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Rotating Strings, Glueballs and Exotic Mesons

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Abstract

The relativistic string equation is solved for motion corresponding to rigid body rotation about the z-axis. One class of these solutions, namely the planar solutions, allows for the construction of two types of exotic meson and one type of glueball, all of which have asymptotically straight Chew–Frautschi plots. The first type of meson only exists when the number of quarks or antiquarks is 4, 5, 6 or 7, whereas no such restriction applies to the second type. For each of these cases the first type of meson is energetically more favourable than the second type for a given angular momentum.

1. Introduction

The relativistic string model was developed by Nambu (1970) and others (Hara 1971; Goto 1971; Mansouri and Nambu 1972) from dual resonance models as a phenomenological model of hadron structure. The string itself is a one dimensional continuum tracing out a minimal area in space–time. It has been interpreted further as a line of quantized colour flux (Nielsen and Olesen 1973; Tassie 1973, 1974) terminated by quarks acting as colour point charges.

Various authors (Chodos and Thorn 1974; Bars and Hanson 1976; Kikkawa and Sato 1977; Kikkawa *et al.* 1978) have extended the original Nambu string model by including point quarks at the ends of the string. Kikkawa *et al.* (1979) have developed this theme to include baryons and exotic hadrons and have determined Chew–Frautschi plots for hadrons composed of straight string segments bounded by fermionic point quarks.

In this paper we find a certain family of possible motions of the relativistic string, namely the rigidly rotating configurations. We then examine some possible hadrons which can be constructed from those solutions within the framework of the model. In particular, we consider two classes of exotic mesons and also some glueball configurations. Finally we determine the asymptotic slopes of the Chew–Frautschi plots of these hadrons in the limit of large energy and angular momentum.

A preliminary note on exotic mesons and glueballs has been given elsewhere (Burden and Tassie 1982).

2. Rigidly Rotating Strings

The motion of the relativistic string is determined by the condition that it traces out a minimal area in space–time. The action for the string is (see e.g. Kikkawa *et al.* 1978)

$$S = \frac{-1}{2\pi\alpha'} \int \{(X_\tau . X_\sigma)^2 - X_\tau^2 X_\sigma^2\}^{\frac{1}{2}} \, d\sigma \, d\tau, \tag{1}$$

leading to the Euler–Lagrange equations

$$\frac{\partial}{\partial\tau}\left(\frac{(X_\tau . X_\sigma)X_\sigma^\mu - X_\sigma^2 X_\tau^\mu}{\{(X_\tau . X_\sigma)^2 - X_\tau^2 X_\sigma^2\}^{\frac{1}{2}}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{(X_\tau . X_\sigma)X_\tau^\mu - X_\tau^2 X_\sigma^\mu}{\{(X_\tau . X_\sigma)^2 - X_\tau^2 X_\sigma^2\}^{\frac{1}{2}}}\right) = 0, \tag{2}$$

where $X^\mu = X^\mu(\tau, \sigma)$ is the world sheet traced out by the string as it moves through space–time. The subscripts τ and σ stand for $\partial/\partial\tau$ and $\partial/\partial\sigma$ respectively and μ is a Lorentz index running from 0 to 3. We use the metric $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

We shall confine ourselves to solutions of equation (2) which have constant angular velocity ω about the z-axis. Although it is well known in classical non-relativistic mechanics that a rotating rigid body with constant angular momentum does not necessarily have constant angular velocity, it is debatable whether analogous motions should be called 'rigid' in relativistic mechanics. Furthermore, such motions would correspond to excited states and would not be easily observable compared with states on the leading trajectory. We therefore omit motions which do not have constant angular velocity from the calculations.

Working in the time-like gauge $X^\mu(\tau, \sigma) = (\tau, \boldsymbol{X}(\tau, \sigma))$, we use for the space-like parameter σ the radial distance r from the z-axis. With cylindrical coordinates, the coordinates of the string at time τ are

$$\boldsymbol{X}(\tau, r) = (r, \theta(r) + \omega\tau, z(r)), \tag{3}$$

where $\theta(r)$ and $z(r)$ are the initial azimuthal and axial coordinates of the string. We have

$$\boldsymbol{X}_r = \hat{\boldsymbol{r}} + r\phi\hat{\boldsymbol{\theta}} + \zeta\hat{\boldsymbol{k}}, \qquad \boldsymbol{X}_\tau = r\omega\hat{\boldsymbol{\theta}}, \tag{4a, b}$$

where

$$\phi(r) = \mathrm{d}\theta/\mathrm{d}r, \qquad \zeta(r) = \mathrm{d}z/\mathrm{d}r. \tag{5a, b}$$

The four components of the string equation (2) can then be written

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\omega r^2 \phi}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}\right) = 0, \tag{6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1-\omega^2 r^2}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}\right) = \frac{-\omega^2 r(1+\zeta^2)+r\phi^2}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}, \tag{6b}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r\phi}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}\right) = \frac{-\phi}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}, \tag{6c}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{(1-\omega^2 r^2)\zeta}{\{(1+\zeta^2)(1-r^2\omega^2)+r^2\phi^2\}^{\frac{1}{2}}}\right) = 0. \tag{6d}$$

Alternatively, equations (6) can be obtained by noting that the invariant area element of world sheet is given by

$$\mathrm{d}\mathscr{A} = \{\mathrm{d}r^2+(1-\omega^2r^2)^{-1}r^2\mathrm{d}\theta^2+\mathrm{d}z^2\}^{\frac{1}{2}}(1-\omega^2r^2)^{\frac{1}{2}}\mathrm{d}t, \tag{7}$$

and hence the area of the world sheet traced out by the string is

$$\int \mathrm{d}\mathscr{A} = \int \{(1-\omega^2r^2)(1+\zeta^2)+r^2\phi^2\}^{\frac{1}{2}}\,\mathrm{d}r\,\mathrm{d}t. \tag{8}$$

Minimizing (8) with respect to z and θ gives equations (6a) and (6d). The remaining equations, (6b) and (6c), can be derived from these.

There exists a two-parameter family of solutions to equations (6) given by

$$\zeta = \frac{A\lambda r}{\{\lambda^2r^2(1-A^2-\omega^2r^2)-A^2(1-\omega^2r^2)\}^{\frac{1}{2}}}, \tag{9}$$

$$\phi = \frac{A(1-\omega^2r^2)}{r\{\lambda^2r^2(1-A^2-\omega^2r^2)-A^2(1-\omega^2r^2)\}^{\frac{1}{2}}}, \tag{10}$$

where A and λ are arbitrary real constants restricted in their allowable values by the constraint that ζ and ϕ should be real. It is sufficient to consider only the cases $A>0, \lambda>0$ since changing the sign of one or both of these merely changes the sign of ζ or ϕ, which physically corresponds to a mirror image of the string. The term in braces in the denominators of (9) and (10) will be positive provided

$$|A\omega-\lambda| \geqslant \lambda A, \tag{11}$$

that is, provided λ and A lie in the shaded area in Fig. 1.

It turns out that $\omega r \lesseqgtr 1$ according as to whether $A\omega \lesseqgtr \lambda$, that is, solutions lying below the line $A\omega = \lambda$ are tachyonic and those above the line $A\omega = \lambda$ are 'physical' tardyonic solutions. From here on we ignore the tachyonic solutions.

Equation (9) integrates to give

$$z-\text{const} = \int \frac{A\lambda r\,\mathrm{d}r}{\{\lambda^2r^2(1-A^2-\omega^2r^2)-A^2(1-\omega^2r^2)\}^{\frac{1}{2}}}$$

$$= \int \frac{Ar\,\mathrm{d}r}{\omega\{(r_2^2-r^2)(r^2-r_1^2)\}^{\frac{1}{2}}} = -\frac{A}{2\omega}\arccos\left(\frac{r_1^2+r_2^2-2r^2}{r_2^2-r_1^2}\right), \tag{12}$$

where

$$r_{1,2}^2 = (2\omega^2\lambda^2)^{-1}(\{(1-A^2)\lambda^2+A^2\omega^2\}\mp[\{(1-A^2)\lambda^2+A^2\omega^2\}^2-4A^2\omega^2\lambda^2]^{\frac{1}{2}}) \tag{13}$$

are the roots of the quadratic in r^2 in the denominator of the integrand. Rearranging (12) we have

$$r^2 = r_1^2\sin^2\{\omega A^{-1}(z-\text{const})\}+r_2^2\cos^2\{\omega A^{-1}(z-\text{const})\}, \tag{14}$$

so the string lies between two concentric cylinders of radii r_1 and r_2.

The θ dependence is, from equation (10),

$$\theta - \text{const} = \int \frac{A(1-\omega^2 r^2)\,\mathrm{d}r}{r\{\lambda^2 r^2(1-A^2-\omega^2 r^2) - A^2(1-\omega^2 r^2)\}^{\frac{1}{2}}}$$

$$= \frac{A}{\lambda\omega}\int \frac{r\,\mathrm{d}r}{r^2\{(r_2^2-r^2)(r^2-r_1^2)\}^{\frac{1}{2}}} - \frac{A\omega}{\lambda}\int \frac{r\,\mathrm{d}r}{\{(r_2^2-r^2)(r^2-r_1^2)\}^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\arcsin\left(\frac{r_1^2+r_2^2}{r_2^2-r_1^2} - \frac{2r_1^2 r_2^2}{r^2(r_2^2-r_1^2)}\right) + \frac{A\omega}{2\lambda}\arcsin\left(\frac{r_1^2+r_2^2-2r^2}{r_2^2-r_1^2}\right). \qquad (15)$$

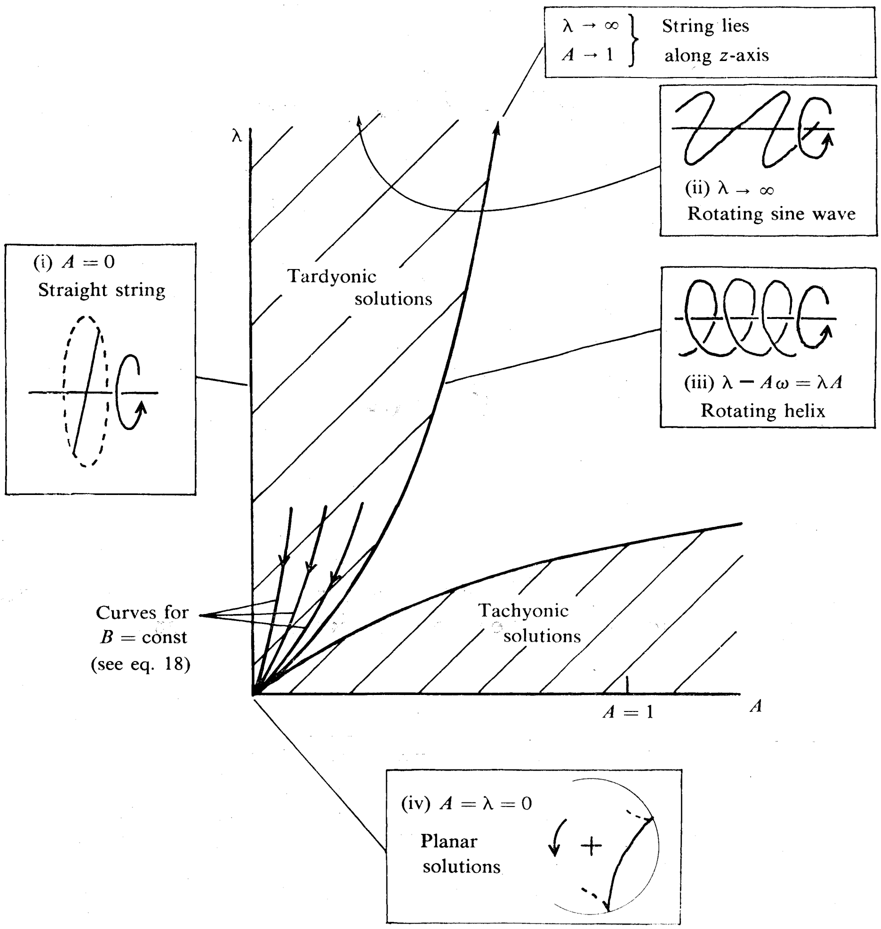

Fig. 1. Parameter space of the rigid rotator string solutions and some specific solutions.

There are certain limiting cases of the above solutions which are easy to interpret. We list them below:

(i) If A tends to zero we have $\phi = \zeta = 0$, giving the usual rotating straight radial string.

(ii) If $\lambda \to \infty$, then $r_1 \to 0$ and $r_2 \to (1-A^2)^{\frac{1}{2}}\omega^{-1}$ giving the solution

$$r = (1-A^2)^{\frac{1}{2}}\omega^{-1}\cos\{\omega A^{-1}(z-\text{const})\}, \qquad \theta = \text{const}, \qquad (16\text{a, b})$$

so the string is in the shape of a sine wave and lies in a rotating plane.

(iii) If $\lambda - A\omega = \lambda A$ then $r_1 = r_2 = (1-A^2)^{\frac{1}{2}}\omega^{-1}$ and the string is the shape of a helix making an angle of

$$\arctan(r^{-1}\,\mathrm{d}z/\mathrm{d}\theta) = \arctan(\zeta/r\phi) = \arctan(1-A^2)^{\frac{1}{2}}, \tag{17}$$

with a plane arranged perpendicular to the z-axis.

(iv) The point $A = 0, \lambda = 0$ in Fig. 1 conceals a one-parameter family of solutions obtainable by approaching the point along the curves shown:

$$B\lambda - A\omega = \lambda A, \qquad 0 \leqslant B \leqslant 1. \tag{18}$$

The region of physical solutions in the $A\lambda$ plane is mapped into the region $0 < B < 1$, $\lambda > 0$ in the $B\lambda$ plane with the point $(A, \lambda) = (0,0)$ now represented by the section $0 < B < 1$ of the B-axis,

Writing the solutions (9) and (10) in terms of the parameters B and λ and taking the limit $\lambda \to 0$ gives the new solutions

$$\mathrm{d}z/\mathrm{d}r = \zeta = 0, \qquad \mathrm{d}\theta/\mathrm{d}r = \phi = B(1-\omega^2 r^2)^{\frac{1}{2}}/r(\omega^2 r^2 - B^2)^{\frac{1}{2}}. \tag{19a, b}$$

Equation (19a) says that the curve lies in the plane $z = \text{const}$, while (19b) integrates to give

$$\theta - \text{const} = \tfrac{1}{2}\arcsin\left(\frac{(1+B^2)\omega^2 r^2 - 2B^2}{(1-B^2)\omega^2 r^2}\right) + \tfrac{1}{2}B\arcsin\left(\frac{1+B^2-2\omega^2 r^2}{1-B^2}\right). \tag{20}$$

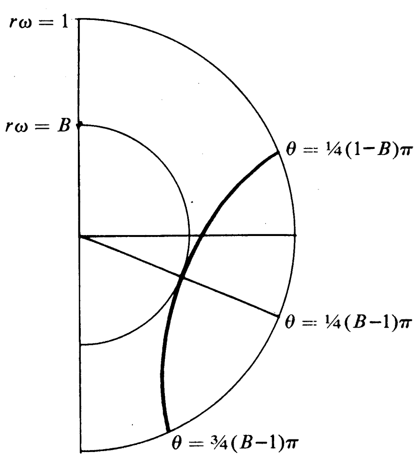

Fig. 2. Planar string solution equation (20) extended by reflection in the line $\theta = \frac{1}{4}(B-1)\pi$.

From (19b) we see that $\mathrm{d}\theta/\mathrm{d}r$ is zero at $\omega r = 1$ and infinite at $\omega r = B$, and also that $B \leqslant \omega r \leqslant 1$. The corresponding limits on θ are $\frac{1}{4}(B-1)\pi \leqslant \theta \leqslant \frac{1}{4}(1-B)\pi$. The shape of the string is shown in Fig. 2. Since replacing ϕ by $-\phi$ gives a legitimate solution to the equations (6) the curve in Fig. 2 has been analytically continued in the line $\theta = \frac{1}{4}(B-1)\pi$. In the limit $B \to 0$ the string becomes the familiar straight string. In the limit $B \to 1$ the string shrinks to a point moving in a circular orbit at the speed of light.

We note that the curves (20) are not the same as the geodesics on a rotating disc (Arzeliés 1966). The difference lies in the factor $(1-\omega^2r^2)^{\frac{1}{2}}\mathrm{d}t$ in (7) which takes care of time dilation in the invariant area element at a radius r from the centre.

The solutions (i) to (iv) are shown schematically in Fig. 1.

3. String Hadrons

For the classical action for the quark string model of a hadron consisting of N strings and I quarks we take (Kikkawa *et al.* 1979)

$$S = \sum_{\kappa=1}^{N} \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \mathscr{L}_{\mathrm{st},\kappa} + \sum_{i=1}^{I} \int L_{\mathrm{q},i}\, \mathrm{d}\tau\,, \tag{21}$$

where $\mathscr{L}_{\mathrm{st},\kappa} = \mathscr{L}_{\mathrm{st},\kappa}(X_{\kappa\tau}, X_{\kappa\sigma})$ is the string Lagrangian density in equation (1) and $L_{\mathrm{q},i}(X_{i\tau}, \psi_i, \psi_{i\tau})$ is the Lagrangian for the ith quark. Strings terminate at a quark or at a string node in configurations consistent with the colour flux conservation law of SU(3), namely that colour flux is conserved modulo 3.

The action (21) leads to the string equation (2) for each string segment and a wave equation for each quark, together with the conditions

$$\mathscr{T}^{\kappa}_{(\mu)} = \mathrm{d}p_{i\mu}/\mathrm{d}\tau\,, \tag{22}$$

where the κth string meets the ith quark and

$$\sum_{\kappa} \mathscr{T}^{(\kappa)}_{\mu} = 0\,, \tag{23}$$

where strings meet at a node. In (22) and (23) we have defined the tension in the κth string by

$$\mathscr{T}^{(\kappa)}_{\mu} = \mp \partial \mathscr{L}_{\mathrm{st},\kappa}/\partial X^{\mu}_{\kappa\sigma}\,, \tag{24}$$

where the minus sign is to be taken if σ increases away from the node and the plus sign if σ decreases. We have also defined

$$p^{\mu}_{i} = \partial L_{\mathrm{q},i}/\partial X_{i\tau\mu} \tag{25}$$

as the momentum of the ith quark conjugate to its coordinate X^{μ}_{i}.

We now try to construct rigidly rotating classical hadrons from this model using the rigid rotator solutions to the string equation found in Section 2. Any such hadrons will contain quarks or antiquarks executing uniform circular motion about the z-axis. For the fermionic quarks modelled by Bars and Hanson (1976) and Kikkawa and Sato (1977) in uniform circular motion the rate of change of momentum given by the right-hand side of (22) is directed along a radial line through the z-axis and has no time component. The tension in the string or strings emanating from such a quark must match this direction.

Substituting the general rigid rotator solution (equations 9 and 10) into the definition (24) of the tension in the string gives, with the help of equations (4),

$$\mathscr{T} = \pm(A/2\pi\alpha')\mathrm{sgn}(\lambda)\{\zeta^{-1}\hat{\boldsymbol{r}} + (r\lambda)^{-1}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{k}}\}\,, \tag{26a}$$

$$\mathscr{T}_0 = \mp A\omega/2\pi\alpha'|\lambda|\,, \tag{26b}$$

where the upper sign applies if r increases moving away from the quark and the lower sign if r decreases, and $\mathscr{T}_\mu = (\mathscr{T}_0, \mathscr{T})$. With the help of (9) and (10) we see that the string never supports a compressive force and that the z component of tension is constant along the string.

For the cases when $A = \lambda = 0$ a similar calculation using the planar solution (19) gives the tension as

$$\mathscr{T} = \pm\frac{1}{2\pi\alpha'}\left(\frac{(1-\omega^2r^2)^{\frac{1}{2}}(\omega^2r^2-B^2)^{\frac{1}{2}}}{\omega r}\hat{\boldsymbol{r}} + \frac{B}{\omega r}\hat{\boldsymbol{\theta}}\right), \tag{27a}$$

$$\mathscr{T}_0 = \mp B/2\pi\alpha'; \qquad \tfrac{1}{4}(B-1)\pi < \theta < \tfrac{1}{4}(1-B)\pi. \tag{27b}$$

For the other half of the string segment, namely $\frac{3}{4}(B-1)\pi < \theta < \frac{1}{4}(B-1)\pi$ in Fig. 2, the θ and time components change sign.

From equations (26) and (27) we see that if a single string emanates from a quark the string tension only acts in the required direction when $B = 0$, that is, when the string is radial. Suppose then that this radial string bifurcates into strings with tensions $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ respectively. The $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{\theta}}$ and time components of $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ must be equal and opposite, which can only happen if the two strings correspond to the same value of λ but opposite values of A. Unless A is zero each of these strings will extend along the z-axis until another node is reached. But at any such node there will be at least one more string extending further along the z-axis to balance the $\hat{\boldsymbol{k}}$ component of tension. Thus the hadron will extend indefinitely along the z-axis, or possibly approach some limiting point after an infinite number of nodes. We reject this last possibility as a serious model of hadron structure, such a state being too massive to be easily observable.

Table 1. Asymptotic Chew–Frautschi slopes for exotic mesons of two types

Number of quarks n	Position of junction x	Asymptotic slopes $\mathrm{d}J/\mathrm{d}E^2$ [A] Fig. 3*a* mesons	Fig. 3*b* mesons
4	0·727	0·292	0·285
5	0·878	0·279	0·278
6	0·949	0·273	0·273
7	0·988	0·269	0·269

[A] In units of α'.

On the other hand, if A is zero the string can bifurcate into two strings of the type described by equation (20). Each of the two strings must have the same value for the parameter B. Balancing the radial components of tension at the junction gives

$$4B^2 = 3x^2, \tag{28}$$

where the string junction is at a radius given by $\omega r = x$.

Using junctions such as these one can construct exotic mesons of the type shown in Fig. 3*a*. If the meson is constructed from n quarks and n antiquarks, the angle between adjacent quarks is π/n. This places a junction at $\theta = \frac{1}{4}(B-1)\pi + \frac{1}{2}\pi/n$, so from (20) we have

$$\tfrac{1}{4}(B-1)\pi+\tfrac{1}{2}\pi/n = \tfrac{1}{2}\arcsin\left(\frac{(1+B^2)x^2-2B^2}{(1-B^2)x^2}\right)+\tfrac{1}{2}B\arcsin\left(\frac{1+B^2-2x^2}{1-B^2}\right). \tag{29}$$

From (28) and (29) we have the position of the junction determined by the transcendental equation

$$x = \frac{2}{\sqrt{3}}\left\{\arcsin\left(\frac{3x^2-2}{4-3x^2}\right)+\tfrac{1}{2}\pi-\pi/n\right\}\Big/\arccos\left(\frac{4-5x^2}{4-3x^2}\right). \tag{30}$$

This equation only has solutions in the range $0 < x < 1$ for $n = 4$, 5, 6 or 7. These solutions are listed in Table 1. For $n = 3$ we have the solution $x = 0$ and the meson assumes a configuration in which six straight radial arms meet at a point. It is easy to see that this should be the case: the infinitesimal hexagon at the centre implicit in this solution will have three strings meeting at equal angles at each of its vertices. In practice however there seems to be nothing to prevent this configuration from becoming three single straight string mesons.

A further set of exotic mesons can be constructed if one allows two strings to emanate from a quark or antiquark, the simplest non-trivial example of which is shown in Fig. 3*b*. In order that there be no net θ or time component of string tension acting on each quark, each string shares the same value for the parameter B. The exact configuration is determined by balancing the centrifugal force of each quark with twice the radial component of tension in (27a).

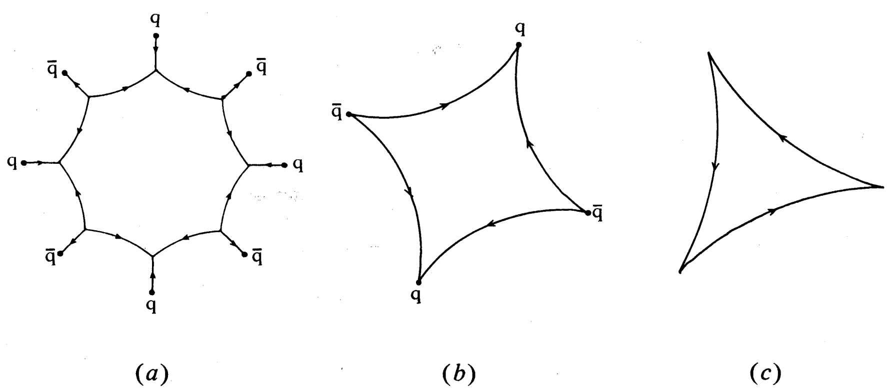

Fig. 3. Exotic mesons (*a*) and (*b*) (q and q̄ are quarks and antiquarks) and glueballs (*c*) constructed from rigidly rotating string segments. The arrows indicate the direction of colour flux.

We also note the existence of glueball solutions such as that shown in Fig. 3*c*. The cusps move at the speed of light, and to balance tensions at the cusps we see from (27) that each curved segment of string must share the same value of the parameter B.

There is a well known result arising from the action (1) that free ends of the string must move at the velocity of light. However, we have just seen from our glueball solutions that there exist points in these strings which move at the speed of light, but these are not free ends and furthermore they cannot be free ends because the string tension is non-zero at these points.

4. Asymptotic Slopes of Chew–Frautschi Plots

At large values of energy and angular momentum the contributions due to massive quarks in quark–string hadrons become negligible compared with the string contributions. For the string hadrons discussed in Section 3 we can determine the asymptotic slope of the Chew–Frautschi plots at large energies by taking the quark masses to be zero. In this limit the string ends which terminate at quarks move at the speed of light.

We first calculate the energy and angular momentum of a segment of planar string described by (20). The energy density is given by

$$\begin{aligned}\mathscr{E} &= X_\tau . \partial\mathscr{L}_{\mathrm{st}}/\partial X_\tau - \mathscr{L}_{st} \\ &= \frac{1}{2\pi\alpha'}\frac{1+\phi^2r^2}{\{\omega^2r^4\phi^2+(1-\omega^2r^2)(1+\phi^2r^2)\}^{\frac{1}{2}}} \\ &= \frac{1}{2\pi\alpha'}\frac{\omega r(1-B^2)}{(\omega^2r^2-B^2)^{\frac{1}{2}}(1-\omega^2r^2)^{\frac{1}{2}}},\end{aligned} \tag{31}$$

using (1) and (19). This integrates to give, for a segment of string $r_1 < r < r_2$,

$$E(r_1\to r_2) = \frac{B^2-1}{4\pi\alpha'\omega}\arcsin\left(\frac{1+B^2-2\omega^2r^2}{1-B^2}\right)\Bigg|_{r_1}^{r_2}. \tag{32}$$

The only non-zero component of angular momentum density is

$$\begin{aligned}\mathscr{J}_z &= (X\times\partial\mathscr{L}_{\mathrm{st}}/\partial X_\tau)_z \\ &= \frac{1}{2\pi\alpha'}\frac{\omega r^2}{\{\omega^2r^4\phi^2+(1-\omega^2r^2)(1+\phi^2r^2)\}^{\frac{1}{2}}} \\ &= \frac{1}{2\pi\alpha'}\frac{r(\omega^2r^2-B^2)^{\frac{1}{2}}}{(1-\omega^2r^2)^{\frac{1}{2}}}.\end{aligned} \tag{33}$$

For a segment of string $r_1 < r < r_2$, equation (33) integrates to

$$\begin{aligned}J_z(r_1\to r_2) = \frac{1}{4\pi\alpha'\omega^2}\Bigg\{&\tfrac{1}{2}(B^2-1)\arcsin\left(\frac{1+B^2-2\omega^2r^2}{1-B^2}\right) \\ &-(\omega^2r^2-B^2)^{\frac{1}{2}}(1-\omega^2r^2)^{\frac{1}{2}}\Bigg\}\Bigg|_{r_1}^{r_2}.\end{aligned} \tag{34}$$

In (32) and (34), if r passes through its minimum value $r_{\min} = B/\omega$, the expressions must be evaluated in two pieces, namely, $E(r_1\to r_2) = E(r_{\min}\to r_1)+E(r_{\min}\to r_2)$, and similarly for J_z.

For the purposes of determining Chew–Frautschi plots, the mesons of the type shown in Fig. 3*b* in their asymptotic limit and the glueballs of Fig. 3*c* can be dealt with together. Consider a string configuration with N cusps separated by equal angles of $2\pi/N$. For each string segment we have

$$B = 1 - 2/N. \tag{35}$$

Using (32), (34) and (35) we obtain the total energy and angular momentum for the configuration as

$$E = 2(1-N^{-1})/\alpha'\omega, \qquad J = (1-N^{-1})/\alpha'\omega^2. \tag{36a, b}$$

Eliminating ω gives the straight Chew–Frautschi plot

$$J = \{\alpha'/4(1-N^{-1})\}E^2. \tag{37}$$

The energetically most favourable configuration is that with the greatest slope of the Chew–Frautschi plot. For the glueballs this will be the $N = 2$ case consisting of two straight strings lying along the same diameter. This configuration has been suggested previously (see e.g. Marinov 1977) and has a Regge slope $\frac{1}{2}\alpha'$. Since the directions of colour flux in the two strings are antiparallel, it is possible that the two strings will annihilate each other making the glueball extremely unstable. This is not the case for the $N = 3$ glueball shown in Fig. 3*c*. For this glueball the Regge slope is $\frac{3}{8}\alpha'$.

For the exotic mesons of the type in Fig. 3*b*, N must be even. For $N = 2$ the meson consists of a quark and an antiquark joined by two straight strings with parallel colour flux, and could be unstable against decay into the usual string meson in which the quarks are joined by a single string. The next least energetic meson of this type is the $N = 4$ configuration shown in Fig. 3*b*. For this meson the Chew–Frautschi plot has an asymptotic slope of $\frac{1}{3}\alpha'$.

A second type of exotic meson, that shown in Fig. 3*a* was considered in Section 3 and it was pointed out that the number of quarks or antiquarks is restricted to be $n = 4$, 5, 6 or 7. In Table 1 we list the values of parameter x determining the position of the junction for each of these values. We also list the asymptotic slope of the Chew–Frautschi plot for each configuration calculated from (32) and (34), taking the quark masses to be zero. For the larger values of n the straight string segments are negligible and the asymptotic slopes approximate those for mesons of the type discussed in the previous paragraph. We note also that the asymptotic slopes are larger for the Fig. 3*a* mesons indicating that their energy is lower for a given angular momentum.

5. Conclusions

We have studied some possible rigidly rotating hadrons arising from the action (21), separate from those considered previously by Kikkawa *et al.* (1979). For these hadrons we have determined classical straight line Chew–Frautschi plots for the limit of high energy and angular momentum.

This is by no means an exhaustive list of hadrons which can be constructed from the planar solutions (20), though other configurations are generally too complex to be likely candidates for genuine particles.

We have yet to address the question of classical stability for these hadrons, that is, whether changes in the solutions remain small at all times given small changes in the initial conditions. Whether the exotics we have discussed above are observable or not will depend on their quantum mechanical stability and how they decay.

One might try a quantum mechanical treatment of these exotics using WKB approximations to the Feynman path integral about the classical solutions. By investigating the classical stability of these hadrons we can determine whether such an attempt could be fruitful.

Note added in proof

Equation (20) describing the shape of the planar string solution in polar coordinates can also be written as

$$\theta - \text{const}' = \frac{1-B}{2}\arccos\left(\frac{r^2-B}{r(1-B)}\right) - \frac{1+B}{2}\arccos\left(\frac{r^2+B}{r(1+B)}\right).$$

This is the equation of a hypocycloid, that is, the locus of a point on the circumference of a circular cylinder as it rolls without slipping on the interior of a larger circular cylinder.

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