerably below that predicted by the linear theory. The present study does not predict this phenomenon. It is seen from equation (32c) that, as $Q \to R$, F approaches $O(1)$, in agreement with the linear theory (Chandrasekhar 1961). The approximation $\tau \gg 1$ made in this paper which eliminates the advection of the magnetic field and the nonlinear interactions of the fluctuating field and the velocity in the induction equation is primarily responsible for ruling out the possibility of subcritical instability. The phenomenon of magnetic flux expulsion by convection (Galloway *et al.* 1978) is also not predicted by the present model, since the magnetic field amplifies weakly by the convective flows in the range $\tau \gg 1$.

Numerical studies for three-dimensional nonlinear magnetic convection have not yet been done, but the two-dimensional problem has recently been analysed numerically by Weiss (1975). He finds, in particular, that for $R = 10^5$, $\sigma = 1$, $\tau = 0 \cdot 2$ and cells of length $0 \cdot 5$ steady convection exists for $Q \leqslant 2 \cdot 16 \times 10^4$ (in agreement with the condition 33b), and for $Q \leqslant 1 \cdot 25 \times 10^2$ the field no longer has any dynamical significance (in agreement with the conditions 14). The present study is based on the maximized nonlinear asymptotic state ($R \to \infty$) for small σ and $\tau \gg 1$, whereas Weiss's analysis was concerned with moderately large values of R and moderate values of τ and σ. It is not expected therefore that there will be many other similarities between the results of these two studies.

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