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Mathematical Theory of Cylindrical Isothermal Blast Waves in a Magnetic Field

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Abstract

An investigation is made of the self-similar flow behind a cylindrical blast wave from a line explosion (situated on $r = 0$, using conventional cylindrical coordinates r, ϕ, z) in a medium whose density and magnetic field both vary as $r^{-\omega}$ ahead of the blast front, with the assumption that the flow is isothermal. The magnetic field can have components in both the azimuthal B_ϕ and longitudinal B_z directions. It is found that:

(i) For $B_\phi \neq 0 \neq B_z$ a continuous single-valued solution with a velocity field representing outflow of material away from the line of explosion does not exist for $\omega < 0$, but only for $\omega \geqslant 0$.

(ii) For $B_z = 0$, but $B_\phi \neq 0$, there is no continuous single-valued solution with a velocity field representing outflow of material away from the line of the explosion for any ω value.

(iii) For $B_\phi = 0$, but $B_z \neq 0$, the behaviour is as follows: (a) for $\omega < 0$ a continuous single-valued solution with outflow of gas everywhere behind the shock does not exist; (b) for $\omega = 0$ the solution is singular and piecewise continuous with an inner region where no fluid flow occurs and an outer region where the fluid flow gradually increases; (c) for $1 > \omega > 0$ the governing equation possesses a set of movable critical points. In this case it is shown that the fluid flow velocity is bracketed between two curves and that the asymptotes of the velocity curve on the shock are intersected by, or are tangent to, the two curves. Thus a solution always exists in the physical domain $r \geqslant 0$.

The overall conclusion from the investigation is that the behaviour of isothermal blast waves in the presence of an ambient magnetic field differs substantially from the behaviour calculated for no magnetic field. These results have an impact upon previous applications of the theory of self-similar flows to evolving supernova remnants without allowance for the dynamical influence of magnetic pressure and magnetic tension.

1. Introduction

In a previous paper (Lerche 1979, hereinafter referred to as Paper I) we pointed out that, despite the extensive application of self-similar flows behind blast waves (Sedov 1959) in the analysis and interpretation of observations of supernova remnants (SNRs) (see e.g. Woltjer 1972; Gorenstein *et al.* 1974; Rappaport *et al.* 1974), nevertheless there exist lacunae in our knowledge of the dynamical evolution of blast wave behaviour which weaken claims concerning the detailed understanding of the observed properties (and inferences drawn from them).

For instance, we pointed out in Paper I, as had others before us (Sedov 1959; Parker 1963; Solinger *et al.* 1975; Lerche and Vasyliunas 1976), that adiabatic models of blast wave behaviour give rise to large temperature gradients which can be inconsistent with the adiabatic assumption that the heat flux can be neglected. Solinger *et al.* (1975) demonstrated quantitatively this inconsistency and, like Sedov

1959) and Parker (1963), advocated the use of isothermal models instead. The above analyses (including that of Lerche and Vasyliunas 1976) give a treatment with a magnetic field determined kinematically, i.e. the flow equations are solved ignoring the field, and the field structure and evolution are then determined from the Lenz law.

However, a potentially more serious problem than just modifying the internal equation of state of the gas behind the shock wave is the neglect of magnetic field effects in influencing the dynamical evolution of the blast wave. Recently, it has been recognized from an analysis of the radio brightness variations across 33 SNRs (Caswell and Lerche 1979*a*, 1979*b*) that the galactic magnetic field plays a dominant role in the evolution of SNRs. Proper consideration must, therefore, be given to the effects of magnetic fields, and their influence on the dynamical evolution of blast waves must be incorporated. While we recognize that the temporal behaviour of SNRs is, presumably, more accurately described by a spherical blast wave, both Cox (1972) and McCray *et al.* (1975) have emphasized that a simplified one-dimensional treatment (ignoring curvature of the shock front) is sufficient to bring out the underlying physics very succinctly. Therefore in Paper I, we investigated the behaviour of a planar isothermal blast wave in a magnetic field in order to provide a vehicle for illustrating the basic dynamical effects of the magnetic field pressure on the evolution of a blast wave.

However, despite the arguments advanced in favour of the basic behaviour being adequately described by a one-dimensional treatment, there remains a slight, nagging, unresolved worry that, since two- and three-dimensional effects (such as oblique magnetic fields and the curvature of the shock front) have not been included in a one-dimensional treatment, there really is no guarantee that the arguments have not overlooked some subtle effect which can only be ascertained by direct calculation. There is the further point, too, that two- and three-dimensional calculations probably provide more realistic models of SNRs than those which have hitherto been available.

Thus, there are strong arguments, both mathematical and physical, for developing the theory of self-similar isothermal flow behind a blast wave beyond the one-dimensional treatment provided in Paper I.

We had intended, as we remarked already in Paper I, to proceed directly in this second paper to a discussion of a spherical isothermal blast wave in a magnetic field, but resolution of this problem is still lacking due principally to the highly complicated nonlinear coupled differential equations governing the evolution of material behind the blast wave. It seems appropriate, however, to investigate here the evolution of a cylindrical blast wave in a magnetic field as the effects of shock-front curvature and oblique magnetic fields can be incorporated into such a discussion. Thus, while this situation is not, perhaps, the problem of direct relevance to the behaviour of SNRs in the galactic magnetic field, nevertheless effects of curvature and oblique magnetic fields are indeed included; in some sense then, discussion of cylindrical blast wave behaviour is a model closer to reality than the planar blast wave discussed in Paper I. While it is true that the governing equations in the cylindrical case are considerably more complex than those in the planar case, it is equally true that they are considerably simpler than their spherical counterparts.

It is on all of the above grounds that we consider an investigation of cylindrical isothermal blast waves to be relevant.

2. Properties of Cylindrical Isothermal Self-similar Blast Waves

(a) *Formulation of Problem*

The general method of constructing the equations describing the self-similar flow is well known and available in standard texts (Landau and Lifschitz 1959; Sedov 1959; Parker 1963). Accordingly, this section will be brief and serves chiefly to introduce notation.

We assume that the density of the cold ambient medium ahead of the blast wave varies with the distance r from the line of explosion as $\rho(r) = \rho_0(a/r)^{\omega}$, where ρ_0 is the density at the reference level (only values of $\omega < 2$ are of physical interest; $\omega \geqslant 2$ would imply an infinite total mass contained within the blast wave). We also assume that the magnetic field imbedded in the ambient medium possesses components in the ϕ and z directions (the usual cylindrical coordinates r, ϕ, z are employed with the origin at the line of the explosion), varying with distance r from the line of explosion as

$$B_\phi(r) = B_{\phi 0}(a/r)^{\Lambda}, \qquad B_z(r) = B_{z0}(a/r)^{\beta}.$$

Note that $\nabla . \boldsymbol{B} = 0$ is identically satisfied. (Only values of β, $\Lambda < 1$ are of physical interest; β, $\Lambda \geqslant 1$ would imply an infinite total magnetic energy contained within the blast wave.)

Let a blast wave move out from $r = 0$ at $t = 0$ so that at time t the blast front is at position $R_s(t)$. The assumption of self-similarity implies that, within the blast wave, the density $\rho(r,t)$, the r-directed flow speed $V_r(r,t)$, the magnetic field components $B_\phi(r,t)$ and $B_z(r,t)$ and the temperature $T(r,t)$ are to be written

$$\rho = \eta\, \rho_0(a/R_s)^{\omega} R(\lambda), \tag{1}$$

$$V_r = (\eta-1)\eta^{-1} V_s\, U(\lambda), \tag{2}$$

$$B_\phi = \eta\, B_{\phi 0}(a/R_s)^{\Lambda} b_\phi(\lambda), \tag{3}$$

$$B_z = \eta\, B_{z0}(a/R_s)^{\beta} b_z(\lambda), \tag{4}$$

$$\begin{aligned} kT = {} & m(\eta-1)\eta^{-2} V_s^2\, \theta(\lambda) - (\eta^2-1)\eta^{-1}(8\pi\rho_0)^{-1}(R_s/a)^{\omega} \\ & \times \{B_{\phi 0}^2(a/R_s)^{2\Lambda} + B_{z0}^2(a/R_s)^{2\beta}\}. \end{aligned} \tag{5}$$

In these equations R, U, θ, b_ϕ and b_z are dimensionless functions of the argument $\lambda \equiv r/R_s$, and $V_s = \mathrm{d}R_s/\mathrm{d}t$. If the constant η is chosen to be the density magnification factor across the shock front, then the equations of mass, momentum and flux conservation across the shock wave are satisfied with $R(1) = U(1) = b_\phi(1) = b_z(1) = \theta(1) = 1$. The assumption of isothermal flow corresponds to setting $\theta(\lambda) = 1$. When this is done the parameter η is determined by the solution to the flow equations and cannot be set to the customary value 4, which is appropriate to adiabatic post-shock flow for a constant speed shock.

Now the equations of continuity, momentum and magnetic induction are respectively

$$\frac{\partial \rho}{\partial t} + r^{-1}\frac{\partial}{\partial r}(\rho r V_r) = 0, \tag{6}$$

$$\rho\left(\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}\right) = -\frac{\partial}{\partial r}\left(\rho k T m^{-1}+(8\pi)^{-1}(B_\phi^2+B_z^2)\right)-(4\pi r)^{-1}B_\phi^2, \tag{7}$$

$$\frac{\partial B_\phi}{\partial t}+\frac{\partial}{\partial r}(V_r B_\phi) = 0, \tag{8}$$

$$\frac{\partial B_z}{\partial t}+r^{-1}\frac{\partial}{\partial r}(r V_r B_z) = 0. \tag{9}$$

Insertion of equations (1)–(5) into (6)–(9) yields four equations for the four functions $R(\lambda)$, $U(\lambda)$, $b_\phi(\lambda)$ and $b_z(\lambda)$:

$$\frac{\mathrm{d}\ln R}{\mathrm{d}\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)-\omega+(\eta-1)\eta^{-1}\left(\frac{\mathrm{d}U}{\mathrm{d}\lambda}+U\lambda^{-1}\right) = 0, \tag{10}$$

$$\frac{\mathrm{d}\ln b_\phi}{\mathrm{d}\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)-\Lambda+(\eta-1)\eta^{-1}\frac{\mathrm{d}U}{\mathrm{d}\lambda} = 0, \tag{11}$$

$$\frac{\mathrm{d}\ln b_z}{\mathrm{d}\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)-\beta+(\eta-1)\eta^{-1}\left(\frac{\mathrm{d}U}{\mathrm{d}\lambda}+U\lambda^{-1}\right) = 0, \tag{12}$$

$$\begin{aligned}(\eta-1)UR_s\ddot{R}_s &+(\eta-1)\dot{R}_s^2\frac{\mathrm{d}U}{\mathrm{d}\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)\\ &= -B_{\phi 0}^2\eta^2\{4\pi\rho_0\lambda R(\lambda)\}^{-1}b_\phi^2(a/R_s)^{2\Lambda-\omega}-R^{-1}\eta\\ &\quad\times\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(R(\lambda)\,k\,T(t)\,m^{-1}+\eta(8\pi\rho_0)^{-1}\{B_{\phi 0}^2 b_\phi^2(a/R_s)^{2\Lambda-\omega}+B_{z0}^2 b_z^2(a/R_s)^{2\beta-\omega}\}\right).\end{aligned} \tag{13}$$

But the self-similar assumption demands that R, U, b_ϕ and b_z be functions of λ *only*. Equation (13) is, therefore, valid only when (i) the temperature T is proportional to V_s^2, (ii) $R_s\ddot{R}_s\dot{R}_s^{-2}$ is constant, (iii) V_s^2 is proportional to $R_s^{\omega-2\Lambda}$ and (iv) V_s^2 is proportional to $R_s^{\omega-2\beta}$. But the conditions (iii) and (iv) can be satisfied only when $\beta=\Lambda$, and then (iv) implies $R_s = R_0 t^{2/\{2(1+\Lambda)-\omega\}}$ where R_0 is constant. In this case

$$R_s\ddot{R}_s\dot{R}_s^{-2} = -(\Lambda-\tfrac{1}{2}\omega),\qquad R_s^{\omega-2\Lambda}\dot{R}^{-2} = R_0^{\omega-2(1+\Lambda)}\{(1+\Lambda)-\tfrac{1}{2}\omega^2\}.$$

Inspection of equations (10) and (12) reveals that to avoid a singularity in *either* R or b_z as U passes through $\lambda\eta(\eta-1)^{-1}$ it is necessary that $\omega=\Lambda$; therefore, we require $\omega<1$ since $\Lambda<1$*. Under these conditions R_s is proportional to $t^{2/(2+\omega)}$.

* As in Paper I, we note that if we take $\omega=\Lambda$ *before* manipulating equations (10)–(13) we find, as will be seen below, that in fact $U(\lambda)$ is everywhere less than $\lambda\eta(\eta-1)^{-1}$. Thus, neither R nor b_z has a singularity. Whether the same is true when $\omega\neq\Lambda$ is unknown. The point is that the structure of the equations determining the post-shock flow properties depends on the parameters ω and Λ. For $\omega\neq\Lambda$, elimination of R, b_z and b_ϕ in favour of U leads to a fourth-order ordinary nonlinear differential equation. The topological nature of the flow pattern is determined by such an equation. However, for $\omega=\Lambda$ the governing equation, while nonlinear, is only third order. This is such an enormous simplification that the present investigation has been restricted to precisely the $\omega=\Lambda$ case. Results of calculations bearing on the more general case would be of considerable interest.

Clearly, only values $\omega > -2$ are of physical interest. Equations (10)–(13) then yield

$$b_z(\lambda) = R(\lambda), \tag{14}$$

$$\frac{db_\phi}{d\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)+b_\phi\left(-\omega+(\eta-1)\eta^{-1}\frac{dU}{d\lambda}\right) = 0, \tag{15}$$

$$\frac{dR}{d\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right)+R\left\{-\omega+(\eta-1)\eta^{-1}\left(\frac{dU}{d\lambda}+U\lambda^{-1}\right)\right\} = 0, \tag{16}$$

$$-\tfrac{1}{2}\omega U+\frac{dU}{d\lambda}\left((\eta-1)\eta^{-1}U-\lambda\right) = -2\eta(1-Y)(\eta^2-1)^{-1}\sin^2\psi\, b_\phi^2(\lambda R)^{-1}$$

$$-R^{-1}\frac{d}{d\lambda}\left(Y\eta^{-1}R+\eta(1-Y)(\eta^2-1)^{-1}(\sin^2\psi\, b_\phi^2+R^2\cos^2\psi)\right), \tag{17}$$

where $Y = \eta^2(\eta-1)^{-1}m^{-1}kTV_s^{-2}$ is a constant less than one, and $B_{z0} = B_0\cos\psi$, $B_{\phi 0} = B_0\sin\psi$. Equation (5) implies $Y \leqslant 1$ with equality if and only if $B_0 = 0$.

It is convenient to define the new variables

$$\varpi = \lambda\eta Y^{-\frac{1}{2}}(\eta-1)^{-\frac{1}{2}}, \qquad u = U(\eta-1)^{\frac{1}{2}}Y^{-\frac{1}{2}}, \tag{18a, b}$$

$$r = 2R\eta^2\cos^2\psi\,(1-Y)Y^{-1}(\eta^2-1)^{-1}, \qquad B_\phi = b_\phi 2\eta^2(1-Y)Y^{-1}(\eta^2-1)^{-1}\sin\psi\cos\psi, \tag{18c, d}$$

in terms of which equations (15)–(17) can be written

$$\frac{dB_\phi}{d\varpi}(u-\varpi)+B_\phi\left(\frac{du}{d\varpi}-\omega\right) = 0, \tag{19}$$

$$\frac{du}{d\varpi}(u-\varpi)-\tfrac{1}{2}\omega u = -B_\phi^2(\varpi r)^{-1}-(1+r^{-1})\frac{dr}{d\varpi}-(2r)^{-1}\frac{dB_\phi^2}{d\varpi}, \tag{20}$$

$$\frac{dr}{d\varpi}(u-\varpi)+r\left(\frac{du}{d\varpi}+u\varpi^{-1}-\omega\right) = 0. \tag{21}$$

Note that the parameter η no longer appears explicitly. Here we explore analytically the nature of the solution to the $B_\phi(\varpi)$, $u(\varpi)$ and $r(\varpi)$ equations in the physical domain $\varpi \geqslant 0$.

The physical requirements that $U(\lambda) = R(\lambda) = b_\phi(\lambda) = b_z(\lambda)$ on $\lambda = 1$ (the shock front) yield

$$\varpi_s = \eta Y^{-\frac{1}{2}}(\eta-1)^{-\frac{1}{2}}, \qquad u_s = Y^{-\frac{1}{2}}(\eta-1)^{\frac{1}{2}}, \tag{22a, b}$$

$$r_s = 2\eta^2\cos^2\psi\,(1-Y)Y^{-1}(\eta^2-1)^{-1}, \qquad B_s = 2\eta^2(1-Y)Y^{-1}(\eta^2-1)^{-1}\cos\psi\sin\psi. \tag{22c, d}$$

Eliminating η in equations (22) gives the shock-curve equations

$$\varpi_s = u_s + (Yu_s)^{-1}, \qquad \eta = 1 + Yu_s^2 \geqslant 1, \tag{23a, b}$$

$$r_s = 2(1-Y)Y^{-2}\cos^2\psi\,(1+Yu_s^2)^2(2+Yu_s^2)^{-1}u_s^{-2}, \tag{23c}$$

$$B_s = 2(1-Y)Y^{-2}\cos\psi\sin\psi\,(1+Yu_s^2)^2(2+Yu_s^2)^{-1}u_s^{-2} = r_s\tan\psi. \tag{23d}$$

Note that the minimum value of ϖ_s is $2Y^{-\frac{1}{2}}$, occurring when $u_s = Y^{-\frac{1}{2}}$, and that at this minimum value we have

$$\eta = 2, \qquad r_s = \tfrac{8}{3}(1-Y)Y^{-1}\cos^2\psi, \qquad B_s = \tfrac{8}{3}(1-Y)Y^{-1}\sin\psi\cos\psi.$$

Now the fluid structure equations (19)–(21) *and* the values of their solutions on the shock equations (23a), (23c) and (23d) are no longer dependent on η explicitly. Hence, the topology of the solutions $u(\varpi)$, $r(\varpi)$ and $B_\phi(\varpi)$ can be discussed independently of the value of η.

Equations (19)–(21) are three first-order ordinary differential equations. They require specification of three boundary conditions. Physically, an obvious requirement is that the flow speed $u(\varpi)$ vanish at the origin $\varpi = 0$; thus an appropriate boundary condition is $u(0) = 0$ (or more precisely $u(\varpi) \to 0$ as $\varpi \to 0$). The second physical boundary condition is that the density $r(\varpi)$ be finite at the origin; $r(\varpi{=}0) = r_0$ say, with $r_0 > 0$ (or more precisely that the normalized mass be zero as $\varpi \to 0$, i.e. $\varpi^2 r(\varpi) \to 0$ as $\varpi \to 0$). The third boundary condition involving the toroidal field component B_ϕ is that at the origin ϖB_ϕ is to be bounded as $\varpi \to 0$.

Equations (19)–(21) are remarkable in that they can be combined into a single second-order ordinary differential equation. An appropriate dependent variable is

$$M(\varpi) = \int_0^{\varpi} \varpi'\, r(\varpi')\,\mathrm{d}\varpi', \tag{24}$$

with

$$r(\varpi) = M_\varpi/\varpi, \qquad u(\varpi) = \varpi - (2-\omega)M/M_\varpi, \tag{25a, b}$$

where a subscript indicates differentiation, i.e. $M_\varpi \equiv \mathrm{d}M/\mathrm{d}\varpi$. Substitution of equations (25a) and (25b) shows that (21) is satisfied automatically. Equation (19) integrates directly to give

$$B_\phi^2(\varpi) = \Gamma^2 M_\varpi^2 M^{-2/(2-\omega)}, \tag{26}$$

where Γ is, so far, an arbitrary constant to be determined by satisfying the boundary conditions on $\varpi = 0$ and/or the shock conditions (23). Thus, on the shock curve $\varpi_s = u_s + (Yu_s)^{-1}$ we have

$$M_\varpi = \varpi_s r_s, \qquad M = \varpi_s r_s\{Y(2-\omega)u_s\}^{-1}, \qquad B_\phi = r_s\tan\psi,$$

so that

$$\Gamma^2 = \tan^2\psi\,\varpi_s^{-2(1-\omega)/(2-\omega)}[r_s\{Yu_s(2-\omega)\}^{-1}]^{2/(2-\omega)}. \tag{27}$$

Use of equations (25) and (26) in (20) gives the equation that M must obey:

$$M_{\varpi\varpi}\{(2-\omega)^2 M^2(\varpi M_{\varpi}^2)^{-1} - \varpi^{-1} - M_{\varpi}\varpi^{-2} - \Gamma^2 M_{\varpi} M^{-2/(2-\omega)}\} + M_{\varpi}(\varpi^{-2} + M_{\varpi}\varpi^{-3}) + \tfrac{1}{2}\omega M_{\varpi} - \tfrac{1}{2}(2-\omega)^2 M\varpi^{-1} - \Gamma^2 M_{\varpi}^2 M^{-2/(2-\omega)}\{\varpi^{-1} - M_{\varpi} M^{-1}(2-\omega)^{-1}\} = 0. \tag{28}$$

The boundary conditions on $M(\varpi)$ are

$$\lim_{\varpi\to 0} \varpi M_{\varpi} \to 0 \quad (r\varpi^2 \to 0); \qquad M M_{\varpi}^{-1} \to 0 \quad (u \to 0) \quad \text{as} \quad \varpi \to 0. \tag{29}$$

With $\tau = \varpi^2$, equation (28) can be cast in the form

$$\tau M_{\tau\tau}\{(1-\tfrac{1}{2}\omega)^2 M^2 - \tau M_\tau^2(1+2M_\tau) - 2\Gamma^2\tau^2 M_\tau^3 M^{-2/(2-\omega)}\} = -\tfrac{1}{4}\omega\tau^2 M_\tau^3 + \tfrac{1}{2}(1-\tfrac{1}{2}\omega)^2 M M_\tau(\tau M_\tau - M) + 2\Gamma^2\tau^2 M_\tau^4 M^{-2/(2-\omega)}\{1 - \tfrac{1}{2}\tau M_\tau M^{-1}(1-\tfrac{1}{2}\omega)^{-1}\}. \tag{30}$$

Note that since $r(\varpi)$ is proportional to the gas density we require $M_{\varpi} \geqslant 0$ everywhere; hence $M(\varpi) \geqslant 0$ everywhere. Further, since the fluid flow speed is required to be radially directed outward (away from $\varpi = 0$), we require $\varpi M_{\varpi} \geqslant (2-\omega)M$ almost surely, almost everywhere. From this inequality it follows that there exists a positive constant M_0 such that $M(\varpi) \geqslant M_0\varpi^{2-\omega}$ everywhere in $\varpi \geqslant 0$. But since $u = \varpi - (2-\omega)M/M_{\varpi}$, and since M and M_{ϖ} are both positive, it follows that $u < \varpi$ everywhere. Hence, as promised in the previous footnote, acceptable solutions must have $U(\lambda) < \eta\lambda(\eta-1)^{-1}$ and so neither R nor b_z has a singularity since $u < \varpi$. Equation (30) is the fundamental equation requiring solution. We shall also find it convenient later to combine properties of equations (19)–(21) with equation (30).

Before considering the topological structure of solutions to equation (30) for arbitrary Γ and $\omega < 1$, we first examine in detail the simpler case $\Gamma = 0 = \omega$. As we shall see, it is already extremely rich in mathematical content.

(b) *Solution of Equation* (30) *for* $\Gamma = 0 = \omega$

For $\Gamma = 0$ (corresponding to $b_\phi = 0$) and $\omega = 0$ (corresponding to constant gas density ahead of the shock), equation (30) reduces to

$$\tau M_{\tau\tau}\{M^2 - \tau M_\tau^2(1+2M_\tau)\} = \tfrac{1}{2}MM_\tau(\tau M_\tau - M), \tag{31}$$

with $\tau = \varpi^2$. A particular solution to equation (31) is $M = m_0\tau$, where m_0 is arbitrary but positive, and then $r = M_{\varpi}/\varpi = 2m_0 > 0$, $u = 0$; so that this solution while satisfying the requisite boundary conditions on $\varpi = 0$ cannot intersect the shock. In order that a continuous post-shock flow exists, it follows that there must be at least one other solution $M_1(\tau)$ to equation (31) which will intersect the shock. In order that this solution patches onto the solution $M_0(\tau) \equiv m_0\tau$ with continuous density and velocity, it would then also follow that there must exist a value $\tau = \tau_*$ at which $M_1(\tau) - M_0(\tau)$ approaches zero with $dM_1(\tau)/d\tau = dM_0(\tau)/d\tau$ on $\tau = \tau_*$, so that $r(\tau)$ and $u(\tau)$ patch on smoothly. Note that *a priori* the value of τ_* may be zero, which would imply that a second solution to equation (31) starts at the origin. We therefore explore the necessary condition that a point $\tau = \tau_*$ be a bifurcation point of the nonlinear equation (31) for the given solution $M_0(\tau) = m_0\tau$.

A condition for the presence to the left (or the right) of the point $\tau = \tau_*$ of (at least) another solution $M_1(\tau)$ of equation (31) such that both $M_1(\tau)-M_0(\tau) \to 0$ and $\mathrm{d}M_1(\tau)/\mathrm{d}\tau - \mathrm{d}M_0(\tau)/\mathrm{d}\tau \to 0$ as $\tau \to \tau_*$, but in which $M_1(\tau)$ does *not* vanish almost everywhere, can be expressed as follows. We set

$$M_1(\tau) = M_0(\tau) + \varepsilon\,\chi(\tau, \varepsilon), \tag{32}$$

subject to the constraints $\chi(\tau = \tau_*, \varepsilon) = 0$ and $\mathrm{d}\chi(\tau, \varepsilon)/\mathrm{d}\tau = 0$ on $\tau = \tau_*$. Then, since both $M_1(\tau)$ and $M_0(\tau)$ are exact solutions of equation (31), we have (in an obvious notation) the identities

$$\tau\,\mathrm{d}^2 M_1/\mathrm{d}\tau^2 = f(\tau, M_1, \mathrm{d}M_1/\mathrm{d}\tau), \qquad \tau\,\mathrm{d}^2 M_0/\mathrm{d}\tau^2 = f(\tau, M_0, \mathrm{d}M_0/\mathrm{d}\tau). \tag{33a, b}$$

Subtracting equation (33b) from (33a) and inserting the form (32) for $M_1(\tau)$, we obtain the equation for χ:

$$\tau\varepsilon\,\mathrm{d}^2\chi/\mathrm{d}\tau^2 = f(\tau, M_0 + \varepsilon\chi, \mathrm{d}M_0/\mathrm{d}\tau + \varepsilon\,\mathrm{d}\chi/\mathrm{d}\tau) - f(\tau, M_0, \mathrm{d}M_0/\mathrm{d}\tau). \tag{34}$$

Upon dividing equation (34) by ε and then taking the limit as $\varepsilon \to 0$, we obtain

$$\tau\frac{\mathrm{d}^2\chi_0}{\mathrm{d}\tau^2} = \chi_0\frac{\partial f}{\partial M}\bigg|_{M=M_0} + \frac{\mathrm{d}\chi_0}{\mathrm{d}\tau}\frac{\partial f}{\partial(\partial M/\partial\tau)}\bigg|_{M=M_0}, \tag{35}$$

where $\chi_0(\tau)$ is the limit function of $\chi(\tau, \varepsilon)$ as $\varepsilon \to 0$ and, in the standard manner, is assumed to be nonzero almost surely, almost everywhere to the left (or the right) of $\tau = \tau_*$, i.e. $\chi(\tau, \varepsilon) = \chi_0(\tau) + \varepsilon\,\chi_1(\tau) + \dots$.

Inserting the functional form for $f(\tau, M, \mathrm{d}M/\mathrm{d}\tau)$ into equation (35), performing the differentiations and then inserting $M_0(\tau) = m_0\,\tau$ yields the equation for χ_0,

$$\tau(\tau - 1 - 2m_0)\,\mathrm{d}^2\chi_0/\mathrm{d}\tau^2 - \tfrac{1}{2}\tau\,\mathrm{d}\chi_0/\mathrm{d}\tau + \tfrac{1}{2}\chi_0 = 0. \tag{36}$$

Equation (36) has the general solution

$$\chi_0 = \tau\left(A + B\int_{\tau_*/(1+2m_0)}^{\tau/(1+2m_0)} z^{-2}(z-1)^{\frac{1}{2}}\,\mathrm{d}z\right), \tag{37}$$

where A and B are constants.

The requirements that $\chi_0(\tau_*) = 0$ and $\mathrm{d}\chi_0(\tau)/\mathrm{d}\tau = 0$ on $\tau = \tau_*$ yield $A = 0$ together with $\tau_* = 1 + 2m_0$. Then, we have

$$\chi_0 = B\tau\int_1^{\tau/\tau_*} z^{-2}(z-1)^{\frac{1}{2}}\,\mathrm{d}z \equiv BF_0(\tau), \tag{38}$$

where B is, at the moment, arbitrary.

The process can be repeated to nth order in ε, yielding $\chi_1(\tau), \chi_2(\tau), \dots, \chi_n(\tau)$. At each stage it becomes progressively more difficult to compute the solution but formally the iterations can be carried out. Now the limit function cannot be valid in the regime $\tau < \tau_*$, for if it was then, as $\tau \to 0$, we would have

$$M_1(\tau) \to m_0\,\tau + \varepsilon B\tau_* \to \varepsilon B\tau_*.$$

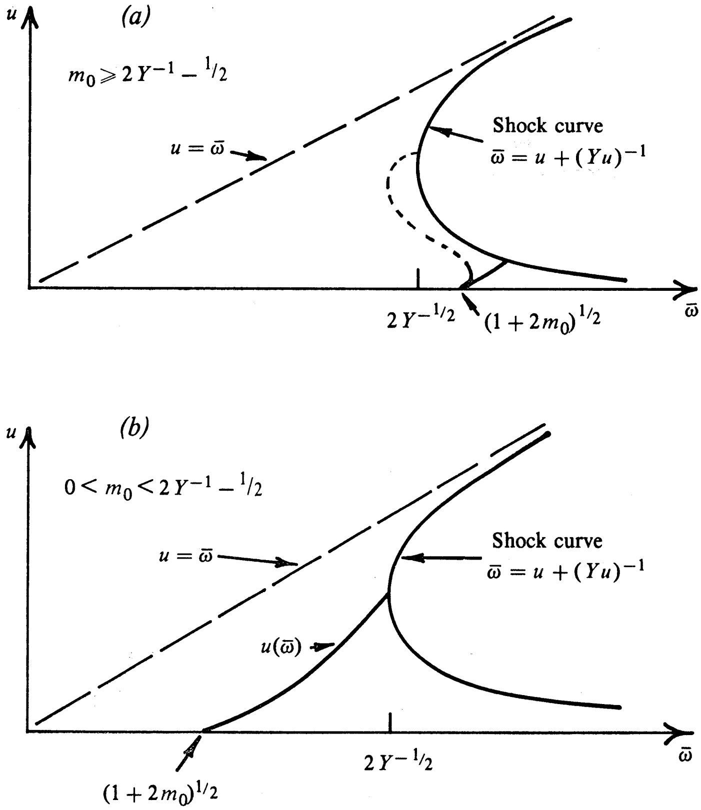

Fig. 1. Schematic representation for the case $\omega = 0$ and $\Gamma = 0$ of the normalized fluid speed u, as a function of normalized 'radius' ϖ, illustrating the behaviour when the bifurcation point $\varpi = (1+2m_0)^{\frac{1}{2}}$ lies to the (*a*) right and (*b*) left of the 'nose' of the shock curve.

But the mass contained in a vanishingly small cylinder centred on $\tau = 0$ must be zero. Hence, the bifurcation point $\tau = \tau_*$ is such that the solution to equation (31) is

$$M(\tau) = m_0\,\tau \qquad \text{in} \quad 0 \leqslant \tau \leqslant \tau_* = 1+2m_0; \tag{39a}$$

$$= m_0\,\tau + \varepsilon\,\chi(\tau, \varepsilon) \quad \text{in} \quad \tau \geqslant \tau_*. \tag{39b}$$

It must be concluded that the interior solution (39a) matches onto the exterior solution (39b) at the bifurcation point $\tau_* = 1+2m_0$, and that for $\tau > \tau_*$ only the exterior solution has the capability of intersecting the shock curve. Since $\tau_* = 1+2m_0$, it follows that the shock crossing position (if it occurs at all) given through $\varpi_s = u_s + (Yu_s)^{-1}$ must be at $\varpi_s^2 \geqslant 1+2m_0$, which implies that either

$$0 \leqslant u_s \leqslant \tfrac{1}{2}\{(1+2m_0)^{\frac{1}{2}} - (1+2m_0-4Y^{-1})^{\frac{1}{2}}\}, \tag{40a}$$

or

$$u_s \geqslant \tfrac{1}{2}\{(1+2m_0)^{\frac{1}{2}} + (1+2m_0-4Y^{-1})^{\frac{1}{2}}\}, \tag{40b}$$

provided only that $m_0 \geqslant 2Y^{-1} - \frac{1}{2} \geqslant \frac{3}{2}$.

For $m_0 < 2Y^{-1}-\frac{1}{2}$, the fact that the shock crossing position must occur in $\tau > \tau_*$ puts no constraint on the shock crossing velocity, only on $\varpi_s \geqslant (1+2m_0)^{\frac{1}{2}}$. A sketch of these two conditions on m_0 is provided in Fig. 1. As can be seen from Fig. 1*a* (representing the case $m_0 \geqslant 2Y^{-1}-\frac{1}{2}$), the only possibility for a shock crossing is when condition (40a) holds because the bifurcation point occurs to the right of the shock 'nose' point $\varpi_s = 2Y^{-\frac{1}{2}}$ of the shock curve $\varpi_s = u_s+(Yu_s)^{-1}$. If the exterior solution $M_1(\tau)$ were to cross the shock with $\varpi_s > 2Y^{-\frac{1}{2}}$ then the fluid velocity would have to double back on itself as shown by the short dashed curve in Fig. 1*a*. But this would imply two values of the velocity at finite ϖ; on physical grounds, this is forbidden. Therefore, it must be concluded that for $m_0 \geqslant 2Y^{-1}-\frac{1}{2}$ the shock crossing of the exterior solution must occur in the u_s regime given by condition (40a).

Consider, then, the exterior solution $M_1(\tau) = m_0\tau+\varepsilon\chi(\tau,\varepsilon)$ where $\chi(\tau,\varepsilon) = \chi_0(\tau)+\varepsilon\chi_1(\tau)+\ldots$, with $\chi_0(\tau)$ given by equation (38). By definition, we know that near $\tau = \tau_*$, $\chi(\tau,\varepsilon)$ is accurately approximated by $\chi_0(\tau)$. In order to see whether the approximate solution $m_0\tau+\varepsilon\chi_0(\tau)$ accurately reflects the behaviour of the exterior solution $M_1(\tau)$ for all $\tau > \tau_*$, up to and including the shock values, we proceed as follows.

Consider the behaviour of $M_1(\tau)$ at the shock front (on the assumption that M_1 intersects the shock). On $\varpi = \varpi_s$ we require that

$$M_1 = \varpi_s r_s(2Yu_s)^{-1}, \qquad dM_1/d\varpi = \varpi_s r_s, \tag{41a, b}$$

where $r_s = 2(1-Y)(1+Yu_s^2)^2(2+Yu_s^2)^{-1}(Yu_s)^{-2}$. To terms $O(\varepsilon B)$ these conditions become

$$m_0\varpi_s^2+\varepsilon BF_0(\tau=\varpi_s^2) = \varpi_s r_s(2Yu_s)^{-1}, \qquad m_0+\varepsilon B\{dF_0(\tau)/d\tau\}_{\tau=\varpi_s^2} = \tfrac{1}{2}r_s. \tag{42a, b}$$

For consistency on $\tau = \varpi_s^2$, equation (42a) requires that

$$\varepsilon B = \varpi_s\{r_s(2Yu_s)^{-1}-m_0\varpi_s\}/F_0(\tau=\varpi_s^2), \tag{43a}$$

while from equation (42b) we obtain

$$\varepsilon B = (\tfrac{1}{2}r_s-m_0)/(dF_0/d\tau). \tag{43b}$$

Inserting

$$F_0(\tau) = \tfrac{1}{2}\tau\{\arctan(\tau/\tau_*-1)^{\frac{1}{2}}-(\tau_*/\tau)(\tau/\tau_*-1)^{\frac{1}{2}}\} \tag{44}$$

into equations (43) and eliminating εB yields to $O(\varepsilon B)$

$$r_s+1-\tau_* = r_s u_s^2 Y\{2\tau_*(1+Yu_s^2)\}^{-1}(\varpi_s^2\tau_*^{-1}-1)^{-\frac{1}{2}}\arctan\{(\omega_s^2\tau_*^{-1}-1)^{\frac{1}{2}}\}, \tag{45}$$

which determines τ_* in terms of the shock crossing velocity u_s for a given Y (or which determines u_s in terms of τ_*). Hence, from equations (43) εB is also determined as a function solely of u_s (or τ_*) with

$$\varepsilon B = r_s u_s Y(\varpi_s^2\tau_*^{-1}-1)^{-\frac{1}{2}}\{2\tau_*(1+Yu_s^2)\}^{-1}. \tag{46}$$

Suppose, for example, that we have $m_0 \gg 2Y^{-1}-\frac{1}{2}$ ($\tau_* \gg 4/Y$), so that from condition (40a)

$$0 \leqslant u_s \lesssim Y\tau_*^{-\frac{1}{2}} < \tfrac{1}{2}Y^{3/2} \ll 1, \tag{47}$$

with $\varpi_s \approx (Yu_s)^{-1}$ and $r_s \approx (1-Y)(Yu_s)^{-2}$. Equation (45) then yields

$$\tau_* \approx (\tfrac{1}{4}\pi)^{2/3}(Y^{-2}u_s)^{2/3}. \tag{48}$$

Inequality (47) is then satisfied provided $u_s \lesssim (4/\pi)Y^{5/4}$, and so

$$\varepsilon B \approx \tfrac{1}{2}(4/\pi)^{1/3}(1-Y)(Yu_s)^{2/3} > 0. \tag{49}$$

In the range $\varpi_s^2 \geqslant \tau \geqslant \tau_*$ it then follows that

$$|M_1(\tau) - m_0\tau|(m_0\tau)^{-1} = \varepsilon m_0^{-1}F_0(\tau)/\tau \leqslant 2Y^2(1-Y) \leqslant \tfrac{8}{27}, \tag{50}$$

so that provided $Y \ll 1$, $\varepsilon B\chi_0(\tau)$ makes a very small correction to $m_0\tau$ in $\varpi_s^2 \geqslant \tau \geqslant \tau_* \gg 4/Y$. The corresponding normalized velocity is

$$u = \varpi - 2M_1(\varpi)/(\mathrm{d}M_1/\mathrm{d}\varpi) \approx \varepsilon B(m_0\varpi^3)^{-1}(\varpi^2\tau_*^{-1}-1)^{\frac{1}{2}}, \tag{51}$$

while the corresponding normalized density is

$$r(\omega) = \varpi^{-1}\mathrm{d}M_1/\mathrm{d}\varpi \approx 2[m_0 + \varepsilon B\arctan\{(\varpi^2\tau_*^{-1}-1)^{\frac{1}{2}}\}]. \tag{52}$$

The normalized fluid flow velocity is then considerably more sensitive to the change over at the bifurcation point from the interior to the exterior solution than is the normalized gas density.

We consider, as a second illustration, that m_0 takes on the particular value $2Y^{-1}-\frac{1}{2}$ ($\tau_* = 4/Y$). Inequality (40a) then implies $0 \leqslant u_s \leqslant Y^{-\frac{1}{2}}$, while if inequality (40b) is operative we have $u_s \geqslant Y^{-\frac{1}{2}}$. Inserting $u_s = \Lambda Y^{-\frac{1}{2}}$ into equation (45) and writing $\tau_* = 4/Y$, we obtain

$$\begin{aligned} &2(1-Y)(1+\Lambda^2)^2\{\Lambda^2Y(2+\Lambda^2)\}^{-1}+1-4Y^{-1} \\ &\qquad = (1-Y)(1+\Lambda^2)\{2(2+\Lambda^2)|1-\Lambda^2|\}^{-1}\arctan\{(2\Lambda)^{-1}|1-\Lambda^2|\}. \end{aligned} \tag{53}$$

First note that equation (53) does not have a solution with $\Lambda = 1$, for if it did this would imply $Y = -1$, whereas we have $0 \leqslant Y \leqslant 1$; hence $\Lambda = 1$ is not a possible solution. For $Y \ll 1$, an approximate solution of equation (53) is provided by

$$\Lambda = \Lambda_0 + \Lambda_1 Y + O(Y^2), \tag{54}$$

with $\Lambda_0 = (2^{\frac{1}{2}}-1)^{\frac{1}{2}} < 1$ and $\Lambda_1 = 2^{-7/2}\Lambda_0^{-1}(2^{\frac{1}{2}}+1)\{5-2^{-3/2}\Lambda_0\arctan(2^{\frac{1}{2}}\Lambda_0)\}$. It follows that

$$\varepsilon B \approx (1-Y)(1+\Lambda^2)\{2(2+\Lambda^2)(1-\Lambda^2)\} > 0, \tag{55}$$

so that

$$\begin{aligned} |\varepsilon BF_0(\tau)(m_0\tau)^{-1}| &\leqslant \tfrac{1}{8}Y\pi\Lambda(1-Y)(1+\Lambda^2)(2+\Lambda^2)^{-1}(1-\Lambda^2)^{-1} \\ &\approx Y(1-Y)\pi/16 \ll 1, \end{aligned} \tag{56}$$

and, hence, $\varepsilon BF_0(\tau)$ is indeed a small correction to $m_0\tau$ throughout the domain $\varpi_s^2 \geqslant \tau \geqslant 4/Y$ for $Y \ll 1$.

For $Y \approx 1$, an approximate solution to equation (53) is provided by

$$\Lambda^2 = (3Y)^{-1}(1-Y) \ll 1. \tag{57}$$

It follows that

$$\varepsilon B \approx \{3Y(1-Y)\}^{\frac{1}{2}} > 0,$$

so that

$$|\varepsilon B F_0(\tau)(m_0\tau)^{-1}| \leqslant \pi Y(4-Y)^{-1}\{3Y(1-Y)\}^{\frac{1}{2}} \approx \pi 3^{-\frac{1}{2}}(1-Y)^{\frac{1}{2}} \ll 1. \tag{58}$$

Hence, $\varepsilon B F_0(\tau)$ is indeed a small correction to $m_0\tau$ throughout the domain $\varpi_s^2 \geqslant \tau \geqslant 4/Y$ for $|1-Y| \ll 1$.

It follows, then, that in order to obtain a self-similar blast wave with a continuous post-shock fluid flow in the case $\omega = 0 = \Gamma$, and for a given value of Y (<1), it is necessary to match the two solution branches $M_0(\tau)$ and $M_1(\tau)$ at a precise value of τ—the bifurcation point. The matching has to be done with discontinuous slopes for $\mathrm{d}u/\mathrm{d}\varpi$ and $\mathrm{d}r/\mathrm{d}\varpi$. All parameters (u_s, r_s, η) of the solution are then uniquely determined as specified functions of the bifurcation point value (which is related to the gas density on $\varpi = 0$) by the requirement that the exterior solution branch must pass through the shock. There is no other self-similar solution with continuous post-shock velocity and density.

(*c*) *Solution of Equation* (30) *for* $\omega = 0$ *and* $\Gamma \neq 0$

In this particular case equation (30) yields

$$\tau M M_{\tau\tau}\{M^3 - \tau M_\tau^2 M(1+2M_\tau) - 2\Gamma^2\tau^2 M_\tau^3\}$$
$$= \tfrac{1}{2}M^3 M_\tau(\tau M_\tau - M) + \Gamma^2\tau^2 M_\tau^4(2M - \tau M_\tau). \tag{59}$$

Consideration of this equation near $\tau = 0$ yields the solution

$$M(\tau) \approx -\tfrac{1}{2}\Gamma^2\tau\{\ln\tau - 1 - \tau(2+\Gamma^{-2}) + \tau^2(\ln\tau)^{-1}(2+\Gamma^{-2}) + \dots\}, \tag{60}$$

with $M(\tau) \to 0$ as $\tau \to 0$. It follows that near $\tau = 0$ we have

$$r(\varpi) = 2M_\tau \approx -\Gamma^2 \ln\tau > 0. \tag{61}$$

Note that equation (61) yields a singular peak in gas density as $\tau \to 0$, but that the mass contained within a small radius is bounded and becomes vanishingly small as $\tau \to 0$. Of a more serious nature is the behaviour of the normalized flow speed:

$$u = \varpi^{-1}(\tau - M/M_\tau) \approx \tfrac{1}{2}\varpi(\ln\varpi)^{-1}\{1 + 3\varpi^2(2+\Gamma^{-2})\}. \tag{62}$$

As $\varpi \to 0$, we have $u \to 0$ *but* from negative values corresponding to an *inflow* of gas towards $\varpi = 0$. However, our original premise was that the flow solution must represent an outflow of gas behind a shock.

It therefore must be concluded that the situation $\omega = 0$ and $\Gamma \neq 0$ does not admit of physically realistic flow characteristics with continuous post-shock variation of gas density and velocity. There is no self-similar flow pattern available in this situation.

In light of these results for the special cases $\omega = 0$, it is interesting to see how nonzero values of ω modify these conclusions.

(*d*) *Solution of Equation* (30) *for* $\Gamma = 0$ *and* $\omega < 0$

On $\Gamma = 0$ ($B_\phi = 0$) equation (30) reduces to

$$\tau M_{\tau\tau}\{(1-\tfrac{1}{2}\omega)^2 M^2 - \tau M_\tau^2(1+2M_\tau)\}$$
$$= -\tfrac{1}{4}\omega\tau^2 M_\tau^3 + \tfrac{1}{2}(1-\tfrac{1}{2}\omega)^2 M M_\tau(\tau M_\tau - M). \qquad (63)$$

With $t = \tau(1-\frac{1}{2}\omega)^2$, $m = \frac{1}{2}M(1-\frac{1}{2}\omega)^2$ and $\Omega = \frac{1}{4}\omega(1-\frac{1}{2}\omega)^{-2}$, equation (63) yields

$$t m_{tt}\{m^2 - t m_t^2(1+m_t)\} = -\Omega t^2 m_t^3 + \tfrac{1}{2} m m_t(t m_t - m). \qquad (64)$$

This equation clearly has movable critical points (Ince 1956). Since the behaviour of solutions is dependent on the structure of the equation at the critical points, and since the structure of movable critical points depends on the values of m and m_t at $t = 0$, an analysis of the topological behaviour of solutions to equation (64) is an extremely difficult problem.*

In the case $\omega = 0$ we were fortunate that an exact analytic solution was available so that the powerful machinery of bifurcation point theory could be brought to bear on the problem. In the more general case ($\omega \neq 0$), it is clear that only a lucky guess or some flash of inspiration will provide an exact analytic solution to equation (64). To date, neither of these eventualities has been reached and, regrettably, we must therefore resort to piecemeal analysis of this equation.

For $t \approx 0$, the solution to equation (64) with $m(t) \approx m_0 t + O(t^2)$ is

$$m(t) \approx m_0 t + \tfrac{1}{2}\Omega m_0^2(1+m_0)^{-1} t^2 + O(t^3), \qquad (65)$$

so that

$$r(\varpi) \approx m_0 + \Omega m_0^2(1+m_0)^{-1} t + O(t^2), \qquad (66)$$

$$u(\varpi) \approx \tfrac{1}{2}\omega\varpi\{1 + \tfrac{1}{4} m_0 \varpi^2(1+m_0)^{-1} + O(\varpi^4)\}. \qquad (67)$$

Consider then the behaviour for $\omega < 0$. In this case we have $\mathrm{d}u/\mathrm{d}\omega < 0$ on $\varpi = 0$ so that u is negative for small ω. But, at least for small ϖ, this represents a fluid flow *towards* the origin, $\varpi = 0$. However, the original premise was that we were dealing with a fluid moving *outward* from a line explosion centred on $\varpi = 0$. It must be concluded that for $\omega < 0$ and $\Gamma = 0$ there is no self-similar flow pattern available with continuous post-shock properties representing an outward flow of gas.

(*e*) *Solution of Equation* (30) *for* $\Gamma \neq 0$ *and* $\omega < 0$

In order to study the behaviour of equation (30) for arbitrary, but finite, values of Γ and ω, it is convenient to remind ourselves that we are interested in outflow

* For a second-order differential equation *not* to have movable critical points, it is necessary (Ince 1956) that it should be of the form

$$\mathrm{d}^2y/\mathrm{d}x^2 = A(x,y)(\mathrm{d}y/\mathrm{d}x)^2 + B(x,y)(\mathrm{d}y/\mathrm{d}x) + C(x,y).$$

This is not the case with equation (64); hence, it has movable critical points.

situations. Suppose then that, near $\varpi = 0$, M was to be proportional to ϖ^{2a}, i.e. $M \approx m_0 \varpi^{2a}$, with $m_0 > 0$ and $a > 0$ in order that, at the least, $r(\omega)$ be positive for $\varpi \approx 0$. Consider the normalized velocity $u = \varpi - (2-\omega)M/M_\varpi \approx \frac{1}{2}\varpi a^{-1}(\omega + 2a - 2)$ near $\varpi = 0$. Thus it is necessary that $a > 1 - \frac{1}{2}\omega$ in order that $u \geqslant 0$. It then follows that $r(\varpi) \propto \varpi^{2a-2}$ near $\varpi = 0$. For $a = 1 - \frac{1}{2}\omega$, we have $r(\varpi) \propto \varpi^{2a-2}$ corresponding to a central *density* peak but one containing a negligible mass as $\varpi \to 0$, since $M \propto \varpi^{2a} \to 0$ as $\varpi \to 0$. Thus, near $\varpi = 0$ any physically acceptable solution of equation (30) must be expressible as a power series in ϖ, with $M \approx m_0 \varpi^{2a}$ and $a \geqslant 1 - \frac{1}{2}\omega$.

Consider the effect of inserting $M = m_0 \tau^a$, with $a \geqslant 1 - \frac{1}{2}\omega$, into equation (30) and retaining each term to its lowest power in τ. The result is

$$\begin{aligned} m_0 a(a-1)\{(1-\tfrac{1}{2}\omega)^2 m_0^2 - m_0^2 a^2 \tau^{-1}(1 + 2am_0 \tau^{a-1}) \\ - 2\Gamma^2 a^3 m_0^3 m_0^{-1/(1-\frac{1}{2}\omega)} \tau^{-\{1+a\omega/(2-\omega)\}}\} \\ \approx -\tfrac{1}{4}\omega a^3 m_0^3 + \tfrac{1}{2}(1-\tfrac{1}{2}\omega)^2 a(a-1) m_0^3 \\ + 2\Gamma^2 m_0^4 a^4 m_0^{-1/(1-\frac{1}{2}\omega)}\{1 - a/(2-\omega)\}\tau^{-\{1+a\omega/(2-\omega)\}}. \end{aligned} \tag{68}$$

Now for $\omega < 0$, inspection of equation (68) reveals that the dominating power on the left-hand side is τ^{-1} (since $a \geqslant 1 - \frac{1}{2}\omega > 1$), while the right-hand side has a dominant power of either τ^0 (if $a > 1 + 2/|\omega|$) or $\tau^{-\{1-|\omega|a/(2+|\omega|)\}}$ (if $a < 1 + 2/|\omega|$). In either eventuality there is no possibility of matching dominating powers from the left- and right-hand sides of equation (68). The alternative is that one can match the dominant powers on both sides of the equation and so determine a, but then a will be less than $1 - \frac{1}{2}\omega$ yielding an inflow of material for small ϖ; on physical grounds this solution would then be discarded. Hence, for $\Gamma \neq 0$ and $\omega < 0$, there is no solution satisfying the required boundary conditions as $\varpi \to 0$. It must be concluded that under these circumstances there is no fluid flow with continuous post-shock variation of its properties.

For the remainder of this paper we therefore restrict ourselves to the only remaining regime capable of supplying physically acceptable solutions, namely $\Gamma \neq 0$ and $\omega > 0$ (the cases with $\omega = 0$ having already been discussed in Sections 2*b* and 2*c* above and found to yield acceptable solutions provided $\Gamma = 0$).

(*f*) *Solution of Equation* (30) *for* $\Gamma \neq 0$ *and* $\omega > 0$

In this case, the dominant powers on the left- and right-hand sides of equation (68) do balance at small ϖ provided that $a = 1 - \frac{1}{2}\omega$, and then the coefficients of the dominant powers yield the value

$$m_0 = \{\Gamma^2(\omega^{-1} - 1)\}^{(1-\frac{1}{2}\omega)} > 0\,. \tag{69}$$

However, this lead term $M \approx m_0 \varpi^{2-\omega}$ is not sufficient to determine the behaviour of u in the vicinity of $\varpi = 0$ since it yields $u = 0$. We must, therefore, obtain the next term in the series expansion for M around $\varpi = 0$ in order that we can determine whether u is positive or negative near $\varpi = 0$. The simplest way to determine the correction is to write $M = m_0 \varpi^{2-\omega} V$ in equation (30) with m_0 given by equation (69).

The result is

$$V \approx 1+(m_0\,\omega)^{-1}(1-\omega)\varpi^{\omega}, \tag{70}$$

and the corresponding normalized velocity is

$$u \approx \omega\{m_0(2-\omega)\}^{-1}\varpi^{1+\omega} > 0. \tag{71}$$

Note that u is positive increasing near $\varpi = 0$. Suppose that at some value of ϖ, ϖ_* say, u was to cross the line $\sigma\varpi$ (where σ is constant) with $M(\varpi_*) = M_* > 0$. Then from

$$u = \varpi - (2-\omega)M/M_{\varpi}, \tag{72}$$

we have

$$\tau_* M_{\tau_*} = (2-\omega)M_*(1-\sigma)^{-1}, \tag{73}$$

where M_{τ_*} is M_τ evaluated on $\tau = \tau_* = \varpi_*^2$. Since the gas density is required to be positive, it follows that equation (73) already demands $\sigma \leqslant 1$.

Now, since $du/d\varpi$ is proportional to ϖ^{ω} as $\varpi \to 0$, it follows that if u is indeed to cross the line $u = \sigma\varpi$ then it must do so with a slope greater than σ. But on this line, we write

$$du/d\varpi = \omega - \sigma + (2-\omega)MM_{\tau\tau}M_\tau^{-2} = u'_*. \tag{74}$$

From equations (30) and (73) we then obtain

$$\begin{aligned}(u'_*+\sigma-\omega)&\{1-\sigma+M_*\,\tau_*^{-1}(2-\omega)\}\\ &= (2-\omega)\Gamma^2(\omega-u'_*)M_*^{-\omega/(2-\omega)} + \tfrac{1}{2}\sigma\tau_*(1-\sigma)^2(2+\omega-\sigma).\end{aligned} \tag{75}$$

Suppose first that u was about to cross the line $u = \varpi$ so that we can set $\sigma = 1$ in equation (75). Then we have

$$u'_* = \omega - \{1+\Gamma^2\tau_*\,M_*^{-2/(2-\omega)}\}^{-1} < 1, \tag{76}$$

so that $u'_* < 1$; hence u cannot cross the line $u = \varpi$. Consider now that u was about to cross the line $u = 0$; we set $\sigma = 0$ in equation (75) to obtain

$$(u'_*-\omega)\{1+M_*\,\tau_*^{-1}(2-\omega)+(2-\omega)\Gamma^2M_*^{-\omega/(2-\omega)}\} = 0. \tag{77}$$

Since the factor in the braces is intrinsically positive it follows that if u crosses zero it does so with positive slope $u'_* = \omega$. But since $u(\varpi \to 0) \propto \varpi^{1+\omega}$, it follows that the first time u crosses zero it would have to do so from above with *negative* slope. But on the line $u = 0$, we have $u'_* = \omega > 0$; hence it must be concluded that u is constrained for all ϖ to lie in the range $\varpi \geqslant u > 0$.

Consider now the shock curve $\varpi_s = u_s+(Yu_s)^{-1}$. Its asymptotes are $\varpi_s = u_s$ and $u_s = 0$. Thus the solution curve to equation (30) for $\Gamma \neq 0$ and $\omega > 0$ is constrained to lie between the asymptotes of the shock curve; hence it must intersect the shock somewhere.

We conclude that in the case $\Gamma \neq 0$ and $\omega > 0$ there is a solution curve, starting at the origin, representing continuous post-shock outflow and positive density with continuous variation of flow parameters.

normalized gas density be positive. Further, since X is proportional to the mass contained in a region of vanishingly small radius as $\chi \to 0$, it follows that we require $X \to +0$ as $\chi \to 0$. Thus, near $\chi = 0$ any physically acceptable solution must be expressible in the form $X \approx X_0 \chi^a$ with $a \geqslant 1-\frac{1}{2}\omega$. But insertion of $X \approx X_0 \chi^a$ near $\chi = 0$ into equation (88) yields the two requirements

$$a = 1-2/\omega < 0, \qquad X_0 = \{\tfrac{1}{2}(1+\omega)\}^{(2-\omega)/\omega}.$$

Thus, a is negative contrary to the requirement that it exceeds $1-\frac{1}{2}\omega$. Hence, in the case $\omega > 0$ and $b_z = 0$ ($\Gamma = \infty$) no physically acceptable solution exists with continuous post-shock variation of flow properties and representing outflow of gas from all points behind the shock.

It must, therefore, be concluded that the only remaining possibilities for a solution with continuous post-shock flow parameters are encompassed by $\gamma \geqslant 0$, $\omega = 0$; $\Gamma = 0$, $\omega > 0$; and $\Gamma \geqslant 0$, $\omega = 0$ (as already discussed in Section 2*c*).

We now consider the case $\gamma \neq 0$ and $\omega = 0$ for which equations (81) and (82) are appropriate. From the requirements $M(\varpi \to 0) \to 0$ and $u \geqslant 0$ as $\varpi \to 0$, equations (81) yield the information that S must tend to a positive constant value as $\tau \to 0$. From equation (87) this value is the positive root of $\gamma^2 S^2 + S - 1 = 0$, i.e.

$$S = (2\gamma^2)^{-1}\{(1+4\gamma^2)^{\frac{1}{2}} - 1\} < 1. \tag{89}$$

But consider the normalized flow velocity near $\varpi = 0$:

$$u = \varpi - 2M/M_{\varpi} = \varpi(1 - M/\tau M_\tau) = \varpi(1 - S^{-1}). \tag{90}$$

The requirement that u represents an outflowing gas translates into the requirement $S \geqslant 1$. But inspection of equation (89) reveals that $S < 1$ for $\gamma \neq 0$. It follows that the solution of equation (87) yields an *inflow* velocity near $\varpi = 0$.

It must be concluded then that in the presence of purely azimuthal magnetic fields the equations of mass conservation, momentum balance and the Lenz law do not permit the existence of self-similar blast waves with continuous post-shock flow velocity and density representing outflow from the line of the explosion.

The only case that remains to be investigated is a purely longitudinal magnetic field ($b_\phi = 0$ and $b_z \neq 0$) for a density ahead of the blast wave proportional to $r^{-\omega}$, with $1 > \omega > 0$.

(*h*) *Solution Properties of Equation* (30) *for* $\Gamma = 0$ *and* $\omega > 0$

On $\Gamma = 0$ and $\omega \neq 0$, equation (30) yields

$$\begin{aligned}\tau M_{\tau\tau}\{M^2(1-\tfrac{1}{2}\omega)^2 - \tau M_\tau^2(1+2M_\tau)\} \\ = -\tfrac{1}{4}\omega\tau^2 M_\tau^3 + \tfrac{1}{2}(1-\tfrac{1}{2}\omega)^2 M M_\tau(\tau M_\tau - M).\end{aligned} \tag{91}$$

The mathematical arguments of equations (65)–(68) remain valid so that equation (66) is in force, but now with $\omega > 0$ and $\Omega > 0$. Then, near $\varpi = 0$, we have

$$r(\varpi) \approx m_0 + \Omega m_0^2(1-\tfrac{1}{2}\omega)^2(1+m_0)^{-1}\varpi^2 + O(\varpi^4), \tag{92a}$$

$$u(\varpi) \approx \tfrac{1}{2}\omega\varpi\{1 + \tfrac{1}{4}m_0\varpi^2(1+m_0)^{-1} + O(\varpi^4)\}. \tag{92b}$$

The important difference in the transformation properties is the equations ...

But since $\omega > 0$, equation (92b) yields a positive normalized flow velocity near $\varpi = 0$, and so the case $\Gamma = 0$ and $\omega > 0$ *is* a candidate for supplying a solution with a fully self-consistent continuous flow behind a blast wave.

To analyse the flow behaviour in more detail, it is convenient to utilize not only equation (91) (or equation 63), but also equations (20) and (21) (with $B_\phi = 0$) in the form

$$2\tau^{\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}\tau}(u-\tau^{\frac{1}{2}}) - \tfrac{1}{2}\omega u = -(1+r^{-1})2\tau^{\frac{1}{2}}\frac{\mathrm{d}r}{\mathrm{d}\tau}, \tag{93a}$$

$$2\tau^{\frac{1}{2}}\frac{\mathrm{d}r}{\mathrm{d}\tau}(u-\tau^{\frac{1}{2}}) = -r\left(2\tau^{\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}\tau} + u\tau^{-\frac{1}{2}} - \omega\right), \tag{93b}$$

together with

$$r(\tau) = 2M_\tau, \qquad u(\tau) = \tau^{\frac{1}{2}}\{1-(1-\tfrac{1}{2}\omega)M/\tau M_\tau\}. \tag{93c, d}$$

We note from equation (92b) that on $\varpi = 0$, $\mathrm{d}u/\mathrm{d}\varpi = \frac{1}{2}\omega > 0$.

Consider all straight lines $u = \sigma\varpi$ emanating from the origin. We first ask for what value of σ are solutions to equations (93) tangent to $u = \sigma\varpi$, so that u cannot cross the line $\sigma\varpi$. Let there be a tangent point at $\varpi = \varpi_*$, with $\mathrm{d}u/\mathrm{d}\varpi = \sigma$ there, and $u = \sigma\varpi_*$, $r = r_* > 0$. Substitution of these requirements into equations (93a, b) yields for σ,

$$\sigma(\sigma-1)(\sigma-1-\tfrac{1}{2}\omega) = \varpi_*^{-2}(1+r_*)(2\sigma-\omega), \tag{94}$$

which always has a root in the range $\frac{1}{2}\omega < \sigma < 1$, which we denote by σ_{L}.

It is easy to show that u never crosses the line $u = \varpi$, for if it did then the line with *positive* slope would have to be crossed first (since $u \to 0$ and $\mathrm{d}u/\mathrm{d}\varpi \to \frac{1}{2}\omega$ on $\varpi \to 0$). But insertion of $u = \varpi$ into equations (93a) and (93b) yields

$$\mathrm{d}u/\mathrm{d}\varpi = -(1-\omega) < 0, \qquad \mathrm{d}r/\mathrm{d}\varpi = \omega\varpi r/2(1+r), \tag{95a, b}$$

i.e. $\mathrm{d}u/\mathrm{d}\varpi < 0$; hence, u never crosses the line $u = \varpi$.† Solutions of equations (93) lie wholly below $u = \varpi$.

Consider then the behaviour of u in the vicinity of $\sigma_{\mathrm{L}}\varpi_*$ where $\mathrm{d}u/\mathrm{d}\varpi = \sigma_{\mathrm{L}}$. Differentiation of equations (93) then yields

$$\begin{aligned}\varpi_*^3(\mathrm{d}^2u/\mathrm{d}\varpi^2)_{\varpi_*}\{(\sigma_{\mathrm{L}}-1)^2 - \sigma_{\mathrm{L}}(1-\sigma_{\mathrm{L}})(1+\tfrac{1}{2}\omega-\sigma_{\mathrm{L}})(\omega-2\sigma_{\mathrm{L}})^{-1}\} \\ = -(\omega-2\sigma_{\mathrm{L}})\{1+r_* - r_*(\omega-2\sigma_{\mathrm{L}})(1-\sigma_{\mathrm{L}})^{-1}\}.\end{aligned} \tag{96}$$

Hence, we have $\mathrm{d}^2u/\mathrm{d}\varpi^2 > 0$ for $u = \sigma_{\mathrm{L}}\varpi_*$. Thus, in the vicinity of $\sigma_{\mathrm{L}}\varpi_*$, it follows that

$$2(u-\sigma_{\mathrm{L}}\varpi) \approx (\varpi-\varpi_*)^2\,\mathrm{d}^2u/\mathrm{d}\varpi^2\,|_{\varpi_*} > 0, \tag{97}$$

and therefore u lies wholly above $\sigma_{\mathrm{L}}\varpi$ and does not cross the line $u = \sigma_{\mathrm{L}}\varpi$.

† The alternative possibility is that $\mathrm{d}u/\mathrm{d}\varpi \to +\infty$ below $u = \varpi$ so that $u(\varpi)$ is a double-valued curve, crossing the line at a smaller value of ϖ than that where $\mathrm{d}u/\mathrm{d}\varpi \to +\infty$. But it can be shown that $\mathrm{d}u/\mathrm{d}\varpi$ can only become infinite on $r = r_* = \frac{1}{2}(1+5^{\frac{1}{2}})$ when $u = \varpi - r_*$. It can also be shown that $\mathrm{d}^2u/\mathrm{d}\varpi^2$ is not correctly signed for $u(\varpi)$ to represent a turning point on $r = r_*$. This possibility can therefore be excluded.

Consider now the behaviour of the shock curve $\varpi_s = u_s + (Yu_s)^{-1}$. Its asymptotes are $u_s = \varpi_s$ and $u_s = 0$, but as we have just proven that u is bounded by $\varpi > u > \sigma_L \varpi$ everywhere, then u must steadily progress outward from the origin until it meets the shock. It must then be concluded that for $\Gamma = 0$ ($B_\phi = 0$) and $\omega > 0$, there does exist a self-similar blast wave, with continuous post-shock flow velocity and density, representing outflow from the line of the explosion.

3. Discussion and Conclusions

We have analysed the properties of cylindrically symmetric self-similar blast waves propagating away from a line source into a medium whose density and magnetic field (with components in both the ϕ and z directions) both vary as $r^{-\omega}$ (with $\omega < 1$) ahead of the blast wave. Our main results divide into two classes:

(a) Class I Results: Zero Azimuthal Field ($B_\phi = 0$ and $B_z \neq 0$)

(i) The case $\omega < 0$ corresponds to increasing gas density and magnetic field ahead of the blast front. Here we found that there were no physically acceptable self-similar solutions, with continuous post-shock variations of flow speed and gas density. This is, perhaps, not too surprising as the magnetic field pressure and amount of swept-up material both increase without limit as the shock wave moves out from the origin. It is then to be expected, since the impulsive energy of explosive is finite, that such a blast wave could not possibly compensate for the steadily increasing amount of work that the fluid must perform in order to continue moving outward into an ever denser medium against an ever increasing magnetic pressure. Accordingly, that no self-similar flow pattern exists in this case is, in hindsight, to be expected.

(ii) The case $\omega = 0$ corresponds to constant gas density and magnetic field ahead of the blast front. Here we found an interesting situation of bifurcation with a constant density, constant magnetic field, zero velocity solution interior to the bifurcation point matching (with discontinuous derivative in the velocity gradient) onto a second solution (exterior to the bifurcation point), which then had continuous post-shock variations of flow speed and gas density out to its intersection with the shock curve. In the light of the results for the $\omega < 0$ case it is relatively easy to figure out the physics of the $\omega = 0$ case. The gas density and magnetic field ahead of the shock are constant for $\omega = 0$. Thus the magnetic field exerts no pressure gradient on the gas. If all the gas swept up by the shock were to be outward moving, eventually so much mass would accumulate that the bulk energy of motion would exceed the initial energy of explosion. Thus only a fraction of the swept-up gas can move. Hence, the interior solution represents the material which passed through the shock and which is then left behind in its wake. The resulting constant magnetic field, interior to the bifurcation point, exerts no pressure gradient on the constant density immobile material.

(iii) The case $\omega > 0$ corresponds to decreasing gas density and magnetic field ahead of the shock. Here we found that physically acceptable solutions exist with continuous post-shock variations of flow speed and gas density. Guided by the results obtained for one-dimensional flow (Lerche 1979) in the presence of a transverse (to the shock front) magnetic field, together with the results above for the cases $\omega \leqslant 0$, it is perhaps not too surprising to see that the investigation of this case also provides physically acceptable solutions. Both the gas density and magnetic field decline ahead of the shock front and, as the shock radius becomes increasingly large, its curvature can be neglected, so that intuitively one expects the results in this case to mirror, in virtually a quantitative fashion, the planar results (Lerche 1979). And this is, in fact, the case. The mass swept up by the shock is finite, and the magnetic pressure continually declines so that all of the fluid partakes of the outward motion.

(*b*) *Class II Results: Zero Longitudinal Field* ($B_z = 0$ *and* $B_\phi \neq 0$)

The results of this class of situations are simple: for all ω, negative, zero or positive, there are no physically acceptable solutions with continuous post-shock variations of flow speed and gas density. The reason seems to be that not only does the azimuthal magnetic field provide a pressure gradient on the gas, but also a tension ($\propto B_\phi^2/\varpi$) which tends to also 'squeeze' the gas—much as the tension in a rubber band confines a rolled-up newspaper. This 'extra' confinement of the gas is difficult to overcome. For instance, a constant value of B_ϕ implies no pressure gradient but still provides a tension of the magnetic field which confines the gas. When coupled with the *a priori* demand that the gas flow be self-similar, this tension then forbids a solution with continuous outflow behind the shock.

When a 'mix' of field components is present ahead of the shock ($B_\phi \neq 0 \neq B_z$), a similar situation prevails as the tension is present in the azimuthal component of field. There is the further effect, too, that the magnetic field lines are helices (of the form $z \propto r\phi$, for r constant) wrapped around the cylinder. It is thus to be doubted, ahead of any detailed calculations, that acceptable solutions should exist in this case for arbitrary ω. And this conviction is substantiated by the computations, for only the regime $\omega > 0$ permits physically acceptable solutions for $B_\phi \neq 0 \neq B_z$.

The calculations reported here were undertaken to investigate the role of two-dimensional effects (such as curvature of the shock front and the influence of magnetic field pressure and tension) on the behaviour of self-similar shocks.

The point here is that it has been argued (Cox 1972; McCray *et al.* 1975) that a simplified one-dimensional treatment is sufficient to elucidate the underlying physics very succinctly. Yet, in the case of supernova remnants in particular (for which $\omega \approx 0$ is often considered appropriate), it is an observational fact that their shape is more akin to a sphere and should, presumably, be represented more accurately by blast wave models which do allow for shock-front curvature. There is, then, always the concern that one-dimensional calculations do not accurately portray the evolution of such a blast wave. But in the absence of detailed calculations such a concern remains unquantified.

The investigation we have given of the evolution of cylindrical isothermal self-similar blast waves into a surrounding magnetized medium demonstrates that some considerable degree of caution should be attached to arguments that claim the shock-front curvature can be neglected. It also suggests further lines of investigation to improve our understanding of blast-wave expansion into media in the presence of magnetic fields:

Firstly, the presence of an azimuthal magnetic field drastically alters the one-dimensional inference, as the tension in such a field forbids the occurrence of an isothermal self-similar blast wave with continuous post-shock outflow of gas for $\omega \leqslant 0$. The case of a zero azimuthal field must be regarded as the analogue of the one-dimensional result. Thus, the presence of shock-front curvature does influence, in both a qualitative and a quantitative sense, inferences deduced from planar shock-front calculations.

In some sense then, the presence of an azimuthal magnetic field is a destabilizing influence. It destroys the capability of the system to possess precisely self-similar solutions with physically acceptable behaviours for the fluid flow behind the shock. But there remains the question: In those cases where acceptable self-similar solutions can be constructed is the magnetic field a stabilizing or destabilizing influence? The point here is that it is known that in the absence of an external magnetic field, three-dimensional isothermal blast waves are both linearly and nonlinearly unstable (Lerche and Vasyliunas 1976; Bernstein and Book 1978). Further, what hydromagnetic and hydrodynamic stability properties result for an isothermal blast wave when magnetic pressure plays a role in determining the overall structure of the self-similar flow?

Secondly, what modifications to the flow behaviour result, even within the cylindrical framework, when the variations of density and magnetic field ahead of the blast wave are allowed to vary differently, that is, $\rho \propto r^{-\omega}$, $B_z \propto r^{-\Lambda}$ and $B_\phi \propto r^{-\Lambda}$ ($\omega \neq \Lambda$; see the footnote in Section 2*a*)? The point here is that for $\omega \neq \Lambda$, the governing differential equation for the fluid flow is nonlinear and of third order, while for $\omega = \Lambda$ the relevant equation is only second order. Is it precisely the constraint $\omega = \Lambda$ which forbids acceptable self-similar solutions when $B_\phi \neq 0$, or is the argument more general?

Thirdly, even if self-similar flows are not rigorously acceptable, is it possible, as Chevalier (1977) has argued might be the case, that nevertheless they represent an accurate enough approximation to the true flow pattern throughout most of the lifetime of a supernova remnant in particular, so that deviations from models based on self-similar flow solutions can, in some sense, be considered small?

Since our understanding of the dynamical evolution of supernova remnants is closely tied to knowledge of the properties of blast-wave behaviour, in our opinion, it is of some importance to ascertain the requirements on the equations of mass conservation, momentum balance and magnetic field evolution which will permit the presence of a stable self-similar flow, so that we can move ahead to their application to supernova remnants and other astrophysical objects with a greater degree of confidence than would otherwise be the case.

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