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An Algebraic Description of Perturbation Theory in Quantum Electrodynamics

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Abstract

We use an algebraic formulation of the electromagnetic field, in which various quantization procedures can be described, to discuss perturbation calculations. We show that the Feynman rules and the second order calculation of the self-energy of the electron can be developed on the basis of the Fermi method of quantization. The algebraic approach clarifies the problems in defining the vacuum and other states, which are associated with calculations in terms of field algebra operators. We demonstrate that the 'vacuum' state defined on the field algebra by Schwinger leads to incorrect results in the self-energy calculation.

1. Introduction

The quantization of the free electromagnetic field does not fit into the standard axiomatic descriptions of quantum field theory, such as the algebraic formulation given by Doplicher *et al.* (1969). While all the usual field theory properties cannot be satisfied simultaneously (Strocchi 1967), there is a certain amount of freedom in deciding which of them are to be violated. As a result, vastly different quantization procedures have been developed. Well-known examples are the Gupta–Bleuler method (Bogoliubov and Shirkov 1959) in which an indefinite metric is introduced, and the radiation gauge method (Bjorken and Drell 1965) in which explicit covariance is given up.

There is an algebraic description of the quantum theory of the electromagnetic field that provides a common framework in which the different procedures can be formulated (Carey *et al.* 1977). Starting from the field algebra $\mathscr{F}$ of the free electromagnetic field, a certain quotient algebra $\mathfrak{A}_{\mathrm{phys}}$ of a subalgebra of $\mathscr{F}$ is selected, which describes the physical degrees of freedom of the electromagnetic field. The various methods of quantization can be regarded as techniques for finding a representation of this physical algebra. In particular it is possible to find representations that correspond to the Fermi method of quantization, used in early perturbation calculations (Fermi 1932; Belinfante 1949; Schwinger 1948, 1949; Coester and Jauch 1950). In its original form, this method had normalization difficulties that are not shared by the Gupta–Bleuler and radiation gauge methods. However, once the Fermi method is re-formulated as a representation of the physical quotient algebra, it provides a rigorous quantization procedure at least for the free electromagnetic field, which can be formulated as a Weyl system. This was established by Carey *et al.* (1977). We shall show that the usual level of rigour for the justification of perturbation calculations can be achieved using a similar algebraic description of the interacting electromagnetic field. For example, the calculations of Belinfante (1949) and Schwinger

(1948, 1949) can be given a sounder mathematical basis by using the revised version of the Fermi method.

In Section 2, we summarize the formulation of the free electromagnetic field given by Carey *et al.* (1977) and discuss the algebraic description of the various quantization procedures, and of time evolution. We show that time evolution is an inner automorphism of the quotient algebra $\mathfrak{A}_{\text{phys}}$ which represents physical quantities. In Sections 3 and 4, we consider how interactions can be dealt with in the algebraic framework. Again, time evolution is an inner automorphism of the physical algebra. We examine perturbation calculations in the radiation gauge and the Lorentz gauge. In the radiation gauge description of the electromagnetic field, physical and non-physical components can be written separately. In the algebraic formulation, the advantages of this separation appear as follows. First, the physical algebra can be described in terms of physical components without reference to other field variables. Also the vacuum has a simple definition as the state which is annihilated by the positive frequency components of the physical variables. The definition of the vacuum does not make any reference to unphysical components of the field. It is completely defined without reference to these components because it is a vector in the representation space of $\mathfrak{A}_{\text{phys}}$ and not the field algebra. This is how the algebraic approach avoids the non-normalizable vacuum pointed out by Belinfante (1949).

We demonstrate that the derivation of the Feynman rules by Coester and Jauch (1950) and the radiation gauge calculations of Schwinger and Belinfante are valid just as they are written if the physical field operators are reinterpreted as representatives of elements of the quotient algebra $\mathfrak{A}_{\text{phys}}$, rather than elements of the field algebra $\mathscr{F}$.

Finally, we shall discuss the Lorentz gauge formulation of perturbation calculations. This gauge is often chosen because it allows a unified treatment of all four components of the electromagnetic field. Thus the calculations are not only in a different gauge, but they are written in terms of elements of the field algebra rather than the physical algebra. One consequence is that four apparently independent components of the electromagnetic field are used, rather than the two components which would be available in $\mathfrak{A}_{\text{phys}}$. The vacuum should refer only to these two components, and hence there is no natural way of identifying a state on the field algebra which corresponds to the physical vacuum. More generally, states corresponding to the presence of a given number of physical particles can only be found in the representation space of the quotient algebra. Hence the calculation of expectation values makes sense only in the physical algebra.

Our main aim, then, is to use the Fermi quantization procedure to develop perturbation calculations. Certain aspects of any known mathematical basis for such calculations are ill-defined. For this reason, we are justified in ignoring various topological questions and concentrating on the algebraic structure. In particular, we shall, without further comment, refer to unbounded generators, such as the Hamiltonian, as belonging to the algebras appropriate to their commutation properties.

2. Free Electromagnetic Field

We write the classical equations of motion of the free electromagnetic field in terms of the 4-vector potential A^μ:

$$\Box A^\mu = 0, \tag{1a}$$

where A^μ is required to satisfy the supplementary condition

$$\partial_\mu A^\mu = 0, \tag{1b}$$

which leaves the choice of A^μ arbitrary up to a gauge transformation of the form

$$A^\mu \to A^\mu + \partial^\mu \lambda, \tag{2}$$

where $\Box\lambda = 0$. If the four components are quantized independently according to the canonical procedure, then the corresponding operators, again denoted by A^μ, satisfy

$$[A^\mu(x), A^\nu(y)] = -\mathrm{i}\hbar c g^{\mu\nu} D(x-y). \tag{3}$$

The algebra of the electromagnetic potential which the A_μ generate describes more degrees of freedom than are available physically: it is necessary to take account of the supplementary condition and gauge invariance. Since classically the supplementary condition has a fixed value (zero) it follows that in the quantum theory all observables will be expected to commute with it. (Operators with this property are precisely the gauge invariant ones.) Also to account for the fact that adding multiples of $\partial^\mu A_\mu$ should not alter the physical meaning of an expression, operators which differ by a multiple of the supplementary condition should be identified. These considerations can be used to formulate a definition of the physical algebra of observables. We shall briefly describe this algebraic structure (omitting a number of technical details which can be found in Carey *et al.* 1977) and then consider the various quantization methods, which can be regarded as techniques for finding representations of the physical algebra.

The Weyl algebra of the electromagnetic potential may be constructed as a Manuceau C* algebra $\Delta_c(M)$ over a complex Hilbert space M of solutions $\phi = (\phi_\mu)$ of the wave equation

$$\Box\phi_u = 0, \qquad \mu = 0,1,2,3.$$

The imaginary part of the inner product in M is given by

$$B(\phi_1, \phi_2) = -\int \mathrm{d}^3x\{\phi_1^\mu(\boldsymbol{x},0)\,\dot\phi_{2\mu}(\boldsymbol{x},0) - \dot\phi_1^\mu(\boldsymbol{x},0)\,\phi_{2\mu}(\boldsymbol{x},0)\}. \tag{4}$$

Heuristically, this algebra may be thought of as being generated by operators

$$W(\phi) = \exp\{\mathrm{i}\,A(\phi)\}, \qquad \text{for } \phi \in M, \tag{5}$$

where $A(\phi)$ is an appropriate smearing of the vector potential,

$$A(\phi) = \int A^\mu(x) f_\mu(x)\,\mathrm{d}x, \tag{6}$$

and where the convolution

$$D * f_\mu = \int D(x-y) f_\mu(y)\,\mathrm{d}y = \phi_\mu.$$

The multiplication law for the $W(\phi)$ is given by

$$W(\phi_1)\,W(\phi_2) = \exp\{\tfrac{1}{2}\mathrm{i}\,B(\phi_1,\phi_2)\}\,W(\phi_1+\phi_2). \tag{7}$$

There are two Lorentz invariant subspaces of M:

$$N = \{\phi \in M \colon \partial\phi^{\mu}/\partial x^{\mu} = 0\}, \tag{8}$$

$$T = \{\partial\lambda/\partial x^{\mu} \in M \colon \Box\lambda = 0\}. \tag{9}$$

If S is defined by

$$S = \{\phi \in M \colon \phi_0 = 0, \nabla \cdot \boldsymbol{\phi} = 0\}, \tag{10}$$

then $N = S \oplus T$ and the corresponding C* algebras satisfy

$$\Delta_c(N) = \Delta_c(S) \otimes \Delta_c(T). \tag{11}$$

Here $\Delta_c(T)$ is the centre of $\Delta_c(N)$, and $\Delta_c(N)$ is the commutant of $\Delta_c(T)$. It can be checked that the transformation

$$W(\phi) \to W(\psi)\, W(\phi)\, W(-\psi) \qquad \text{for } \psi^{\mu} = \partial^{\mu}\lambda \in T \tag{12}$$

is equivalent under the heuristic correspondence (5) and (6) to the gauge transformation (2). Hence $\Delta_c(T)$ provides the Lorentz gauge transformations. It is also the algebra of the supplementary condition: if $\psi_{\mu} = \partial_{\mu}\lambda \in T$, choose f_{μ} and g such that

$$\psi_{\mu} = D * f_{\mu}, \qquad \partial g/\partial x^{\mu} = f_{\mu}. \tag{13a, b}$$

Then $\lambda = D * g$, and we have

$$A(\psi) = \int \mathrm{d}x\, A^{\mu}(x)\, \partial g(x)/\partial x^{\mu} = \int \mathrm{d}x\, \partial_{\mu} A^{\mu}(x) g(x). \tag{14}$$

From this interpretation of $\Delta_c(T)$, it follows that $\Delta_c(N)$ must contain the gauge invariant operators, and hence all quantities of physical interest. In fact, physical quantities are described by $\Delta_c(S)$, and equation (11) shows that $\Delta_c(N)$ also describes unphysical quantities associated with the supplementary condition. There is a natural homomorphism π projecting $\Delta_c(N)$ onto $\Delta_c(S)$: since $N = S \oplus T$, any ϕ in N can be written uniquely as $\phi_s + \phi_t$, where $\phi_s \in S$ and $\phi_t \in T$. The homomorphism is determined by

$$\pi(W(\phi)) = W(\phi_s). \tag{15}$$

It can be deduced that there is an isomorphism between $\Delta_c(S)$ and the quotient algebra $\Delta_c(N)/I$, where I is the kernel of π and is thus the ideal in $\Delta_c(N)$ generated by the supplementary condition operator. The quotient $\Delta_c(N)/I$ is the physical algebra of the electromagnetic field, for which representations are provided by the various quantization procedures.

Let $\mathscr{H}$ be a Hilbert space carrying the Fock representation π_F of the algebra of the electromagnetic potential $\Delta_c(M)$. A representation appropriate for quantization in the radiation gauge is obtained by restricting π_F to $\Delta_c(S)$. This provides a representation of $\Delta_c(N)/I$ through the isomorphism (15) between $\Delta_c(S)$ and $\Delta_c(N)/I$. The definition (10) of S shows that it corresponds to the heuristic formulation in which A_0 and $\nabla \cdot A$ are set equal to zero.

In the Gupta–Bleuler method, an indefinite metric is defined on $\mathscr{H}$ in such a way that the inequality

$$\langle \Phi, \Phi \rangle \geqslant 0 \tag{16}$$

holds for exactly those states Φ which satisfy

$$(\partial_\mu A^\mu)^{(+)} \Phi = 0 . \tag{17}$$

We denote the space of such states by $\mathscr{H}'$ and denote by $\mathscr{H}''$ the subspace of states of zero norm.

Using the fact that physical states satisfy (17) and gauge invariance arguments, the following properties can be established (see e.g. Bogoliubov and Shirkov 1959, Sect. 13). The expectation value of a gauge invariant operator K (one which commutes with $\partial_\mu A^\mu$) in a state $\Phi \in \mathscr{H}'$ depends only on the transverse components of K (those corresponding to transverse photon operators in the Fock representation). Also, the expectation value of K will not be altered if an element of $\mathscr{H}''$ is added to Φ. The space $\mathscr{H}'$ is invariant under K. Hence the quotient space

$$\mathscr{H}_{\text{phys}} = \mathscr{H}' / \mathscr{H}'' \tag{18}$$

inherits a (non-faithful) representation of these operators from their representation on $\mathscr{H}$. Its kernel is the ideal I.

This means that a calculation in a representation of the physical algebra $\Delta_c(N)/I$ acting on $\mathscr{H}_{\text{phys}}$ can be written in terms of a calculation in the algebra of the electromagnetic potential acting on $\mathscr{H}$, so long as the constraint (17) is borne in mind. Thus the indefinite metric is a mechanism for defining a representation of $\Delta_c(N)/I$ in such a way that explicitly covariant expressions involving the electromagnetic potential may be used to calculate physical quantities.

The Fermi method can be formulated as a representation of $\Delta_c(N)/I$ in the following way. Consider again the Fock representation π_F of $\Delta_c(M)$ on $\mathscr{H}$. A direct integral decomposition of $\mathscr{H}$ with respect to $T^\perp$, the spectrum of the supplementary condition operator, will diagonalize any operator in $\Delta_c(N)$. We introduce the following notation for this decomposition:

$$\mathscr{H} = \int_{\oplus T^\perp} \mathrm{d}\mu(\zeta)\, \mathscr{H}_\zeta , \tag{19}$$

where μ is the measure on $T^\perp$ with characteristic function $\psi \to \exp\{-\frac{1}{4}(\psi, J_F \psi)\}$, with J_F an appropriate complex structure. For each ζ we denote the component representation of $\Delta_c(N)$ acting on $\mathscr{H}_\zeta$ by π_ζ:

$$\pi_F = \int_{\oplus T^\perp} \mathrm{d}\mu(\zeta)\, \pi_\zeta . \tag{20}$$

These representations are determined by their action on $W(\phi)$, $\phi \in N$:

$$\pi_0\big(W(\phi)\big) = \exp\Big(-\tfrac{1}{2} \int \bar{\phi}^\mu(k)\, \phi_\mu(k)\, \mathrm{d}^3k/2|k|\Big), \tag{21a}$$

$$\pi_\zeta\big(W(\phi)\big) = \exp\{\mathrm{i}\, B(\zeta, \phi)\}\, \pi_0\big(W(\phi)\big), \tag{21b}$$

and the mapping from π_0 to π_ζ is essentially a displacement:

$$A^\mu \to A^\mu - \zeta^\mu. \tag{22}$$

In the representation π_0 of $\Delta_c(N)$, the degrees of freedom are just the physical ones, since the supplementary condition operators vanish. In fact, the two-sided ideal I of the supplementary condition operators in $\Delta_c(N)$ is just $\ker \pi_0$, so that π_0 determines a representation $\tilde{\pi}_0$ of $\Delta_c(N)/I$.

The representation π_0 is distinguished from the other π_ζ by the fact that it is stable under Lorentz transformations. The corresponding cosets will be indicated by a superscript L: for C in $\Delta_c(N)$ we have

$$C + \ker \pi_0 = C^{\mathrm{L}}. \tag{23}$$

Further, π_0 acting on $\mathscr{H}_0$ is a description of the free electromagnetic field in the Lorentz gauge.

Given the above description of the physical algebra, it is surprising to find that the usual form of the Hamiltonian

$$H = \int \mathrm{d}^3 k \, \frac{1}{2|\boldsymbol{k}|} a_\mu^*(\boldsymbol{k})\, a^\mu(\boldsymbol{k}) \tag{24}$$

(where a_μ and a_μ^* are photon annihilation and creation operators) does not commute with the supplementary condition operator. However, we can demonstrate that the time evolution automorphism it generates is an inner automorphism of $\Delta_c(N)/I$ so that the noncommuting part of H has no physical significance. This discussion is expounded at greater length by Carey and Hurst (1977).

The time evolution of an operator C in $\Delta_c(N)$ is given by

$$C(t) = \exp(\mathrm{i}\, Ht)\, C \exp(-\mathrm{i}\, Ht), \tag{25}$$

and lies in $\Delta_c(N)$ for each t. Since also I is time-translation invariant, it follows that time evolution is an automorphism of $\Delta_c(N)/I$. To demonstrate that the automorphism is inner, we shall find an operator G in $\Delta_c(N)$ such that $\exp(\mathrm{i}\, G^{\mathrm{L}} t)$ is a unitary operator in $\Delta_c(N)/I$ which implements

$$C^{\mathrm{L}} \to [C(t)]^{\mathrm{L}}. \tag{26}$$

In order to do this, we introduce some notation from the heuristic formulation of the electromagnetic field. In the algebra of the vector potential, we denote the operators conjugate to A^μ by Π^μ, and let their three-dimensional Fourier transforms be q^μ and p^μ:

$$q^\mu(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \; A^\mu(\boldsymbol{x}) \exp(-\mathrm{i}\, \boldsymbol{k} . \boldsymbol{x}), \tag{27a}$$

$$p^\mu(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 k \; \Pi^\mu(\boldsymbol{x}) \exp(\mathrm{i}\, \boldsymbol{k} . \boldsymbol{x}), \tag{27b}$$

so that

$$[q^\mu(\boldsymbol{k}), p^\nu(\boldsymbol{l})] = \mathrm{i}\hbar \delta^{\mu\nu}\, \delta(\boldsymbol{k} - \boldsymbol{l}). \tag{28}$$

In terms of q^μ and p^μ, the Hamiltonian (24) for the free electromagnetic field may be written

$$H = \int \mathrm{d}^3k \, [\tfrac{1}{2}c^2 p^\mu(-\boldsymbol{k}) \, p_\mu(\boldsymbol{k}) + \tfrac{1}{2}k^2(\delta^{jl} - k^j k^l/k^2) q_j(\boldsymbol{k}) \, q_l(\boldsymbol{k}) - \mathrm{i}\, k^j c \{p_0(\boldsymbol{k}) \, q_j(\boldsymbol{k}) + p_j(\boldsymbol{k}) \, q_0(\boldsymbol{k})\}]. \tag{29}$$

Classically, the requirement that $\partial^\mu A_\mu$ should vanish for all time is equivalent to the requirement that χ and $\dot{\chi}$, given by

$$\chi(x) = \Pi_0(x), \qquad \dot{\chi}(x) = \dot{\Pi}_0(x) = \partial \Pi_j / \partial \chi_j, \tag{30a, b}$$

should vanish. We write the positive and negative frequency components of the operators corresponding to (30) in terms of momentum space operators:

$$\chi^+(\boldsymbol{k}) = \mathrm{i}\, c k^\mu p_\mu(-\boldsymbol{k}), \qquad \chi^-(\boldsymbol{k}) = -\mathrm{i}\, c k^\mu p_\mu(\boldsymbol{k}). \tag{31a, b}$$

In order to obtain an operator G with the property that $U^{\mathrm{L}}(t) \equiv \exp(\mathrm{i}\, G^{\mathrm{L}} t)$ is the unitary in $\Delta_{\mathrm{c}}(N)/I$ which implements time translations, we seek to write H in two parts:

$$H = G + K,$$

such that (a) $[K, G] = 0$, (b) $[G, \chi^\pm] = 0$ and (c) $[K, C] \in I$ (i.e. is a multiple of, and commutes with, the supplementary condition operator) for all C in $\Delta_{\mathrm{c}}(N)$.

Suppose (a), (b) and (c) hold, then

$$C(t) = \exp(\mathrm{i}\, Kt)\{\exp(\mathrm{i}\, Gt) \, C \exp(-\mathrm{i}\, Gt)\}\exp(-\mathrm{i}\, Kt) \in \{\exp(\mathrm{i}\, Gt) \, C \exp(-\mathrm{i}\, Gt)\}^{\mathrm{L}},$$

that is, we have

$$C^{\mathrm{L}}(t) \equiv [C(t)]^{\mathrm{L}} = U^{\mathrm{L}}(t) \, C^{\mathrm{L}} \, U^{\mathrm{L}}(-t),$$

as required. We construct G and K as follows: H satisfies

$$[\chi^+(\boldsymbol{k}), H] = \hbar k c \, \chi^+(\boldsymbol{k}), \qquad [\chi^-(\boldsymbol{k}), H] = -\hbar k c \, \chi^-(\boldsymbol{k}),$$

so that for (b) we require

$$[\chi^+(\boldsymbol{k}), K] = \hbar k c \, \chi^+(\boldsymbol{k}), \qquad [\chi^-(\boldsymbol{k}), K] = -\hbar k c \, \chi^-(\boldsymbol{k}).$$

This will be satisfied if

$$K = \int \mathrm{d}^3k \{\chi^+(\boldsymbol{k}) \, \xi_1(\boldsymbol{k}) + \chi^-(\boldsymbol{k}) \, \xi_2(\boldsymbol{k})\},$$

where

$$[\xi_1(\boldsymbol{k}), \chi^+(\boldsymbol{l})] = [\xi_2(\boldsymbol{k}), \chi^-(\boldsymbol{l})] = \hbar k c \, \delta(\boldsymbol{k} - \boldsymbol{l}), \tag{32a}$$

$$[\xi_1(\boldsymbol{k}), \chi^-(\boldsymbol{l})] = [\xi_2(\boldsymbol{k}), \chi^+(\boldsymbol{l})] = 0. \tag{32b}$$

Then (c) will be true if

$$[[\xi_i(\boldsymbol{k}), C], \chi^\pm] = 0 \quad \text{for all } C \text{ in } \Delta_{\mathrm{c}}(N).$$

This follows from (32) by an application of the Jacobi identity to $\chi^{\pm}$, ξ_i and C. Suitable choices for the ξ_i are

$$\xi_1(\boldsymbol{k}) = -(1/2k)k^\mu q_\mu(-\boldsymbol{k}), \qquad \xi_2(\boldsymbol{k}) = (1/2k)k^\mu q_\mu(\boldsymbol{k}), \tag{33a, b}$$

so that

$$K = \int d^3k(-\mathrm{i}\,c/2k)k^\mu k^\nu\{p_\mu(-\boldsymbol{k})\,q_\nu(-\boldsymbol{k}) + p_\mu(\boldsymbol{k})\,q_\nu(\boldsymbol{k})\}\,.$$

It follows that $G = H - K$ is in the same equivalence class as the effective free Hamiltonian

$$\tfrac{1}{2}\int d^3x\,(\boldsymbol{E}^2 + \boldsymbol{B}^2)\,.$$

3. Interaction Picture and Derivation of Feynman Rules

We consider the interaction between the electromagnetic field and the electron field described by a spinor variable ψ. The equations of motion for the interacting system are

$$\{\gamma^\mu(\partial/\partial x^\mu - \mathrm{i}\,eA_\mu) + m\}\psi = 0\,, \tag{34a}$$

$$\bar{\psi}\{\gamma^\mu(\partial/\partial x^\mu + \mathrm{i}\,eA_\mu) - m\} = 0\,, \tag{34b}$$

$$\Box A^\mu(x) = j^\mu(x)\,, \tag{34c}$$

where $j^\mu(x) = -\frac{1}{2}\mathrm{i}\,e\bar{\psi}\gamma^\mu\psi$. These equations correspond to a Hamiltonian given by

$$H = H_\mathrm{e} + H_\gamma - c^{-1}\int d^3x\; j^\mu(x)\,A_\mu(x)\,, \tag{35}$$

where H_e and H_γ are the Hamiltonians for the free electron and free photon fields respectively. The Lorentz gauge transformations are now

$$A^\mu \to A^\mu + \partial^\mu\lambda\,, \qquad \psi \to \exp(\mathrm{i}\,e\lambda/\hbar c)\psi\,, \qquad \bar{\psi} \to \exp(-\mathrm{i}\,e\lambda/\hbar c)\bar{\psi}\,, \tag{36a, b, c}$$

where $\Box\lambda = 0$, and in terms of momentum space operators, the supplementary condition operators are

$$\chi^+(-\boldsymbol{k}) = \mathrm{i}\,c\{k^\mu p_\mu(-\boldsymbol{k}) - \mathrm{i}\,c^{-2}j_0(-\boldsymbol{k})\}\,, \tag{37a}$$

$$\chi^-(\boldsymbol{k}) = -\mathrm{i}\,c\{k^\mu p_\mu(\boldsymbol{k}) + \mathrm{i}\,c^{-2}j_0(-\boldsymbol{k})\}\,. \tag{37b}$$

We write the field algebra of the interacting system as a product algebra $\mathfrak{A}_\gamma \times \mathfrak{A}_\mathrm{e}$, where $\mathfrak{A}_\gamma$ is the algebra of the electromagnetic potential and $\mathfrak{A}_\mathrm{e}$ that of the electron field. In order to identify an algebra which describes physical quantities, it will be necessary to select those elements of $\mathfrak{A}_\gamma \times \mathfrak{A}_\mathrm{e}$ which commute with $\chi^{\pm}$, and construct a quotient algebra of this commutant with respect to the ideal I generated by χ^+ and χ^-, involving both electromagnetic and electron field operators. On a purely algebraic level, it is possible to apply an automorphism to $\mathfrak{A}_\gamma \times \mathfrak{A}_\mathrm{e}$ which decouples the supplementary condition from the electron field. Formally, this automorphism may be written as (Hurst 1961)

$$a \to a^S = Sa\bar{S}\,, \tag{38a}$$

where

$$S = \exp\left(\frac{1}{\hbar c}\int \frac{j_0(-\boldsymbol{k})\,k^l q_l(\boldsymbol{k})}{k^2}\,\mathrm{d}^3k\right) \tag{38b}$$

(cf. the operator $G[\sigma]$ in Schwinger 1948). Although this looks formally like a unitary operator, the exponent is unbounded, and (38a) is really a shorthand for an automorphism which is not implemented in any representation:

$$\begin{aligned} q_\mu^S &= q_\mu\,, \qquad p_0^S = p_0\,, \\ p_l^S(\boldsymbol{k}) &= p_l(\boldsymbol{k}) + \mathrm{i}\,k_l\,j_0(-\boldsymbol{k})/c^2k^2\,, \qquad l = 1,2,3\,, \\ \psi_\sigma^S(\boldsymbol{x}) &= \exp(-\alpha)\,\psi_\sigma(\boldsymbol{x})\,, \qquad \pi_\sigma^S(\boldsymbol{x}) = \exp(\alpha)\,\pi_\sigma(\boldsymbol{x})\,, \end{aligned} \tag{39}$$

where

$$\alpha = \frac{\mathrm{i}\,e}{4\pi\hbar c}\int \frac{\operatorname{div} A(x')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}^3x'\,.$$

In terms of the transformed operators, χ^+ and χ^- are independent of the electron field:

$$\chi^+(-\boldsymbol{k}) = \mathrm{i}\,ck^\mu p_\mu^S(-\boldsymbol{k})\,, \qquad \chi^-(\boldsymbol{k}) = -\mathrm{i}\,ck^\mu p_\mu^S(\boldsymbol{k})\,. \tag{40a, b}$$

Let $\mathfrak{A}_\mathrm{e}^S$ and $\mathfrak{A}_\gamma^S$ be the algebras generated by the S-transformed electron and electromagnetic field operators respectively. Then the commutant of the supplementary condition algebra is most conveniently expressed in terms of S-transformed operators:

$$\mathscr{C} = \mathfrak{A}_\mathrm{e}^S \times \Delta_\mathrm{c}(N)^S\,,$$

where $\mathfrak{A}_\mathrm{e}^S$ is generated by ψ^S and $\bar{\psi}^S$, and $\Delta_\mathrm{c}(N)^S$ is the set of elements commuting with χ^+ and χ^- in the algebra generated by S-transformed electromagnetic operators. This set is algebraically identical to $\Delta_\mathrm{c}(N)$. Clearly, there is an isomorphism:

$$\mathscr{C}/I \cong \mathfrak{A}_\mathrm{e}^S \times \left(\Delta_\mathrm{c}(N)^S/I\right), \tag{41}$$

and this is the physical algebra $\mathfrak{A}_{\mathrm{phys}}$. Its formulation in terms of S-transformed operators clarifies the correspondence with the radiation gauge formulation, since the second term in the product (41) is isomorphic to $\Delta_\mathrm{c}(S)$ (cf. equation 15). Finding a representation of the algebra (41) (by the Gupta–Bleuler method, by the Fermi method or by representing $\mathfrak{A}_\mathrm{e} \times \Delta_\mathrm{c}(S)$ on the Fock space of the electron field and electromagnetic potential) is equivalent to quantization in the radiation gauge.

In order to show that time translation can be implemented in this algebra, we exhibit the Hamiltonian as a sum of commuting operators:

$$H = G + K\,,$$

where

$$[G, \chi^\pm] = 0\,, \qquad [K, G] = 0\,, \tag{42a}$$

$$[K, C] \in I \quad \text{for all } C \in \mathfrak{A}_\mathrm{e}^S \times \Delta_\mathrm{c}(N)^S\,. \tag{42b}$$

This decomposition of H can be achieved in the same way as for the free field. A suitable operator G is given by

$$G = H_e + G_\gamma + \int d^3k\left(-c^{-1}j_m(-\boldsymbol{k})\,q^{m\mathrm{tr}}(\boldsymbol{k}) + \frac{1}{2k^2c^2}j_0(\boldsymbol{k})j_0(-\boldsymbol{k})\right)^S, \tag{43}$$

where $q_m^{\mathrm{tr}}(\boldsymbol{k}) = (\delta_{lm} - k_l k_m/k^2)q^l(\boldsymbol{k})$, and G_γ is of the same form as the generator G found for the free case in Section 2. The coset it represents, G^{L} say, is the generator of time translations for the quotient algebra.

To set up the interaction picture, we divide up the Hamiltonian as follows:

$$G = G_0 + G_1,$$

where $G_0 = H_e + G_\gamma$, and then G_1 has the form of the usual interaction Hamiltonian in the radiation gauge. In terms of x-space operators, we have

$$G_1(t) = \int d^3x\left(-c^{-1}j_l(x,t)\,\mathscr{A}^l(x,t) + j_0(x,t)\int d^3x'\,\frac{j_0(x',t)}{8\pi c^2|\boldsymbol{x}-\boldsymbol{x}'|}\right), \tag{44}$$

where $\mathscr{A}^l$ denotes the transverse part of the electromagnetic field (cf. Bjorken and Drell 1965). The G_0 and G_1 separately commute with the supplementary condition operators.

Because G is just the usual radiation gauge Hamiltonian, the derivation of the Feynman rules will be algebraically identical to the radiation gauge proof given by Bjorken and Drell (1965). We need only add a few comments about the description in terms of the quotient algebra, and representation-dependent aspects.

Consider the representation of the radiation gauge algebra (41) obtained by the Fermi method. The usual Fock representation of electron field operators is used, and the quotient algebra $\Delta_c(N)/I$ is represented on the component $\mathscr{H}_0$ of the direct integral decomposition of the Fock space $\mathscr{H}$ given in (18). In the interaction picture, operators in the field algebra vary with time according to

$$A \to A(t) = \exp(\mathrm{i}\,G_0 t/\hbar)\,A\exp(-\mathrm{i}\,G_0 t/\hbar), \tag{45}$$

and since $\chi^\pm$ commute with G_0, this determines a time evolution for elements of the physical algebra: if A commutes with $\chi^\pm$ then

$$(A(t))^{\mathrm{L}} = A^{\mathrm{L}}(t).$$

The S matrix is an element of the physical algebra, and may be written

$$S^{\mathrm{L}} = 1 + \sum_{n=1}^{\infty}\left(\frac{-\mathrm{i}}{\hbar c}\right)^n \frac{1}{n!}\int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n\, T\big(G_1^{\mathrm{L}}(t_1)\ldots G_n^{\mathrm{L}}(t_n)\big). \tag{46}$$

The first step in evaluating S-matrix elements is to apply Wick's theorem. This involves the definition of normal ordering, the associated definition of the vacuum state Ω_0, and the commutation relations used in the definition of time-ordered vacuum expectation values. In order to give these definitions in the radiation gauge, we introduce notation which distinguishes the transverse, longitudinal and timelike components of the electromagnetic field.

In a frame where the timelike component is in the direction $n^\mu = (1,0,0,0)$, the potential A^μ may be written

$$A^0(x) = -\partial\Lambda(x)/\partial x_0, \qquad A^l(x) = \mathscr{A}^l(x) - \partial\Lambda'(x)/\partial x_l, \tag{47a, b}$$

with $\Box\Lambda = \Box\Lambda' = \Box\mathscr{A}^l = 0$, where $\mathscr{A}^l$ satisfies $\partial\mathscr{A}^l/\partial x^l = 0$, and denotes the transverse components, and $-\partial\Lambda'/\partial x_l$ is the longitudinal component (a particular case of the notation used by Schwinger).

From (3) we deduce the commutation relations

$$[\mathscr{A}_l(x), \mathscr{A}_m(x')] = -\mathrm{i}\hbar c g_{lm} D(x-x') + \mathrm{i}\hbar c \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^m}\mathscr{D}(x-x'), \tag{48}$$

where $\mathscr{D}$ is determined by

$$\Box\mathscr{D}(x) = 0, \qquad (\partial/\partial x^{02})\mathscr{D}(x) = D(x),$$

which apply both to the operators and to the equivalence classes they represent. In addition, for the longitudinal and timelike components, we have

$$[\Lambda(x), \Lambda(x')] = -[\Lambda'(x), \Lambda'(x')] = -\mathrm{i}\hbar c\mathscr{D}(x-x'), \tag{49a}$$

$$[\Lambda(x), \Lambda'(x')] = [\Lambda(x), \mathscr{A}^\mu(x')] = [\Lambda'(x), \mathscr{A}^\mu(x')] = 0, \tag{49b}$$

and this also holds for the S-transformed quantities. The vacuum vector Ω_0 is defined by

$$\bar{\psi}^{\mathrm{L}(+)}(x)\,\Omega_0 = \psi^{\mathrm{L}(+)}(x)\,\Omega_0 = 0, \qquad \mathscr{A}_l^{\mathrm{L}(+)}(x)\,\Omega_0 = 0, \tag{50a, b}$$

in the representation space of the quotient algebra (41) corresponding to the spectral value zero for the supplementary condition operator, which can be written

$$\{\Lambda(x) - \Lambda'(x)\}^S; \tag{51}$$

note that we have

$$\partial_\mu A^\mu(x) = (-\partial^2/\partial x_0^2)\{\Lambda(x) - \Lambda'(x)\}. \tag{52}$$

The definition of the vacuum is relativistically invariant, since the operators $\mathscr{A}_\mu^{(+)}(x)$ corresponding to a different choice of timelike vector n^μ lie in the same equivalence class $\mathscr{A}_\mu^{\mathrm{L}(+)}(x)$.

The vacuum vector Ω_0 is an element of the representation space of the quotient algebra $\mathfrak{A}_{\mathrm{phys}}$, and normal ordering of operators in $\mathfrak{A}_{\mathrm{phys}}$ is defined in accordance with this vacuum. It follows, for example, that

$$T\big(\mathscr{A}_l^{\mathrm{L}}(x)\,\mathscr{A}_m^{\mathrm{L}}(x')\big) - :\mathscr{A}_l^{\mathrm{L}}(x)\,\mathscr{A}_m^{\mathrm{L}}(x'): = \langle T\big(\mathscr{A}_l^{\mathrm{L}}(x)\,\mathscr{A}_m^{\mathrm{L}}(x')\big)\rangle, \tag{53}$$

with

$$\langle T\big(\mathscr{A}_l^{\mathrm{L}}(x)\,\mathscr{A}_m^{\mathrm{L}}(x')\big)\rangle = \tfrac{1}{2}\hbar c\left(g_{lm} D_{\mathrm{F}}(x-x') - \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^m}\mathscr{D}_{\mathrm{F}}(x-x')\right), \tag{54}$$

where

$$D_{\mathrm{F}}(x) = D^{(+)}(x)\,\theta(x^0) - D^{(-)}(x)\,\theta(-x^0),$$

$$\mathscr{D}_{\mathrm{F}}(x) = \mathscr{D}^{(+)}(x)\,\theta(x^0) - \mathscr{D}^{(-)}(x)\,\theta(-x^0).$$

The Wick expansion for the S matrix, and its evaluation, are now precisely as described in Bjorken and Drell (1965) for the standard radiation gauge formulation.

The derivation of the Feynman rules on the basis of the Fermi method was discussed by Coester and Jauch (1950). They defined the vacuum in the radiation gauge as a state in the space $\mathscr{H}_{\mathrm{e}} \times \mathscr{H}_{\gamma}$ which carries a representation of the field algebra $\mathfrak{A}_{\mathrm{e}} \times \mathfrak{A}_{\gamma}$, with the electromagnetic component satisfying

$$\mathscr{A}_l^{(+)}(x)\, \Omega_0 = 0, \tag{55}$$

$$\partial^\mu A_\mu(x)\, \Omega_0 = 0. \tag{56}$$

This is the same vacuum definition as used by Schwinger (1948, 1949) in the radiation gauge. As Coester and Jauch pointed out, it is equivalent to the state used by Belinfante (1949) which he showed to be non-normalizable. The condition (56) provides the troublesome combination of annihilation and creation operators which allows an indefinite number of unphysical photons. In calculating the self-energy of the electron, Belinfante showed how this poorly defined vacuum led to ambiguous divergent summations, which yielded the correct answer as one alternative among many. One resolution is the introduction of the indefinite metric, so that cancellation of terms involving unphysical photons leads to a normalizable vacuum and eliminates the possibility of incorrect contributions to the self-energy. The indefinite metric does not have a physical interpretation, and from the algebraic viewpoint, it is a mechanism for obtaining a representation of the physical quotient algebra.

As another mechanism for finding such a representation, the Fermi method provides an alternative resolution of these difficulties. The quotient algebra is represented on $\mathscr{H}_{\mathrm{e}} \times \mathscr{H}_0$ and the vacuum (50) is a well-defined element of this space.

It is now possible to see, at least for their radiation gauge calculations, why Schwinger (1948, 1949) and Coester and Jauch (1950) obtained the correct results. Since the interaction Hamiltonian (44) commutes with the supplementary condition operators, so does the S matrix and its components of any order. It follows that all operators used in perturbation calculations may be replaced by the equivalence classes to which they belong in the quotient algebra (41). If we do this, and also replace the Coester–Jauch definition of the vacuum (55) and (56) by (50), then all their calculations may be interpreted as taking place in the representation of the quotient algebra specified by the Fermi method. In other words, the condition (56) is not necessary in order to obtain the correct Feynman rules, and its imposition only leads to needless ambiguities.

We have chosen to use an interaction Hamiltonian formally the same as that in the radiation gauge quantization method, by writing all physical quantities in terms of S-transformed field operators. But we describe the interacting system as a representation of the algebra (41), and hence the electromagnetic degrees of freedom are described in terms of a quotient algebra rather than $\Delta_{\mathrm{c}}(S)$. The radiation gauge description is obtained by picking a representative of each element of the quotient algebra; we have not made such a choice of gauge. This leads to a difference in the description of the Lorentz transformations, which are inner automorphisms of $\Delta_{\mathrm{c}}(N)/I$, whereas $\Delta_{\mathrm{c}}(S)$ is not stable under these transformations. *Thus, calculations using the radiation gauge method can be re-interpreted as taking place in a covariant representation of $\Delta_{\mathrm{c}}(N)/I$, and this explains why they lead to covariant results.*

4. Self-energy of Electron

We wish to show how calculations involving states at finite times can be described in the algebraic picture. We consider a second order calculation of the self-energy of the electron in a radiation gauge formulation. This can be completely described in a representation of the physical quotient algebra $\mathfrak{A}_{\text{phys}}$. We shall then look into the meaning of the Lorentz gauge formulation.

In the radiation gauge, the interaction Hamiltonian is given by

$$G_1(t) = \int \mathrm{d}^3x\{-c^{-1}j_l(x)\mathscr{A}^l(x)\} + J(t), \tag{57}$$

where the Coulomb term $J(t)$ is given by

$$J(t) = -c^{-1}\int \mathrm{d}^3x \int \mathrm{d}^3x' \, \tfrac{1}{2}(\partial\mathscr{D}(x-x')/\partial x^0)\, j^0(x)\, j^0(x'). \tag{58}$$

Here $G_1(t)$ is a representative of the equivalence class $G_1^{\mathrm{L}}(t)$ in $\mathfrak{A}_{\text{phys}}$. The interaction picture Schrödinger equation describes the time evolution of a vector Ψ in the representation space of $\mathfrak{A}_{\text{phys}}$:

$$\mathrm{i}\hbar\,\partial\Psi(t)/\partial t = G_1^{\mathrm{L}}(t)\,\Psi(t). \tag{59}$$

First order processes do not contribute to observable quantities directly, and we make a transformation to eliminate first order terms:

$$\Psi(t) \rightarrow \exp(-\mathrm{i}\,T)\,\Psi(t), \tag{60}$$

where

$$T(t) = -\frac{1}{2\hbar c^2}\int \mathrm{d}^4x' \, \varepsilon(x-x')\, j_l^{\mathrm{L}}(x')\,\mathscr{A}^{l\mathrm{L}}(x'); \tag{61}$$

$$\begin{aligned} \varepsilon(x-x') &= 1, \qquad x^0 > x^{0\prime}, \\ &= -1, \quad x^0 < x^{0\prime}. \end{aligned}$$

The Schrödinger equation becomes

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\Psi(t)}{\partial t} = J(t)^{\mathrm{L}} - \frac{\mathrm{i}}{8\hbar c^3}\int \mathrm{d}^3x \int \mathrm{d}^4x' \, \varepsilon(x-x')\big(\{j_l(x), j^m(x')\} \\ \times [\mathscr{A}_m(x'), \mathscr{A}^l(x)] + [j^m(x'), j_l(x)]\{\mathscr{A}_m(x'), \mathscr{A}^l(x)\}\big)^{\mathrm{L}}\Psi(t). \end{aligned} \tag{62}$$

We wish to determine the contribution to the Hamiltonian from processes whose initial and final states contain one electron and no photons.

The vacuum Ω_0 is defined by

$$\psi^{(+)\mathrm{L}}(x)\,\Omega_0 = \bar{\psi}^{(+)\mathrm{L}}(x)\,\Omega_0 = \mathscr{A}^{(+)\mathrm{L}}(x)\,\Omega_0 = 0, \tag{63}$$

and the vacuum expectation value of electromagnetic factors in the Hamiltonian density is given by

$$\begin{aligned} \langle[\mathscr{A}_m^{\mathrm{L}}(x'), \mathscr{A}^{l\mathrm{L}}(x)]\rangle_0 &= [\mathscr{A}_m^{\mathrm{L}}(x'), \mathscr{A}^{l\mathrm{L}}(x)] \\ &= \mathrm{i}\hbar c\delta_m^l D(x-x') - \mathrm{i}\hbar c\frac{\partial}{\partial x_l}\frac{\partial}{\partial x^m}\mathscr{D}(x-x'), \end{aligned} \tag{64}$$

$$\langle\{\mathscr{A}^{\mathrm{L}}_m(x'), \mathscr{A}^{l\mathrm{L}}(x)\}\rangle_0 = -\hbar c\delta^l_m D^{(1)}(x-x') + \hbar c \frac{\partial}{\partial x_l}\frac{\partial}{\partial x^m}\mathscr{D}^{(1)}(x-x'), \tag{65}$$

where $D^{(1)}(x) = \mathrm{i}\{D^{(+)}(x) - D^{(-)}(x)\}$, and similarly for $\mathscr{D}^{(1)}$. Equation (65) depends on the definition of the vacuum, but (64) does not.

The electron field terms all consist of products of pairs of currents:

$$\begin{aligned} j_\mu(x)\,j_\nu(x') &= :\bar\psi_\alpha(x)\,\gamma^{\alpha\beta}_\mu\,\psi_\beta(x)::\bar\psi_\gamma(x')\,\gamma^{\gamma\delta}_\nu\,\psi_\delta(x'): \\ &= \gamma^{\alpha\beta}_\mu\,\gamma^{\gamma\delta}_\nu\{\langle\bar\psi_\alpha(x)\,\psi_\delta(x')\rangle:\psi_\beta(x)\,\bar\psi_\gamma(x'): + \langle\psi_\beta(x)\,\bar\psi_\gamma(x')\rangle:\bar\psi_\alpha(x)\,\psi_\delta(x'): \\ &\quad + :\bar\psi_\alpha(x)\,\psi_\beta(x)\,\psi_\gamma(x')\,\psi_\delta(x'): + \langle\bar\psi_\alpha(x)\,\psi_\delta(x')\rangle\langle\psi_\beta(x)\,\bar\psi_\gamma(x')\rangle\}. \end{aligned} \tag{66}$$

Only the first two terms contribute in a one-electron state; we denote these by $(j_\mu(x)\,j_\nu(x'))_1$. Substituting expressions of this form for all quadratic current terms in the Hamiltonian (62), and also using (64) and (65), we find that the Coulomb term combines with the terms involving electromagnetic operators to yield the second order self-energy term:

$$\begin{aligned} -\frac{\mathrm{i}}{8\hbar c^3}\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\,\gamma^{\alpha\beta}_\mu\,\gamma^{\gamma\delta}_\nu\{&(:\psi_\beta(x)\,\bar\psi_\gamma(x'):\langle\bar\psi_\alpha(x), \psi_\delta(x')\rangle \\ &+ :\bar\psi_\alpha(x)\,\psi_\delta(x'):\langle\{\psi_\beta(x), \psi_\gamma(x')\}\rangle)\,\mathrm{i}\,\hbar g^{\mu\nu}\,D(x-x') \\ &+ (:\psi_\beta(x)\,\bar\psi_\gamma(x'):\langle[\bar\psi_\alpha(x), \psi_\delta(x')]\rangle + :\bar\psi_\alpha(x)\,\psi_\delta(x'):\langle[\psi_\beta(x), \bar\psi_\gamma(x')]\rangle) \\ &\times(-\hbar c)g^{\mu\nu}D^{(1)}(x-x')\}. \end{aligned} \tag{67}$$

After some further rearrangement, this term may be interpreted as contributing the usual logarithmically divergent addition δm to the mass of the electron.

The Lorentz gauge is often preferred for perturbation calculations because it allows explicit covariance to be maintained. In this gauge, many steps of the self-energy calculation can be carried through in terms of apparently unconstrained 4-vector potentials. We attempt to explain the success of this procedure in terms of the algebraic formulation.

In the Lorentz gauge, the usual interaction Hamiltonian has the form

$$H_1(t) = -c^{-1}\int \mathrm{d}^3x\ j_\mu(x)\,A^\mu(x), \tag{68}$$

and this does not commute with the supplementary condition operators (37). Thus it cannot be regarded directly as a representative of an element of the Lorentz gauge physical algebra $\mathfrak{A}_{\mathrm{phys}}$. (However, as discussed in Section 3, the total Hamiltonian $H = H_0 + H_1$ is known to generate the same time evolution as an operator G, which does correspond to an element of the physical algebra.)

A transformation to eliminate first order terms is now generated by the operator

$$-\frac{1}{2\hbar c^2}\int \mathrm{d}^4x'\ j_\mu(x')\,A^\mu(x')\,\varepsilon(x-x'), \tag{69}$$

and leads to the new Hamiltonian

$$\int \mathrm{d}^4x \, \frac{1}{8c^2}\{j_\mu(x), j^\mu(x')\} D(x-x')\,\varepsilon(x-x')$$

$$-\int \mathrm{d}^4x \, \frac{\mathrm{i}}{8\hbar c^3}[j_\mu(x), j_\nu(x')]\{A^\mu(x), A^\nu(x')\}\varepsilon(x-x'). \tag{70}$$

We also note that the supplementary condition is transformed to

$$\partial_\mu A^\mu(x) - \frac{\mathrm{i}}{4\hbar c^3}\int_{x^{0'}=x^0} \mathrm{d}^3x' \int \mathrm{d}^4x'' \; D(x-x')[j_\mu(x'), j_\nu(x'')]A^\nu(x'')\,\varepsilon(x-x''). \tag{71}$$

In terms of Λ and Λ' this can be written

$$\Lambda(x) - \Lambda'(x) - \frac{\mathrm{i}}{4\hbar c^3}\int_{x^{0'}=x^0} \mathrm{d}^3x' \int \mathrm{d}^4x'' \; \mathscr{D}(x-x')[j_\mu(x'), j_\nu(x'')]A^\nu(x'')\,\varepsilon(x-x''). \tag{72}$$

We wish to evaluate the electromagnetic term $\{A^\mu(x), A^\nu(x')\}$ in the photon vacuum, and the current terms in the one-electron state. The photon vacuum is known to be the state in the representation space of $\mathfrak{A}_{\text{phys}}$ which is annihilated by positive frequency components of the transverse electromagnetic field. We cannot require that it should also be a vacuum for timelike and longitudinal photons, as this would be incompatible with the supplementary condition, and hence with its definition as a state on $\mathfrak{A}_{\text{phys}}$. Thus it appears that, before evaluating operators in a given state, it will be necessary to adopt gauge invariant expressions and interpret them as elements of the physical algebra. This transformation to the physical algebra can be done at any stage before expectation values are calculated. Expectation value calculations could only be done in terms of the field algebra if an appropriate vacuum state could be defined on the field algebra in such a way that evaluation of operators gave the same answer as the corresponding calculation in the physical algebra. There has been a certain amount of confusion on this point. Schwinger's (1949, p. 669) calculation appears to suggest that the state defined on the field algebra Fock space by

$$A_\mu^{(+)}(x)\,\Omega_0 = 0 \tag{73}$$

has the required properties. It is the state satisfying the physical requirement of no transverse photons,

$$\mathscr{A}_l^{(+)}(x)\,\Omega_0 = 0, \tag{74}$$

and in addition

$$\Lambda^{(+)}(x)\,\Omega_0 = \Lambda'^{(+)}(x)\,\Omega_0 = 0. \tag{75}$$

In fact his calculation uses this definition, without recourse to the supplementary condition in any form. As previously noted, the only place where the electromagnetic vacuum is used is in finding the expectation value of the anticommutator:

$$\begin{aligned}\langle\{A^\mu(x), A^\nu(x')\}\rangle_0 &= -\mathrm{i}[A^{\mu(1)}(x), A^\nu(x')] + 2\langle A^{\mu(-)}(x)\,A_\nu(x') + A_\nu(x')\,A_\mu^{(+)}(x)\rangle_0 \\ &= \hbar c g^{\mu\nu} D^{(1)}(x-x') + 2\langle A^{\mu(-)}(x)\,A_\nu(x') + A_\nu(x')\,A_\mu^{(+)}(x)\rangle_0,\end{aligned} \tag{76}$$

where

$$A^{\mu(1)}(x) = \mathrm{i}\{A^{\mu(+)}(x) - A^{\mu(-)}(x)\} \tag{77}$$

($\mathscr{A}^{l(1)}$, $\Lambda^{(1)}$ and $\Lambda'^{(1)}$ can be defined analogously). With the definition of the vacuum (73), the expectation value on the right-hand side vanishes. If we do not assume the unphysical conditions (75), we still find that some terms vanish by (74), and others vanish to second order by the supplementary condition. In order to identify these terms, we rewrite the separation of A^μ into longitudinal, timelike and transverse components given in (47):

$$A^0(x) = -(\partial/\partial x_0)\{\Lambda(x)-\Lambda'(x)\}-(\partial/\partial x_0)\Lambda'(x), \tag{78a}$$

$$A^l(x) = \mathscr{A}^l(x)-(\partial/\partial x_l)\Lambda'(x). \tag{78b}$$

Using (74) we find

$$\langle \mathscr{A}_l^{(-)}(x)\,A_\nu(x') + A_\nu(x')\,\mathscr{A}_l^{(+)}(x)\rangle_0 = 0. \tag{79}$$

Equation (74) provides no information about

$$\begin{aligned}&\langle(\partial/\partial x_0)\{\Lambda^{(-)}(x)-\Lambda'^{(-)}(x)\}(\partial/\partial x_0')\{\Lambda(x')-\Lambda'(x')\}\\ &+(\partial/\partial x_0')\{\Lambda(x')-\Lambda'(x')\}(\partial/\partial x_0)\{\Lambda^{(+)}(x)-\Lambda'^{(+)}(x)\}\rangle_0,\end{aligned} \tag{80}$$

but this is the expectation value of a gauge invariant operator which may be replaced by another representative of its coset in the physical algebra. By equation (72), $\Lambda-\Lambda'$ is thereby replaced by a second order current term. The whole expression is to be multiplied by further current factors, and therefore (80) does not contribute to second order.

However, we are still left with terms of the form

$$\begin{aligned}&\langle(\partial/\partial x^\mu)\Lambda'^{(-)}(x)(\partial/\partial x_0')\{\Lambda(x')-\Lambda'(x')\}\\ &+(\partial/\partial x_0')\{\Lambda(x')-\Lambda'(x')\}(\partial/\partial x^\mu)\{\Lambda'^{(+)}(x)\}\rangle_0,\\ &\langle(\partial/\partial x_\mu)\Lambda'^{(-)}(x)(\partial/\partial x_\nu')\Lambda'(x')\\ &+(\partial/\partial x_\nu')\Lambda'(x')(\partial/\partial x_\mu)\Lambda'^{(+)}(x)\rangle_0,\end{aligned}$$

which are not gauge invariant and cannot be proved to vanish on physical grounds. Therefore we cannot reproduce all the consequences of the vacuum definition (73).

We now examine the consequences of using the field algebra state (73) in the self-energy calculation, and compare the results obtained with those in the radiation gauge calculation. The contribution to the self-energy term from the two terms in (70) can be compared with the terms of similar form in (62). The contribution from

$$\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \frac{1}{8c^2}\{j_\mu(x),j^\mu(x')\}_1\,D(x-x')\,\varepsilon(x-x')$$

differs from the corresponding radiation gauge term by

$$\frac{1}{8c^2}\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\,D(x-x')\{j_0(x),j^0(x')\}_1 \tag{81a}$$

$$+\frac{1}{8c^2}\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\{j_l(x),j^m(x')\}_1\frac{\partial}{\partial x_l}\frac{\partial}{\partial x^m}\mathscr{D}(x-x'). \tag{81b}$$

The first term (81a) just balances the contribution from the Coulomb term. Application of integration by parts to (81b) yields nonzero boundary terms, since the boundary is at finite time.

The contribution

$$\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\left(\frac{-\mathrm{i}}{8\hbar c^3}\right)[j_\mu(x), j_\nu(x')]_1 \langle\{A^\mu(x), A^\nu(x')\}\rangle_0$$

(using the vacuum defined in equation 73) differs from

$$\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\left(\frac{-\mathrm{i}}{8\hbar c^3}\right)[j_\mu(x), j_\nu(x')]_1 \langle\{\mathscr{A}^\mu(x), \mathscr{A}^\nu(x')\}\rangle_0$$

(using the vacuum definition 63) by

$$\frac{\mathrm{i}}{8c^2}\int \mathrm{d}^3x \int \mathrm{d}^4x'\ \varepsilon(x-x')\Big([j_l(x), j_m(x')]_1 \frac{\partial}{\partial x_l}\frac{\partial}{\partial x_m}\mathscr{D}^{(1)}(x-x') + [j^0(x), j_0(x')]_1\, D^{(1)}(x-x')\Big). \tag{82}$$

This difference between the two calculations of the self-energy occurs after the electromagnetic operators have been evaluated in a state, hence it cannot be eliminated by any kind of transformation of the operators.

This implies that Schwinger's vacuum definition (73) was not correct. As a result, terms were obtained corresponding to (81) and (82) (see Schwinger (1949), equations (3.29) and (3.44)) which, contrary to his assertion, cannot be gauged away. We conclude that there is no simple definition of the field algebra vacuum. Hence, the method of calculation that involves converting to a formulation in the physical algebra before calculating expectation values is no more complicated (or less covariant) than any other.

In the indefinite metric formalism, Bleuler (1950) introduced a field algebra vacuum definition appropriate for calculations at finite time:

$$A_0^{(+)}(x)\,\Omega_0 = 0, \tag{83a}$$

$$\left(A_j^{(+)}(x) + \frac{\partial}{\partial x^j}\int \mathrm{d}^3x'\ \mathscr{D}^{(+)}(x-x')j^0(x)\right)\Omega_0 = 0, \tag{83b}$$

and showed that it yielded the same results as the radiation gauge scheme in which longitudinal and timelike photons are eliminated. Clearly, calculations of expectation values in this state will not treat all four components of the electromagnetic field covariantly, any more than radiation gauge calculations do. Furthermore, it is not clear whether an indefinite metric operator exists for the interacting case, and this would be needed in order to complete the justification of the expectation value evaluations described by Bleuler.

In summary, calculations involving states at finite time can be done in the radiation gauge, whose covariance is understood through the algebraic approach. The formal field algebra calculations in Lorentz gauge with vacuum state (83) are justified by the fact that they give the same results as the radiation gauge calculations.

5. Conclusions

In the algebraic picture, all quantization schemes are interpreted as providing representations of the physical algebra $\mathfrak{A}_{\text{phys}}$. It is then reasonable to expect that they should all lead to the same calculational results. We have demonstrated that Fermi method calculations can be justified by appealing to the algebraic description of this quantization method, and in particular, the Feynman rules can be established in this way.

Also using the algebraic picture, we have seen that the physical content of the field algebra description of time evolution can be understood in terms of the inner automorphism of the observable algebra $\mathfrak{A}_{\text{phys}}$ which it generates. It is in this context that equations written in terms of the field algebra can be understood. However, physical states are defined as states on the quotient algebra $\mathfrak{A}_{\text{phys}}$, and there are no physical grounds for defining a state on the field algebra corresponding to a given physical situation. Thus to evaluate expectation values, it is necessary to make use of the mechanism provided by the quantization scheme for finding a representation of $\mathfrak{A}_{\text{phys}}$, and then use states defined on the physical algebra. Covarient field algebra calculation schemes are justified by demonstrating that they agree with calculations in the physical algebra. In the Gupta–Bleuler approach, this is precisely the demonstration that the indefinite metric does not affect the calculation of physical quantities. Alternatively, the cancellation of non-covariant terms may be demonstrated more directly.

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