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Thermosolutal Instability of a Hall Plasma

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Abstract

The thermosolutal convection in a layer of viscous plasma heated from below and subjected to a stable solute gradient is investigated in the presence of Hall currents. For the case of stationary convection, the stable solute gradient and the Hall currents are found to have stabilizing and destabilizing effects respectively on the system. The question of the onset of instability as overstability is also discussed.

1. Introduction

The problem of thermal convection under varying assumptions of hydrodynamics and hydromagnetics has been treated in detail by Chandrasekhar (1961). The conditions under which convective motions are important in stellar atmospheres are usually far removed from considerations of single component fluids and rigid boundaries, and therefore it is desirable to consider a fluid acted on by a solute gradient (or two-component fluid) and free boundaries. In such situations, buoyancy forces can arise not only from density differences due to variations in temperature, but also from those due to variations in solute concentration. Veronis (1965) has investigated the problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient, whereas the problem of thermohaline convection in a horizontal layer of viscous fluid heated from below and salted from above has been studied by Nield (1967). Gupta (1967) has studied the problem of thermal instability in the presence of Hall currents (see also Sharma and Sharma 1978).

The problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its application to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layers of the solar atmosphere (Spiegel 1969). As Hall effects are likely to be important in these regions, we decided to reconsider the thermosolutal convection problem including these effects.

2. Formulation of Problem and Perturbation Equations

We consider an infinite horizontal fluid layer of thickness d heated from below and subjected to a stable solute gradient so that the temperatures and solute concentrations at the bottom surface $z = 0$ are T_0 and C_0 and at the upper surface $z = d$ are T_1 and C_1, the z-axis being taken as vertical. This layer is acted on by a gravitational force $\boldsymbol{g}(0, 0, -g)$ and a magnetic field $\boldsymbol{H}(0, 0, H)$. The layer is heated and soluted from below such that a uniform temperature gradient β $(=|\mathrm{d}T/\mathrm{d}z|)$ and a solute concentration gradient β' ($=|\mathrm{d}C/\mathrm{d}z|$) are maintained. Let N and e denote respectively the number density and charge of an electron. Then the equations governing the motion of a fluid, following the Boussinesq approximation, are

$$\partial \boldsymbol{v}/\partial t + (\boldsymbol{v}\,.\,\nabla)\boldsymbol{v} = -\rho_0^{-1}\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{g}(1+\delta\rho/\rho_0) + (\mu_e/\rho_0)(\nabla\times\boldsymbol{H})\times\boldsymbol{H}, \tag{1a}$$

$$\nabla\,.\,\boldsymbol{v} = 0, \tag{1b}$$

$$\partial T/\partial t + (\boldsymbol{v}\,.\,\nabla)T = \kappa\nabla^2 T, \tag{1c}$$

$$\partial C/\partial t + (\boldsymbol{v}\,.\,\nabla)C = \kappa'\nabla^2 C, \tag{1d}$$

$$\rho = \rho_0\{1-\alpha(T-T_0)+\alpha'(C-C_0)\}, \tag{1e}$$

where ρ, p, α, α' and $\boldsymbol{v}(u, v, w)$ are respectively the density, the pressure, a thermal coefficient of expansion, an analogous solvent coefficient of expansion and the velocity. The magnetic permeability μ_e, the kinematic viscosity ν, the thermal diffusivity κ and the solute diffusivity κ' are all assumed to be constant. Equations (1a)–(1d) express the conservation of momentum, mass, temperature and solute mass concentration respectively. Equation (1e) represents the equation of state. The suffix zero refers to values at the reference level $z = 0$.

Maxwell's equations give

$$\mathrm{d}\boldsymbol{H}/\mathrm{d}t = (\boldsymbol{H}\,.\,\nabla)\boldsymbol{v} + \eta\nabla^2\boldsymbol{H} - (1/Ne)\nabla\times\{(\nabla\times\boldsymbol{H})\times\boldsymbol{H}\}, \tag{2a}$$

$$\nabla\,.\,\boldsymbol{H} = 0, \tag{2b}$$

where η is the 'resistivity' and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v}\,.\,\nabla$. (Rationalized MKS units are used throughout.)

The steady state solution is

$$\boldsymbol{v} = 0, \quad T = T_0 - \beta z, \quad C = C_0 - \beta' z, \quad \rho = \rho_0(1+\alpha\beta z - \alpha'\beta' z). \tag{3}$$

Let us consider a small perturbation to the steady state solution and let $\delta\rho, \delta p, \theta, \gamma, \boldsymbol{v}$ and $\boldsymbol{h}(h_x, h_y, h_z)$ denote respectively perturbations in density, pressure, temperature T, concentration C, velocity and magnetic field $\boldsymbol{H}$, so that the change in density caused by the perturbations in temperature and concentration is given by

$$\delta\rho = -\rho_0(\alpha\theta - \alpha'\gamma). \tag{4}$$

The linearized perturbation forms of equations (1a)–(1d) and (2) become

$$\partial\boldsymbol{v}/\partial t = -\rho_0^{-1}\nabla\delta p + \nu\nabla^2\boldsymbol{v} - \boldsymbol{g}(\alpha\theta - \alpha'\gamma) + (\mu_e/\rho_0)(\nabla\times\boldsymbol{h})\times\boldsymbol{H}, \tag{5a}$$

$$\nabla\,.\,\boldsymbol{v} = 0, \tag{5b}$$

$$\partial\theta/\partial t = \beta w + \kappa\nabla^2\theta, \tag{5c}$$

$$\partial\gamma/\partial t = \beta' w + \kappa'\nabla^2\gamma, \tag{5d}$$

$$(\partial/\partial t-\eta\nabla^2)\boldsymbol{h} = (\boldsymbol{H}.\nabla)\boldsymbol{v}-(1/Ne)\nabla\times\{(\nabla\times\boldsymbol{h})\times\boldsymbol{H}\}, \tag{6a}$$

$$\nabla.\boldsymbol{h} = 0. \tag{6b}$$

Here we consider the case in which both the boundaries are free, the most appropriate case for stellar atmospheres (Spiegel 1965), and the adjoining medium is electrically non-conducting. The boundaries are assumed to be perfect conductors of both heat and solute concentration. The appropriate boundary conditions are

$$w = \partial^2 w/\partial z^2 = \theta = \gamma = \xi = 0, \tag{7}$$

and the components of $\boldsymbol{h}$ are continuous.

3. Dispersion Relation

Analysing the disturbance into normal modes, we assume that the perturbation quantities are of the form

$$[w,\theta,\gamma,h_z,\zeta,\xi] = [W(z),\Theta(z),\Gamma(z),K(z),Z(z),X(z)]\exp(\mathrm{i}k_x x+\mathrm{i}k_y y+nt), \tag{8}$$

where $\zeta = \partial v/\partial x-\partial u/\partial y$ and $\xi = \partial h_y/\partial x-\partial h_x/\partial y$ denote respectively the z components of vorticity and current density, k_x and k_y are the wave numbers in the x and y directions $\left(k = (k_x^2+k_y^2)^{\frac{1}{2}}\right.$ is the resultant wave number$\left.\right)$ and n is the growth rate. Equations (5) and (6) give

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2 w = g\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\alpha\theta-\alpha'\gamma)+\frac{\mu_e H}{\rho_0}\frac{\partial}{\partial z}\nabla^2 h_z, \tag{9a}$$

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\zeta = \frac{\mu_e H}{\rho_0}\frac{\partial\xi}{\partial z}, \tag{9b}$$

$$\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\theta = \beta w, \tag{9c}$$

$$\left(\frac{\partial}{\partial t}-\kappa'\nabla^2\right)\gamma = \beta' w, \tag{9d}$$

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)\xi = H\frac{\partial\zeta}{\partial z}+\frac{H}{Ne}\frac{\partial}{\partial z}\nabla^2 h_z, \tag{10a}$$

$$\left(\frac{\partial}{\partial t}-\eta\nabla^2\right)h_z = H\frac{\partial w}{\partial z}-\frac{H}{Ne}\frac{\partial\xi}{\partial z}. \tag{10b}$$

Using expression (8), equations (9) and (10) in non-dimensional form become

$$(\mathrm{D}^2-a^2)(\mathrm{D}^2-a^2-\sigma)W-(gd^2/\nu)a^2(\alpha\Theta-\alpha'\Gamma)$$
$$+(\mu_e Hd/\rho_0\nu)(\mathrm{D}^2-a^2)\mathrm{D}K = 0, \tag{11a}$$

$$(\mathrm{D}^2-a^2-\sigma)Z = -(\mu_e Hd/\rho_0\nu)\mathrm{D}X, \tag{11b}$$

$$(\mathrm{D}^2-a^2-p_2\sigma)X = -(Hd/\eta)\mathrm{D}Z-(H/Ne\eta d)(\mathrm{D}^2-a^2)\mathrm{D}K, \tag{11c}$$

$$(\mathrm{D}^2-a^2-p_2\sigma)K = -(Hd/\eta)\mathrm{D}W+(Hd/Ne\eta)\mathrm{D}X, \tag{11d}$$

$$(\mathrm{D}^2-a^2-p_1\sigma)\Theta = -(\beta d^2/\kappa)W, \tag{11e}$$

$$(\mathrm{D}^2-a^2-q\sigma)\Gamma = -(\beta' d^2/\kappa')W, \tag{11f}$$

where $p_1 = \nu/\kappa$ is the Prandtl number, $q = \nu/\kappa'$ is the Schmidt number, $p_2 = \nu/\eta$, $a = kd$ and $\sigma = nd^2/\nu$. We have also put the coordinates x, y, z in the new unit of length d and written $\mathrm{D} = \mathrm{d}/\mathrm{d}z$.

Eliminating K, Z, X, Θ and Γ from equations (11), we get

$$\begin{aligned}
&(\mathrm{D}^2-a^2)(\mathrm{D}^2-a^2-\sigma)^2(\mathrm{D}^2-a^2-p_1\sigma)(\mathrm{D}^2-a^2-p_2\sigma)^2(\mathrm{D}^2-a^2-q\sigma)W \\
&\quad -Q\mathrm{D}^2(\mathrm{D}^2-a^2)(\mathrm{D}^2-a^2-p_1\sigma)(\mathrm{D}^2-a^2-q\sigma)\{2(\mathrm{D}^2-a^2-\sigma) \\
&\quad \times(\mathrm{D}^2-a^2-p_2\sigma)-Q\mathrm{D}^2\}W+M\mathrm{D}^2(\mathrm{D}^2-a^2)^2(\mathrm{D}^2-a^2-\sigma)^2(\mathrm{D}^2-a^2-p_1\sigma) \\
&\quad \times(\mathrm{D}^2-a^2-q\sigma)W+\{Ra^2(\mathrm{D}^2-a^2-q\sigma)-Sa^2(\mathrm{D}^2-a^2-p_1\sigma)\}\{(\mathrm{D}^2-a^2-p_2\sigma)^2 \\
&\quad \times(\mathrm{D}^2-a^2-\sigma)-Q\mathrm{D}^2(\mathrm{D}^2-a^2-p_2\sigma)+M\mathrm{D}^2(\mathrm{D}^2-a^2)(\mathrm{D}^2-a^2-\sigma)\}W = 0,
\end{aligned} \tag{12}$$

where $R = g\alpha\beta d^4/\nu\kappa$ is the thermal Rayleigh number, $S = g\alpha'\beta' d^4/\nu\kappa'$ is the analogous solute Rayleigh number, $Q = \mu_e H^2 d^2/\rho_0 \nu\eta$ is the Chandrasekhar number and $M = (H/Ne\eta)^2$ is a non-dimensional number accounting for the Hall currents.

The boundary conditions (7) transform to

$$W = \mathrm{D}^2 W = \Theta = \Gamma = X = \mathrm{D}Z = 0, \tag{13}$$

and the components of $\boldsymbol{h}$ are continuous. Because the components of the magnetic field are continuous and the tangential components are zero outside the fluid then

$$\mathrm{D}K = 0 \tag{14}$$

on the boundaries. With the boundary conditions (13) and (14), it can be shown that all the even order derivatives of W must vanish for $z = 0$ and 1, and hence the proper solution of (12) characterizing the lowest mode is

$$W = A\sin\pi z, \tag{15}$$

where A is a constant. Substituting (15) in equation (12), we obtain the dispersion relation

$$\begin{aligned}
R_1 x = {} & [(1+x)(1+x+\sigma')^2(1+x+p_1\sigma')(1+x+p_2\sigma')^2+Q_1(1+x)(1+x+p_1\sigma') \\
& \times\{2(1+x+\sigma')(1+x+p_2\sigma')+Q_1\}+M(1+x)^2(1+x+\sigma')^2(1+x+p_1\sigma')] \\
& /[(1+x+p_2\sigma')\{(1+x+p_2\sigma')(1+x+\sigma')+Q_1\}+M(1+x)(1+x+\sigma')] \\
& +S_1 x(1+x+p_1\sigma')/(1+x+q\sigma'),
\end{aligned} \tag{16}$$

where $R_1 = R/\pi^4$, $S_1 = S/\pi^4$, $Q_1 = Q/\pi^2$, $x = a^2/\pi^2$ and $\sigma' = \sigma/\pi^2$.

4. Stationary Convection

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma' = \sigma = 0$, in which case equation (16) reduces to

$$R_1 = \frac{1+x}{x}\frac{[\{(1+x)^2+Q_1\}^2+M(1+x)^3]}{(1+x)^2+Q_1+M(1+x)}+S_1\,. \tag{17}$$

In order to investigate the effect of Hall currents and stable solute gradient, we examine the behaviour of $\mathrm{d}R_1/\mathrm{d}M$ and $\mathrm{d}R_1/\mathrm{d}S_1$ analytically. Equation (17) yields

$$\frac{\mathrm{d}R_1}{\mathrm{d}M} = -\frac{1+x}{x}\frac{Q_1(1+x)\{(1+x)^2+Q_1\}}{\{(1+x)^2+Q_1+M(1+x)\}^2}, \tag{18}$$

which is negative, and the Hall currents have, therefore, a destabilizing effect on the thermosolutal convection. Equation (17) also yields

$$\mathrm{d}R_1/\mathrm{d}S_1 = +1\,, \tag{19}$$

which implies that the stable solute gradient has a stabilizing effect on the system.

5. Possibility of Overstable Convection

Here we discuss the possibility of whether instability may occur as overstability, i.e. as oscillations of increasing amplitude. Once again we shall restrict ourselves to the case of two free boundaries. We put $\sigma' = \sigma/\pi^2 = \mathrm{i}\sigma_1$ in equation (16), remembering that σ may be complex. Since for overstability we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillation, it suffices to find conditions for which (16) will admit of solutions with σ_1 real. By equating real and imaginary parts of (16) and eliminating R_1 between them, we obtain

$$A_1'\,c^4+B_1'\,c^3+C_1'\,c^2+D_1'\,c+E_1' = 0\,, \tag{20}$$

where we have put $c = \sigma_1^2$ and (with $b = 1+x$)

$$A_1' = bp_2^2\{4p_1\,q(1-p_2)+p_2(p_1+q)\}\,, \tag{21a}$$

$$\begin{aligned} E_1' = {} & (p_1+q)b^9+2M(p_1+q)b^8+[Q_1\{3(p_1+q)-p_2-1\}+M^2(p_1+q)]b^7 \\ & +Q_1\,M[3(p_1+q)+p_2-1]b^6+S_1(p_1-1)(b^7-b^6)+Q_1^2[3(p_1+q) \\ & -2(p_2+1)]b^5+2MS_1(p_1-1)(b^6-b^5)+MQ_1^2(p_1+q-2)b^4 \\ & +(M^2S_1+2S_1\,Q_1)(p_1-1)(b^5-b^4)+Q_1^3(p_1+q-p_2-1)b^3 \\ & +2MS_1\,Q_1(p_1-1)(b^4-b^3)+S_1\,Q_1^2(p_1-1)\,. \end{aligned} \tag{21b}$$

As σ_1 is real for overstability, the four values of c have to be positive. Now the product of the roots of (20) is E_1'/A_1', which is positive if $E_1' > 0$ and $p_2 < 1$ (since from (21a) $A_1' > 0$ for $p_2 < 1$). Equation (21b) shows that E_1' is always positive if $p_1 > 1$ and $q > 1$. Thus $p_1 > 1$, $q > 1$ and $p_2 < 1$, i.e. $\kappa < \nu < \eta$ and $\kappa' < \nu$, are the necessary conditions for the existence of overstability.

6. Discussion

To study the convective stability of a hydrostatic radiative model, one considers the behaviour of a portion of gas which is slightly displaced from equilibrium. If the portion has a lower density than its new surroundings, after an outward displacement, it continues to rise and so the equilibrium is unstable. Since any heat conduction will tend to equilibrate the density, the relative density change in the portion is at most that resulting from an adiabatic displacement. One may assume therefore that the density change is proportional to the adiabatic density gradient $(\mathrm{d}\rho/\mathrm{d}r)_{\mathrm{ad}}$. Since density decreases outward and the change in ambient density is proportional to $\mathrm{d}\rho/\mathrm{d}r$, one may conclude that the equilibrium is stable if

$$(\mathrm{d}\rho/\mathrm{d}r)-(\mathrm{d}\rho/\mathrm{d}r)_{\mathrm{ad}} < 0. \tag{22}$$

If the chemical composition is constant, and if motions are subsonic so that pressure equilibrates instantaneously, we obtain the well-known (K) Schwarzschild criterion in terms of temperature gradients, as used by Schwarzschild and Härm (1958),

$$(\mathrm{d}T/\mathrm{d}r)-(\mathrm{d}T/\mathrm{d}r)_{\mathrm{ad}} > 0. \tag{23}$$

When the chemical composition varies, as in semiconvection, the variation in density has a contribution from the variation of molecular weight μ, so that the stability criterion (22) becomes

$$\frac{1}{T}\left\{\frac{\mathrm{d}T}{\mathrm{d}r}-\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}}\right\}-\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}r} > 0. \tag{24}$$

This criterion was used by Ledoux (1947). Kato (1966) has objected to the validity of this criterion, a view which has received support from recent work in oceanography. On the whole the oceanographic evidence supports the Schwarzschild–Härm prescription.

We now turn our attention to the Veronis (1965) thermohaline configuration in which the total density decreases upwards but water at the top is relatively cool and fresh. It has been shown by Veronis and experimentally demonstrated by Shirtcliffe (1967) that even though infinitesimal adiabatic perturbations may be stable, there are overstable modes when the effects of heat conduction are included. Turner (1968) performed an experiment in which a stable salinity gradient in isothermal water was heated from below. Within the principal convecting layer the water was observed to be isothermal and isohaline, indicating that the fluid had mixed to the point of neutrality. This situation is neutral according to the Schwarzschild criterion, since $(\mathrm{d}T/\mathrm{d}r)_{\mathrm{ad}} \approx 0$ in water.

The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. In close analogy with the above theoretical and experimental findings for the thermohaline configuration, the stable solute gradient has a stabilizing effect and overstable modes can occur. If the oscillations just outside the core amplify, excursions of gas into the core seem possible and the hydrogen-rich portion is quickly replaced by a helium-rich one. Ledoux (1947) also considered a situation in which a helium gradient arose from accretion of interstellar hydrogen, as suggested by Hoyle and Lyttleton.

In summary, the Hall effect, which is likely to be important in atmospheric physics and astrophysics, especially in the ionosphere and the outer layers of the solar atmosphere, has a destabilizing effect on thermosolutal convection, similar to its role in the problem of thermal instability without solute gradient (Gupta 1967).

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References

Chandrasekhar, S. (1961). 'Hydrodynamic and Hydromagnetic Stability' (Clarendon: Oxford).
Gupta, A. S. (1967). *Rev. Roum. Math. Pures Appl.* **12**, 665.
Kato, S. (1966). *Publ. Astron. Soc. Jpn* **18**, 374.
Ledoux, P. (1947). *Astrophys. J.* **105**, 305.
Nield, D. A. (1967). *J. Fluid Mech.* **29**, 545.
Schwarzschild, M., and Härm, R. (1958). *Astrophys. J.* **128**, 348.
Sharma, R. C., and Sharma, K. C. (1978). *Aust. J. Phys.* **31**, 181.
Shirtcliffe, T. G. L. (1967). *Nature* **213**, 489.
Spiegel, E. A. (1965). *Astrophys. J.* **141**, 1068.
Spiegel, E. A. (1969). *Comments Astrophys. Space Sci.* **1**, 57.
Turner, J. S. (1968). *J. Fluid Mech.* **33**, 183.
Veronis, G. (1965). *J. Mar. Res.* **23**, 1.

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