Aust. J. Phys., 1981, **34**, 317–31

Boundary-layer Acceleration and Particle Mirroring in Pulsar Magnetospheres

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Abstract

Where the number density of a species becomes very small, inertial development of vorticity occurs; so a magnetospheric zone in which a species is contained must be enclosed by a vortical boundary layer. Where zones of corotating electrons and ions abut, there exists a large local non-corotational electric field, directed so as to force a merging of the electron and ion boundary layers. The poloidal accelerations and azimuthal drift velocities generated in these layers are estimated here. Ions are accelerated to nonrelativistic or mildly relativistic poloidal speeds, then penetrate into the electron corotation zones where they are centrifugally decelerated as they travel approximately along magnetic field lines. They mirror between points above the stellar surface and the boundary layer, presumably moving to lower magnetic field lines until they reach the star. Electrons are accelerated to poloidal speeds that are relativistic for distances from the axis of rotation exceeding about 1/30 of the radius of the light cylinder. They enter the ion corotation zone where they are further accelerated as they travel approximately along outgoing portions of the closed magnetic field lines, and are then decelerated on ingoing portions. They mirror between the northern and southern boundary layers, presumably moving to lower magnetic field lines until they reach the star. The electrons in the outer parts of the ion zone are very highly relativistic and emit gamma radiation which, in the case of the Crab pulsar, might create electron–positron pairs.

1. Introduction

This paper is one in a series following up a paper (Burman 1980) in which I gave a careful analysis of flow dynamics in steadily rotating neutron star magnetospheres, using the exact dynamical equations expressing balance between the Lorentz force and relativistic inertia. The purpose of this series is to deduce the implications for model building of the results of that analysis.

In my previous paper (Burman 1981*a*), I concentrated on magnetospheric regions in which the particles have azimuthal speeds that are close to the local speed of corotation. I showed that, in order to understand the flow dynamics and avoid absurdities, it is essential to use the correct functional form for the non-corotational electric potential Φ everywhere outside the star—including zones of corotation in which Φ had invariably been neglected. The implications include a potential difference between zones of corotating electrons and ions, resulting in a large local $\nabla\Phi$ field in vortical boundary layers separating those zones, directed so as to force a mixing of electrons and ions there. This contrasts sharply with the well-known work of Holloway (1973), based on use of the zero-inertia approximation with Φ piecewise constant on magnetic field lines: He argued that the electron and ion zones are separated by evacuated regions, but I have contended (Burman 1981*a*) that this conclusion was reached by an unjustifiable extrapolation of the results of a thought experiment. A further consequence of my analysis is a mirroring of electrons between the northern and southern boundary layers.

The purpose of this paper is to initiate study of the physics of the boundary layers separating the electron and ion corotation zones, and to consider the consequences of that physics. The poloidal accelerations and azimuthal drift velocities generated in those layers will be estimated for both ions and electrons. The behaviour of the accelerated ions and electrons as they subsequently penetrate into the electron and ion zones, respectively, will be deduced. Implications of the results for model building will be discussed.

2. Basic Formalism

Let ϖ, ϕ and z be cylindrical polar coordinates with the z axis as the rotation axis of the star. The system is steadily rotating at angular frequency Ω. The electric field is the sum of a part originating in the rotation of the magnetic field structure and a non-corotational part $-\nabla \Phi$ (Mestel 1971); the potential Φ is defined in terms of the familiar scalar and vector potentials as $\phi-(\Omega\varpi/c)A_{\phi}$, where c is the vacuum speed of light (Endean 1972*a*); Φ is gauge invariant.

The magnetospheric plasma is taken to be cold and non-dissipative: the equation of motion of each species expresses balance of the Lorentz force by relativistic inertia. Let $\boldsymbol{v}_k$ and γ_k denote the flow velocity and corresponding Lorentz factor of species k, and let e_k and m_k denote the charge and rest mass of its particles. Also, $\boldsymbol{u}_k$ will denote the flow velocity of species k reduced by the local speed of corotation with the star: $\boldsymbol{u}_k \equiv \boldsymbol{v}_k - \Omega\varpi\boldsymbol{t}$, where $\boldsymbol{t}$ is the unit toroidal vector.

By using the Endean (1972*a*, 1972*b*) integral of the motion—which follows from the steady rotation constraint—together with a fluxoid conservation theorem (Buckingham *et al.* 1972, 1973), Burman and Mestel (1978) reduced the equation of motion to a simple form. They went on to point out that, if all particles of species k are nonrelativistic when arbitrarily near the star, then the Endean integral of the motion becomes

$$1-\frac{e_k\Phi}{m_k c^2} = \gamma_k\left(1-\frac{\Omega\varpi}{c}\frac{v_{k\phi}}{c}\right), \tag{1}$$

and the equation of motion reduces to a very simple form, which is just a generalized isorotation law; this law, which is a differential equation relating the flow velocity to the magnetic field $\boldsymbol{B}$, states that the reduced flow velocity $\boldsymbol{u}_k$ is along the magnetoidal field $\boldsymbol{B}+(cm_k/e_k)\nabla\times(\gamma_k\boldsymbol{v}_k)$. For these two results to be valid, the particles concerned must, if leaving the star, be emitted with nonrelativistic speeds; if returning to the star, or accreted by it, they must be decelerated so as to be nonrelativistic on impact; in any case it is necessary for the particles to be nonrelativistic in only an arbitrarily thin neighbourhood of the stellar surface.

I have shown (Burman 1981*b*) that, when the number density of a species becomes sufficiently small, a process of 'inertial development of vorticity' occurs. For example, if a species is contained within a zone, then that zone must be surrounded by a skin or boundary layer of vortical flow in which the 'inertial drift' term in the generalized isorotation law is important.

Let $-e$ denote the electronic charge while m_e and m_p are the electron and proton rest masses. Let $\Phi(e)$ and $\Phi(i)$ represent Φ in a zone of corotating electrons and in a zone of corotating ions of atomic number Z and mass number A. Combining the corotational Lorentz factor with the integral (1) of the motion gives (Burman 1980)

$$-e\,\Phi(\mathrm{e})/m_{\mathrm{e}}c^2 = 1-(1-x^2)^{\frac{1}{2}}, \tag{2}$$

$$Ze\,\Phi(\mathrm{i})/Am_{\mathrm{p}}c^2 = 1-(1-x^2)^{\frac{1}{2}}, \tag{3}$$

where x is the dimensionless cylindrical polar radial coordinate $\Omega\varpi/c$. These equations demonstrate the existence of a strong local gradient of Φ across the boundary layers separating zones of electron and ion corotation, directed from the electron zones to the ion zone (Burman 1981*a*). The force it exerts on positive ions is towards the electron zones, while that on electrons is towards the ion zone: the zones must merge to form a boundary layer of mixed plasma. The effects of these forces will now be estimated.

3. Poloidal Acceleration of Ions

Consider a positive ion, of atomic number Z and mass number A, with negligible poloidal speed, entering a boundary layer between zones of corotating electrons and ions. For simplicity, in this first approach to the calculation of boundary-layer acceleration, it will be assumed that the azimuthal speed of the ion remains about $\Omega\varpi$; this assumption will be checked in Section 7 below.

The integral (1) of the motion becomes, dropping the subscript k,

$$\gamma \approx \frac{1-Ze\Phi/Am_{\mathrm{p}}c^2}{1-x^2}, \tag{4}$$

while the definition of the Lorentz factor becomes $\gamma^{-2} \approx 1-x^2-v_{\mathrm{p}}^2/c^2$, where v_{p} denotes the poloidal speed of the ion. Eliminating γ between these two equations gives

$$\frac{v_{\mathrm{p}}^2}{c^2} \approx (1-x^2)\left(1-\frac{1-x^2}{(1-Ze\Phi/Am_{\mathrm{p}}c^2)^2}\right). \tag{5}$$

In the ion zone, Φ is $\Phi(\mathrm{i})$, as given by equation (3); hence the integral (4) reduces to the equation $\gamma \approx (1-x^2)^{-\frac{1}{2}}$ for the corotational Lorentz factor, while (5) reduces to $v_{\mathrm{p}} = 0$, both as specified. But, so long as v_ϕ remains about $\Omega\varpi$, equations (4) and (5) for γ and v_{p} remain true as the ion traverses the boundary layer, where Φ decreases from $\Phi(\mathrm{i})$ to $\Phi(\mathrm{e})$; they continue to hold, with $\Phi(\mathrm{e})$ for Φ, as the ion subsequently penetrates into the electron corotation zone.

When the ion has crossed the layer, we have $\Phi = \Phi(\mathrm{e})$ so equation (5) gives

$$v_{\mathrm{p}} \approx \Omega\varpi(1-x^2)^{\frac{1}{2}}. \tag{6}$$

Thus, the local non-corotational electric field in the boundary layer accelerates ions to poloidal speeds that, near the star, are similar to their (original) azimuthal speeds $\Omega\varpi$, but that are rather less than $\Omega\varpi$ further from the star, becoming small near the light cylinder. According to equation (6), the generated poloidal speed is greatest at $x \approx \sqrt{\frac{1}{2}}$, where it is about $\frac{1}{2}c$; of course, the position and value of this maximum may well be significantly affected by departure of the azimuthal speed from its corotational value.

With equation (6) for v_p and $\Omega\varpi$ for v_ϕ, the definition of the Lorentz factor gives

$$\gamma \approx (1-x^2)^{-1}. \tag{7}$$

Equations (6) and (7) for v_p and γ remain valid as the accelerated ions penetrate into the electron zone.

Equation (7) shows that the accelerated ions will be no more than mildly relativistic, except very close to the light cylinder. In the axisymmetric case, the corotating ion zone terminates where the corotational Lorentz factor $(1-x^2)^{-\frac{1}{2}}$ is approximately equal to $(-ZeB_z/Am_p c\Omega)^{1/3}$, as shown by Wang (1978). The magnetic field near the light cylinder might be 10^6 G ($\equiv 10^2$ T) for the Crab pulsar and 1 G for a three-second pulsar (Ruderman 1972, see p. 444), so the maximum corotational Lorentz factor in the ion zone is perhaps of order $10^{2\pm\frac{1}{2}}$. Equation (7) indicates that, in this vicinity, the boundary-layer acceleration mechanism will increase the Lorentz factor of the ions to perhaps $10^{4\pm1}$.

In the boundary layer, the process of inertial development of vorticity occurs (Burman 1981*b*): the poloidal motions of the particles cannot be along, or nearly along, the poloidal magnetic field lines. But once the ions have entered the electron zone, their reduced flow will lie approximately along magnetic field lines. In the electron zone they will be acted on by the centrifugal effect and the non-corotational electric force $-Ze\nabla\Phi(\mathrm{e})$. The force $e\nabla\Phi(\mathrm{e})$ on the corotating electrons acts inwards balancing the centrifugal effect. Since the centrifugal effect is much greater for the ions than for the electrons—their azimuthal speeds being similar—the net unbalanced force on the ions as they sweep through the electron zone is close to the centrifugal 'force'. That is, the ions are centrifugally decelerated as they travel approximately along magnetic field lines toward the star. This is represented by equations (6) and (7) for v_p and γ, which remain valid as the ions propagate through the electron zone.

Equation (6) would indicate that the centrifugal deceleration is insufficient to stop the poloidal motion of the ions, suggesting that they hit the star as slow non-relativistic particles with $v_p \sim \Omega\varpi$. But when the additional azimuthal speed given to the ions in the boundary layer (see Section 7) is taken into account, together with the dissipation that undoubtedly occurs there, it is clear that the acceleration developed in the layer is rather less than has been calculated here. Consequently, the centrifugal deceleration will stop the poloidal motion of the ions above the stellar surface and return them to the boundary layer. They will mirror between the boundary layer and points above the stellar surface. It is likely that energy losses will cause the ions to move gradually to lower magnetic field lines until they eventually reach the star, but this remains to be checked in a more complete study.

4. Poloidal Acceleration of Electrons

Consider an electron, with negligible poloidal speed, entering a boundary layer between zones of corotating electrons and ions. As with the ions, it will be assumed for the present that the azimuthal speed of the electron remains about $\Omega\varpi$, an assumption which will be checked in Section 7. The integral (1) of the motion becomes, dropping the subscript k,

$$\gamma \approx \frac{1+e\Phi/m_e c^2}{1-x^2}, \tag{8}$$

while the definition of the Lorentz factor becomes $\gamma^{-2} \approx 1 - x^2 - v_p^2/c^2$. Eliminating γ between these two equations gives

$$\frac{v_p^2}{c^2} \approx (1-x^2)\left(1 - \frac{1-x^2}{(1+e\Phi/m_e c^2)^2}\right). \tag{9}$$

In the electron zone, Φ is Φ(e), as given by equation (2); hence the integral (8) gives the corotational value for γ while (9) reduces to $v_p = 0$, both as specified. But, so long as v_ϕ remains about $\Omega\varpi$, equations (8) and (9) for γ and v_p remain true as the electron traverses the boundary layer, where Φ increases from Φ(e) to Φ(i); they continue to hold, with Φ(i) for Φ, as the electron subsequently penetrates into the ion corotation zone.

When the electron has crossed the layer, $\Phi = \Phi$(i) so the integral (8) becomes

$$\gamma \approx \frac{1 + (Am_p/Zm_e)\{1-(1-x^2)^{\frac{1}{2}}\}}{1-x^2}, \tag{10}$$

while (9) gives

$$\frac{v_p^2}{c^2} \approx (1-x^2)\left(1 - \frac{1-x^2}{[1+(Am_p/Zm_e)\{1-(1-x^2)^{\frac{1}{2}}\}]^2}\right). \tag{11}$$

Since these expressions are a little complicated, it is illuminating to break them down into simpler ones valid in particular ranges of x.

For $x^2 \ll 1$, equations (10) and (11) show that

$$\gamma \approx 1 + \frac{Am_p}{Zm_e}\frac{x^2}{2}, \tag{12}$$

with

$$1 - \frac{v_p^2}{c^2} \approx \left(1 + \frac{Am_p}{Zm_e}\frac{x^2}{2}\right)^{-2} + x^2. \tag{13}$$

In particular, for $x \ll (2Zm_e/Am_p)^{\frac{1}{2}} \approx 1/30$, the electrons in the ion zone are non-relativistic with

$$v_p \approx (Am_p/Zm_e)^{\frac{1}{2}}\Omega\varpi \approx 40\,\Omega\varpi\,; \tag{14}$$

that is, the boundary layers accelerate electrons to poloidal speeds of about 40 times their (initial) azimuthal speeds. For $x \approx 1/30$, the boundary layers cause the electrons to become mildly relativistic, with γ of order 1 and v_p close to c. For $x \gtrsim 1/30$ but $x^2 \ll 1$, equations (12) and (13) show that the accelerated electrons are mildly to moderately relativistic:

$$\gamma \approx \frac{Am_p}{Zm_e}\frac{x^2}{2} \approx 1\times 10^3 x^2, \tag{15}$$

with

$$1 - \frac{v_p^2}{c^2} \approx \left(\frac{Am_p}{Zm_e}\frac{x^2}{2}\right)^{-2} + x^2 \approx (1\times 10^3 x^2)^{-2} + x^2. \tag{16}$$

In the region $1/30 \lesssim x < 1$, the boundary layers produce only relativistic electrons:

$$\gamma \approx \frac{Am_{\rm p}}{Zm_{\rm e}} \frac{1-(1-x^2)^{\frac{1}{2}}}{1-x^2}, \tag{17}$$

with

$$1 - \frac{v_{\rm p}^2}{c^2} \approx \left(\frac{Zm_{\rm e}}{Am_{\rm p}} \frac{1-x^2}{1-(1-x^2)^{\frac{1}{2}}}\right)^2 + x^2, \tag{18}$$

which reduce to equations (15) and (16) for $x^2 \ll 1$ but $x \gtrsim 1/30$. Close to the light cylinder, or more precisely for $(1-x^2)^{\frac{1}{2}} \ll 1$, equations (17) and (18) simplify to

$$\gamma \approx \frac{Am_{\rm p}/Zm_{\rm e}}{1-x^2} \approx \frac{1\times 10^3}{1-x}, \tag{19a, b}$$

$$v_{\rm p}^2/c^2 \approx 1-x^2. \tag{20}$$

According to (18), the greatest value of $v_{\rm p}$ is achieved for $x \approx 2^{\frac{1}{2}}(Zm_{\rm e}/Am_{\rm p})^{1/3} \approx 1/10$, where $1 - v_{\rm p}/c \approx 1\times 10^{-2}$. The location and value of this maximum may well be significantly affected by departure of the azimuthal speed from $\Omega\varpi$.

All of the equations (10)–(20) remain valid as the electrons sweep through the ion zone.

The above analysis shows that the accelerated electrons will be highly relativistic for $x \gg 1/30$. In the axisymmetric case, the corotating electron zone terminates where the corotational Lorentz factor is approximately equal to $(eB_z/m_{\rm e}c\Omega)^{1/3}$ (Wang 1978); near the light cylinder this is perhaps of order $10^{3\pm\frac{1}{2}}$. But the boundary layer with the ion zone will terminate where the ion zone terminates, at perhaps $(1-x^2)^{-\frac{1}{2}} \sim 10^{2\pm\frac{1}{2}}$. Equations (19) indicate that, in this vicinity, the boundary-layer acceleration mechanism will increase the Lorentz factor of the electrons to perhaps $10^{7\pm1}$ or so.

In the boundary layers, the poloidal motions of the particles must depart from the poloidal magnetic field lines. But once the electrons have entered the ion zone proper (excluding its outer boundary layer), the inertia of even the most highly accelerated ones is insufficient to cause the reduced flow velocity to depart significantly from magnetic field lines, as will now be shown.

The relative importance of the inertial and magnetic effects in the fluxoid law for species k is estimated by the magnetic Rossby number ε_k, introduced by Wright (1978): $\varepsilon_k = \gamma_k v_k/\omega_{Bk} L_k$, where ω_{Bk} is the magnitude of the nonrelativistic angular gyrofrequency of species k and L_k is a length scale over which the macroscopic properties of species k vary significantly. For corotating particles, $v_k \approx \Omega\varpi$ and L_k is typically of order ϖ, so

$$\varepsilon_k \sim \gamma_k \Omega/\omega_{Bk}. \tag{21}$$

This estimate shows that the ratio of inertial to magnetic effects is of the order of the ratio of a macroscopic rotational frequency to a microscopic gyrofrequency, as Mestel (1971) showed by considering the corresponding energy densities.

For relativistic particles traversing the outer parts of a corotating zone, $v_k \approx c$ and $L_k \sim c/\Omega$ so the estimate (21) again applies. It will not apply to particles in the thin outer boundary layer of a corotating zone, since L_k there will be very much smaller than c/Ω. Since $(1-x^2)^{-\frac{1}{2}} \approx (\omega_{Bp}/\Omega)^{1/3}$ where the ion zone terminates near the light cylinder, with ω_{Bp} denoting the nonrelativistic proton angular gyrofrequency, equation (19a) indicates that the maximum Lorentz factor for the accelerated electrons as they traverse the ion zone is about $(m_p/m_e)(\omega_{Bp}/\Omega)^{2/3}$. So, from (21), the magnetic Rossby number for these maximally accelerated electrons is of order $(\Omega/\omega_{Bp})^{1/3}$, which is perhaps of order $10^{-2\pm\frac{1}{2}}$. Thus, even for these electrons, the magnetic effects outweigh the inertial effects in the fluxoid law: their inertia is insufficient to bring about any significant departure of their reduced flow from the magnetic field lines. (This will not be the case for electrons in the outer boundary layer of the ion zone.)

In the ion zone, the accelerated electrons will be acted on by the centrifugal effect and the non-corotational electric force $e\nabla\Phi(\mathrm{i})$. The force $-Ze\nabla\Phi(\mathrm{i})$ on the corotating ions acts inwards balancing the centrifugal effect. The force $e\nabla\Phi(\mathrm{i})$ on the electrons acts outwards, supporting the centrifugal effect. For $x^2 \ll 1$, the centrifugal 'force' on the electrons is much less than that on the ions. So then the centrifugal force on the electrons is small compared with the non-corotational electric force on them, provided Z is small. For $(1-x^2)^{\frac{1}{2}} \ll 1$, equation (19a) shows that the accelerated electrons are just so much more highly relativistic than the corotating ions that their relativistic masses, and hence the centrifugal forces on them, are approximately equal. So then the centrifugal force on the electrons is approximately Z times the non-corotational electric force on them.

As each electron traverses the ion zone, it will experience a total force directed outwards, perpendicularly to the stellar rotation axis. The component of this force parallel to the magnetic field will act to accelerate the electron further as it travels approximately along outgoing portions of the closed magnetic field lines and to decelerate it as it travels along ingoing portions. This is represented by equations (10)–(20) for v_p and γ, which remain valid as the electrons propagate through the ion zone. The electrons will be rapidly decelerated on reaching the boundary layer in the opposite hemisphere. In the second boundary layer, their poloidal motion will be essentially stopped and they will be accelerated again back into the ion zone: the electrons will mirror between the northern and southern boundary layers. Dissipation will occur both in the boundary layers and as the electrons traverse the ion zone. This will presumably cause the electrons to move to lower magnetic field lines until they reach the star.

5. Boundary-layer Thickness

In my note (Burman 1981*b*) demonstrating the existence of vortical boundary layers in pulsar magnetospheres, I made some rough estimates of the thicknesses of the boundary layers separating the electron and ion corotation zones. The information obtained in the last two sections on particle speeds and Lorentz factors in those layers now enables me to make improved estimates.

As I pointed out (Burman 1981*b*), the inertial and magnetic contributions to the magnetoidal field of a species must cancel where its number density becomes very small; this means that the magnetic Rossby number ε_k is of order one there. So a boundary layer between two species has a thickness which can be estimated to be of the order of the greater of the quantities $\gamma_k v_k/\omega_{Bk}$ for the two species. These two quantities will be called L_i and L_e for the ions and electrons respectively.

It was seen above that the ions in the boundary layers have speeds of about $\Omega\varpi$ and Lorentz factors reaching about $(1-x^2)^{-1}$. For $x \ll 1/30$, the electrons in the boundary layers are nonrelativistic with speeds reaching about $40\,\Omega\varpi$. For $1/30 \lesssim x < 1$, they are relativistic with Lorentz factors reaching that given by equation (17). Hence we have

$$L_{\rm i}/L_{\rm e} \sim 50 \qquad \text{for} \qquad x \ll 1/30\,, \tag{22a}$$

$$L_{\rm i}/L_{\rm e} \sim x/\{1-(1-x^2)^{\frac{1}{2}}\} \quad \text{for} \quad 1/30 \lesssim x < 1\,. \tag{22b}$$

In particular, for $x \gtrsim 1/30$ but $x^2 \ll 1$, the expression (22b) shows that $L_{\rm i}/L_{\rm e} \sim 2/x$, which drops from about 50 to a few as x increases in this range. For $(1-x^2)^{\frac{1}{2}} \ll 1$, (22b) shows that $L_{\rm i} \sim L_{\rm e}$. So, for all x, $L_{\rm i} \gtrsim L_{\rm e}$, the two quantities being similar near the light cylinder.

Hence, the thickness L of the boundary layers between the electron and ion corotation zones can be taken to be $L_{\rm i}$. Thus we have

$$\frac{L}{\varpi} \sim \frac{\Omega/\omega_{B{\rm p}}}{1-x^2}\,. \tag{23}$$

Near the star, the thickness is microscopically small, indicating that fluid dynamics will not provide there an adequate description of behaviour in the boundary layer (Burman 1981*b*). Because of the ϖ factor and the rapid decline of magnetic field strength, the boundary layer increases rapidly in thickness away from the star, roughly as ϖ^4 until near the light cylinder where it flares out because of the $(1-x^2)^{-1}$ factor. With the boundary layer terminating near the light cylinder where $(1-x^2)^{-\frac{1}{2}} \approx (\omega_{B{\rm p}}/\Omega)^{1/3}$, it follows from the estimate (23) that L/ϖ varies from being of order $\Omega/\omega_{B{\rm p}}$ near the star to being of order $(\Omega/\omega_{B{\rm p}})^{1/3}$ near the light cylinder. Near the light cylinder of the Crab pulsar, L is perhaps a few kilometres. Near the light cylinder of a three-second pulsar, L is perhaps 10^4 km—about one-tenth of the radius of the light cylinder.

6. Electric Fields

The non-corotational electric fields $-\nabla\Phi(\mathrm{e})$ and $-\nabla\Phi(\mathrm{i})$ in the electron and ion corotation zones follow from equations (2) and (3) of Section 2. The non-corotational electric field in the boundary layers between those zones can be estimated by using (2) and (3) together with the layer thickness just calculated. In this section, these fields will be obtained and their magnitudes compared with each other and with the magnitude $E_{\rm cr}$ of the corotational electric field $-x\boldsymbol{t}\times\boldsymbol{B}$.

Equations (2) and (3) give

$$\frac{-e}{m_{\rm e}c^2}\nabla\Phi(\mathrm{e}) \approx \frac{\Omega}{c}\frac{x}{(1-x^2)^{\frac{1}{2}}}\boldsymbol{i}\,, \tag{24}$$

$$\frac{Ze}{Am_{\rm p}c^2}\nabla\Phi(\mathrm{i}) \approx \frac{\Omega}{c}\frac{x}{(1-x^2)^{\frac{1}{2}}}\boldsymbol{i}\,, \tag{25}$$

where $\boldsymbol{i}$ is the cylindrical polar unit radial vector. The corotational and non-corotational electric fields are both poloidal, with the latter in the $+\boldsymbol{i}$ or $-\boldsymbol{i}$ direction according to whether the corotating particles are negatively or positively charged. The non-corotational electric forces $e\nabla\Phi(\mathrm{e})$ and $-Ze\nabla\Phi(\mathrm{i})$ act inwards, balancing the centrifugal effect on the corotating particles.

From equations (24) and (25) we have

$$|\nabla\Phi(\mathrm{e})|/E_{\mathrm{cr}} \approx (\Omega/\omega_{B\mathrm{e}})/(1-x^2)^{\frac{1}{2}} \sim \varepsilon_{\mathrm{e}}, \tag{26}$$

$$|\nabla\Phi(\mathrm{i})|/E_{\mathrm{cr}} \approx (\Omega/\omega_{B\mathrm{i}})/(1-x^2)^{\frac{1}{2}} \sim \varepsilon_{\mathrm{i}}, \tag{27}$$

where ε_{e} and ε_{i} are the magnetic Rossby numbers of the corotating electrons and ions, for which the estimate (21) has been invoked. Thus, the ratio of non-corotational to corotational electric field strengths in a zone of corotation is of the order of the magnetic Rossby number of the corotating species, which measures the relative importance of the inertial and magnetic contributions to the magnetoidal field of that species; it is the predominance of the magnetic field that enforces corotation, so the magnetic Rossby number is small. With corotation terminating near the light cylinder where $(1-x^2)^{-\frac{1}{2}} \approx (\omega_{Bk}/\Omega)^{1/3}$, it follows that ε_k, and hence $|\nabla\Phi(k)|/E_{\mathrm{cr}}$, vary from being of order Ω/ω_{Bk} near the star to being of order $(\Omega/\omega_{Bk})^{2/3}$ near the light cylinder, remaining small throughout.

Because their function is to provide forces to balance the centrifugal effects on the corotating particles, the potentials $\Phi(\mathrm{e})$ and $\Phi(\mathrm{i})$ are proportional to the mass to charge ratios of those particles. They have the sign of the corresponding particles, and, for similar values of x, $\Phi(\mathrm{i})$ is larger than $|\Phi(\mathrm{e})|$ by a factor of $(2\text{–}4)\times 10^3$—hence the occurrence of a strong non-corotational electric field across the boundary layers separating electron and ion zones. Its magnitude will now be estimated.

From equations (2) and (3), at a given value of x, we have

$$\Phi(\mathrm{i})-\Phi(\mathrm{e}) \approx (Am_{\mathrm{p}}c^2/Ze)\{1-(1-x^2)^{\frac{1}{2}}\}. \tag{28}$$

So, the estimate (23) for the layer thickness implies that

$$|\nabla\Phi|/E_{\mathrm{cr}} \sim \{(1-x^2)/x^2\}\{1-(1-x^2)^{\frac{1}{2}}\}. \tag{29}$$

This function is $\frac{1}{2}$ for $x^2 \ll 1$; it falls as x increases but declines slowly at first reaching $\frac{1}{4}$ slightly beyond $x = \frac{3}{4}$; it falls rapidly for x^2 very close to one, dropping to about $(\Omega/\omega_{B\mathrm{p}})^{2/3}$ where the boundary layer terminates.

The boundary layer $\nabla\Phi$ is always very large compared with nearby values of $\nabla\Phi(\mathrm{e})$, and is large compared with nearby values of $\nabla\Phi(\mathrm{i})$ except near the light cylinder where these two become the same order of magnitude.

In the boundary layer, inertial and magnetic effects are of similar importance. Therefore it is no surprise to have found the non-corotational electric field to be, except near the light cylinder, close to the corotational electric field in magnitude.

7. Toroidal Drifts

An unbalanced poloidal force acting on a particle—unbalanced relative to that required to maintain corotation—will produce two effects: It will, except where it happens to be perpendicular to the particle's trajectory, cause poloidal acceleration; its other effect is a toroidal drift. In this first approach to the study of the physics of pulsar magnetospheric boundary layers, I am proceeding on the assumption that the two effects can be estimated separately. The order-of-magnitude calculation of the boundary-layer $\nabla\Phi$ field in the last section enables the toroidal drifts it generates to be estimated. For the procedure to be valid, it is necessary that these drifts are not so large as to upset seriously the poloidal accelerations previously calculated. I shall show in this section that the drifts are indeed not too large. But first the toroidal drifts of the accelerated particles as they travel through the corotation zones will be calculated and shown to be small.

The approximation $(1-x^2)^{-1}$ for the Lorentz factor of the accelerated ions in an electron zone implies that the centrifugal effect on each ion is greater than that on a corotating electron at the same value of x by a factor of $(Am_p/m_e)(1-x^2)^{-\frac{1}{2}}$. The centrifugal effect on a corotating electron is balanced by the non-corotational electric force $e\nabla\Phi(\mathrm{e})$. So the centrifugal force on an accelerated ion greatly exceeds the non-corotational electric force $-Ze\nabla\Phi(\mathrm{e})$: the unbalanced force on an ion is $\boldsymbol{F}_\mathrm{i}$, where

$$\boldsymbol{F}_\mathrm{i} \approx \gamma_\mathrm{i} Am_\mathrm{p}\Omega^2\varpi\boldsymbol{i}. \tag{30}$$

The toroidal drift velocity produced by a poloidal force $\boldsymbol{F}$ acting on a particle of charge q is $c\boldsymbol{F}\times\boldsymbol{B}_\mathrm{p}/qB^2$, where $\boldsymbol{B}_\mathrm{p}$ is the poloidal part of the magnetic field. Thus $\boldsymbol{F}_\mathrm{i}$ given by equation (30) generates a toroidal drift of the ions, relative to the motion of corotation, with velocity $\boldsymbol{w}_\mathrm{i}$, where

$$\frac{\boldsymbol{w}_\mathrm{i}}{\Omega\varpi} \approx \gamma_\mathrm{i}\frac{\Omega}{\omega_{B\mathrm{i}}}\frac{\boldsymbol{i}\times\boldsymbol{B}_\mathrm{p}}{B}. \tag{31}$$

The accelerated ions have speeds close to $\Omega\varpi$. (For $x^2 \ll 1$ they have velocity components $v_\mathrm{p} \approx \Omega\varpi \approx v_\phi$; for larger x^2 they have $v_\mathrm{p} \ll \Omega\varpi$ and $v_\phi \approx \Omega\varpi$.) Taking their macroscopic properties to vary on a length scale of order ϖ shows that the estimate (21) for the magnetic Rossby number is in fact valid for all the accelerated ions, not just those traversing the outer parts of the electron corotation zones. Hence equation (31) implies that

$$w_\mathrm{i}/\Omega\varpi \lesssim \gamma_\mathrm{i}\Omega/\omega_{B\mathrm{i}} \sim \varepsilon_\mathrm{i}, \tag{32}$$

where ε_i denotes the magnetic Rossby number of the accelerated ions. Taking $\gamma_\mathrm{i} \lesssim (\omega_{B\mathrm{i}}/\Omega)^{2/3}$ implies that ε_i and $w_\mathrm{i}/\Omega\varpi$ are less than about $(\Omega/\omega_{B\mathrm{i}})^{1/3}$, and so are both small. Since ε_i is small, the ions travel through the electron corotation zones with a reduced flow velocity that is essentially along the magnetic field lines.

The centrifugal effect on an accelerated electron in the ion zone will, depending on x, be more or less than that on a corotating ion at the same value of x: the ratio of the effects is $(1-x^2)^{\frac{1}{2}}\gamma_\mathrm{e}m_\mathrm{e}/Am_\mathrm{p}$, and, for sufficiently large x, the electrons are so much more relativistic than the ions that this ratio exceeds one. The centrifugal effect on a corotating ion is balanced by the non-corotational electric force $-Ze\nabla\Phi(\mathrm{i})$. So the centrifugal force on an accelerated electron may be more or less than the non-corotational electric force $e\nabla\Phi(\mathrm{i})$: the unbalanced force is $\boldsymbol{F}_\mathrm{e}$, where

$$\boldsymbol{F}_\mathrm{e} = \left(\frac{Am_\mathrm{p}/Z}{(1-x^2)^{\frac{1}{2}}} + \gamma_\mathrm{e}m_\mathrm{e}\right)\Omega^2\varpi\boldsymbol{i}; \tag{33}$$

equation (25) has been used for $\nabla \Phi(\mathrm{i})$. In $1/30 \lesssim x < 1$, the accelerated electrons are relativistic with Lorentz factors given by equation (17); hence

$$\boldsymbol{F}_{\mathrm{e}} \approx \frac{A}{Z} m_{\mathrm{p}} \frac{\Omega^2 \varpi}{1-x^2} \boldsymbol{i}, \tag{34}$$

with the centrifugal effect predominating in $\boldsymbol{F}_{\mathrm{e}}$ when $x \gtrsim \frac{1}{2}\sqrt{3}$. For $x \ll 1/30$, the electrons are nonrelativistic, the centrifugal contribution to $\boldsymbol{F}_{\mathrm{e}}$ is negligible and equation (33) reduces to (34) with $x^2 \ll 1$. So (34) can be taken to be applicable for all accelerated electrons.

The force $\boldsymbol{F}_{\mathrm{e}}$ given by equation (34) produces a toroidal drift of the electrons, relative to the motion of corotation, with velocity $\boldsymbol{w}_{\mathrm{e}}$, where

$$\frac{\boldsymbol{w}_{\mathrm{e}}}{\Omega \varpi} \approx \frac{1}{1-x^2} \frac{\Omega}{\omega_{B\mathrm{i}}} \frac{\boldsymbol{B}_{\mathrm{p}} \times \boldsymbol{i}}{B}. \tag{35}$$

Hence, with $(1-x^2)^{-\frac{1}{2}} \lesssim (\omega_{B\mathrm{i}}/\Omega)^{1/3}$ in the ion zone, we have

$$\frac{w_{\mathrm{e}}}{\Omega \varpi} \lesssim \frac{1}{1-x^2} \frac{\Omega}{\omega_{B\mathrm{i}}} \lesssim \left(\frac{\Omega}{\omega_{B\mathrm{i}}}\right)^{1/3}. \tag{36}$$

Thus, the toroidal drift speeds of the electrons in the ion zone, like those of the ions in the electron zones, are less than the local speed of corotation by factors which are less than about $(\Omega/\omega_{B\mathrm{i}})^{1/3}$, which is small.

For $x \ll 1/30$, the accelerated electrons are nonrelativistic with speeds of about $40\,\Omega\varpi$. In $1/30 \lesssim x < 1$, they are relativistic with Lorentz factors given by equation (17). So, taking their macroscopic properties to vary on a length scale ϖ implies that the magnetic Rossby number ε_{e} of the accelerated electrons is given by

$$\varepsilon_{\mathrm{e}} \sim \frac{1}{50} \frac{\Omega}{\omega_{B\mathrm{i}}} \qquad \text{for} \qquad x \ll 1/30, \tag{37a}$$

$$\varepsilon_{\mathrm{e}} \sim \frac{1}{x} \frac{1-(1-x^2)^{\frac{1}{2}}}{1-x^2} \frac{\Omega}{\omega_{B\mathrm{i}}} \qquad \text{for} \qquad 1/30 \lesssim x < 1. \tag{37b}$$

Thus ε_{e} varies from about $(1/50)(\Omega/\omega_{B\mathrm{i}})$ for $x \lesssim 1/30$, through about $\frac{1}{2}x(\Omega/\omega_{B\mathrm{i}})$ for $x \gtrsim 1/30$ with $x^2 \ll 1$, to about $(\Omega/\omega_{B\mathrm{i}})^{1/3}$ near the light cylinder: it is always small. So, as mentioned before, these particles—even the most highly relativistic of them—do not have sufficient inertia to untie themselves from the magnetic field lines. For the most energetic of them, the radiation reaction force might be strong enough to make them cross magnetic field lines. But most of the electrons travel through the ion corotation zone with a reduced flow velocity that is essentially along the magnetic field.

The toroidal drift velocity $\boldsymbol{w}$, relative to the motion of corotation, in a poloidal $\nabla \Phi$ field is given by

$$\boldsymbol{w}/c = \{\boldsymbol{B}_{\mathrm{p}} \times (\nabla \Phi)\}/B^2. \tag{38}$$

This velocity, being an electric field drift, is independent of the charge on the drifting particles. From equation (38) we get

$$w \lesssim (|\nabla \Phi|/E_{\mathrm{cr}})\Omega\varpi. \tag{39}$$

In a boundary layer between electron and ion corotation zones, $\nabla\Phi$ is directed from the electron zone to the ion zone. So, in the axisymmetric case, the drift velocity $\boldsymbol{w}$ there always adds to the corotation velocity $\Omega\varpi\boldsymbol{t}$ to produce an increased azimuthal speed; this is true in both the northern and southern boundary layers and, of course, is true for particles when being decelerated as well as when being accelerated.

From the expressions (39) and (29), the drift speed in the boundary layers is limited by

$$u_\phi/\Omega\varpi \lesssim \{(1-x^2)/x^2\}\{1-(1-x^2)^{\frac{1}{2}}\}. \tag{40}$$

Thus, u_ϕ varies from less than roughly $\frac{1}{2}\Omega\varpi$ for $x^2 \ll 1$ to less than roughly $\frac{1}{4}\Omega\varpi$ at $x = \frac{3}{4}$, dropping far below the local corotational speed where the boundary layer ends near the light cylinder.

With $v_{k\phi}$ approximated by $\Omega\varpi$ in the integral (1) of the motion, the resulting expression for γ_k diverges as $x \to 1$ from below, since $e_k\Phi/m_k c^2 < 1$ for $x \leqslant 1$ (Burman 1980). This divergence just corresponds, of course, to the impossibility of having $v_{k\phi} = \Omega\varpi$ on the light cylinder. Because of it, Lorentz factors calculated from (1) will be particularly sensitive to departures of $v_{k\phi}$ from $\Omega\varpi$ near the light cylinder. The above estimates show that, for both ions and electrons, the relative departure of $v_{k\phi}$ from $\Omega\varpi$ in the boundary layers is greatest near the star, becoming very small beyond $x = \frac{3}{4}$ and approaching zero as $x \to 1$. It is indeed helpful that the ratio of $u_{k\phi}$ to $\Omega\varpi$ becomes small just where it would cause most trouble in the calculation of γ_k.

These figures, though only order of magnitude estimates, indicate that neglect of the toroidal $\nabla\Phi$ drifts in the boundary layers when calculating the accelerations produced there is justified in a first approximation. Certainly, a more accurate analysis of the boundary-layer dynamics is warranted—indeed required—but the calculations presented in this paper are quite adequate for a first approach. Refinement of the calculations will probably not change the essential features found here, though inclusion of more physics could do so.

When the reduced flow velocity of species k lies approximately along the magnetic field, it follows that

$$u_{k\phi} \approx v_{k\mathrm{p}} B_\phi/B_\mathrm{p}. \tag{41}$$

Hence, for the ions streaming through the electron corotation zones, the result $v_\mathrm{p} \approx \Omega\varpi(1-x^2)^{\frac{1}{2}}$ implies that

$$v_\phi/\Omega\varpi \approx 1+(1-x^2)^{\frac{1}{2}}B_\phi/B_\mathrm{p}. \tag{42}$$

So the calculations of the behaviour of these particles appear to be at least roughly self-consistent provided

$$(1-x^2)^{\frac{1}{2}}B_\phi/B_\mathrm{p} \lesssim 1, \tag{43}$$

which is likely to be satisfied.

For the electrons streaming through the ion corotation zone, the results $v_\mathrm{p} \approx 40\,\Omega\varpi$ for $x \ll 1/30$ and $v_\mathrm{p} \approx c$ for $1/30 \lesssim x < 1$ imply that

$$v_\phi/\Omega\varpi \approx 1+40B_\phi/B_\mathrm{p} \quad \text{for} \quad x \ll 1/30, \tag{44a}$$

$$v_\phi/\Omega\varpi \approx 1+B_\phi/xB_\mathrm{p} \quad \text{for} \quad 1/30 \lesssim x < 1, \tag{44b}$$

except perhaps for the most energetic electrons. Hence, the calculations of the behaviour of these particles appear to be at least roughly self-consistent provided

$$B_\phi/B_p \lesssim 1/40 \quad \text{for } x \ll 1/30, \qquad B_\phi/B_p \lesssim x \quad \text{for } 1/30 \lesssim x < 1. \qquad (45a,b)$$

Near the light cylinder, where the calculations are most sensitive to departure of v_ϕ from $\Omega\varpi$, the condition (45b) is likely to be satisfied. Near the star, (45a) might be satisfied for models that are close to axisymmetry, but will fail for others.

8. Implications for Model Building

Two realizations have led me to the picture of the pulsar magnetosphere that I am developing in this series of papers. One was the recognition of the existence of boundary layers in which the inertial generation of vorticity occurs (Burman 1981*b*), together with the expectation that they would be likely to be key features of magnetospheric physics (Burman 1981*c*) rather than merely local perturbations. The other was the recognition (Burman 1980, 1981*a*) that it is essential to use the correct functional form for the non-corotational electric potential Φ everywhere outside the star—including zones containing corotating particles, where Φ had invariably been neglected.

There are at least three reasons why it is necessary to account properly for Φ in zones of corotation (Burman 1981*a*): to avoid obtaining absurd results for the azimuthal velocities; to be able to calculate the net force on a species flowing through a zone where another is corotating; and to demonstrate the existence and evaluate the amount of the jump in Φ between zones of corotation, which corresponds to a powerful local $\nabla\Phi$ field. Even where Φ is small, in the sense that $e_k\Phi/m_k c^2 \ll 1$ for the species present, neglect of Φ leads, locally, to absurdities and, globally, to the misunderstanding and even omission of essential physics.

The result of these realizations is a drastic revision of the conventional Goldreich–Julian (1969) picture of the pulsar magnetosphere. Of course, the magnetospheric existence theorem, or Goldreich–Julian mechanism (Goldreich 1969; Goldreich and Julian 1969; Michel 1969) for the extraction of charged particles from the star, remains. But I have shown that their system, with a corotating positively charged zone, possibly containing positive ions only, and corotating negatively charged zones, possibly containing electrons only, is violently unstable. The zones must, in fact, be separated by boundary layers, across which there exist large non-corotational electric fields. These fields are comparable in magnitude, over most of each layer, with the electric field generated by corotation of the magnetic field structure, and are directed so as to accelerate the ions into the electron zones and the electrons into the ion zones. My considerations have thus invalidated Holloway's (1973) idea that the Goldreich–Julian corotation zones are separated by vacuum gaps.

My calculations have indicated that the accelerated electrons will cross the zone of ion corotation. All except perhaps the most highly relativistic of them will travel essentially along the magnetic field lines in the ion zone, and mirror between the northern and southern boundary layers. It is likely that they will gradually lose energy, move to lower magnetic field lines and eventually return to the star. The ions penetrate well into the electron zones, where they travel essentially along magnetic field lines; they are centrifugally decelerated and then centrifugally accelerated outwards, mirroring between the boundary layer and points above the stellar surface. Presumably they, too, will move to lower magnetic field lines as they lose energy and gradually find their way to the star.

Thus, the ion, or ion-dominated, corotation zone of Goldreich and Julian (1969) must be replaced by a zone containing ions that are essentially corotating, having small poloidal speeds, together with poloidally flowing and mirroring electrons. Their electron, or electron-dominated, corotation zones, must be replaced by zones containing electrons that are essentially corotating, having small poloidal speeds, together with poloidally flowing and mirroring ions. The relative numbers of poloidally flowing and corotating particles remain to be determined. The physics of the various boundary layers must be properly included. A first approach to the analysis of the layers between the zones containing corotating particles has been developed in this paper. A more accurate treatment will be given later. Other layers will be reported in future papers in this series.

The electrons entering the zone of ion corotation are further accelerated there as they travel along outgoing sections of magnetic field lines, and are then decelerated as they travel along ingoing sections. For most of them, this acceleration and deceleration is essentially electrical, arising from the $\nabla \Phi$ field; but for the most highly relativistic electrons, the centrifugal effect predominates.

The most energetic of these electrons are those traversing the outer parts of the zone of ion corotation. The calculations indicate that their Lorentz factors reach perhaps 10^6 for a three-second pulsar and perhaps 10^8 for the Crab pulsar. They will emit gamma radiation, so it is to be expected that pulsars, including non-pulsing axisymmetric objects, will emit gamma radiation from the outer parts of the zone of ion corotation. The effect of radiation reaction on the motions of these very high-energy electrons remains to be investigated.

In the case of the Crab pulsar, there is a possibility that the gamma rays will produce electron–positron pairs by the mechanism introduced into pulsar theory by Sturrock (1971), in whose picture the pair production occurs above the 'polar caps'. Thus, the Crab pulsar's zone of ion corotation might terminate near the light cylinder in a highly relativistic ternary ion–electron–positron plasma.

9. Concluding Remarks

In my approach, I am retaining, at least for the present, the idea of having zones in which there are particles that are close to being in corotation with the star, but I regard them as having small, nonzero, poloidal velocities, and to be not the exclusive occupants of those zones: certainly the oft-used assumption of essentially complete charge separation has failed to survive the introduction of inertial effects. In fact, Goldreich and Julian (1969) found positive ions to stream out through their corotating electron zones—this has always been considered to be an unsatisfactory feature, but I have shown that properly accounting for inertial effects implies that the zones of corotation are permeated by fast poloidally streaming and accelerating particles of the opposite sign to those that are close to corotation.

Beyond a few qualitative considerations, I have not considered dissipative effects. These will undoubtedly be important, particularly in the interaction of the species in the boundary layers, but also in the interaction of the streaming and non-streaming species in the zones of near-corotation. It will be interesting to see how many of the proposed magnetospheric features survive the introduction of dissipative forces. But first it is certainly worthwhile obtaining a fair understanding of the implications of the dissipation-free equations.

Many of the ideas being developed appear to be transferable, with ions replaced by positrons, to possible black hole magnetospheres.

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Manuscript received 15 September 1980, accepted 28 April 1981