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A Covariant and Unitary Equation for Single Quantum Exchange*

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Abstract

By requiring the 'bound state' of particle and quantum to have the mass of the particle and be physically indistinguishable from the particle we derive fully covariant and unitary equations for particle–particle scattering; these reduce to the Lippmann–Schwinger equation for Yukawa potential scattering in the nonrelativistic kinematic region and provide a new definition of the 'nuclear potential'.

In quantum electrodynamics the nonrelativistic limit to lowest order taken in the Coulomb gauge leads to the nonrelativistic Schrödinger equation with a Coulomb potential. For a scalar meson theory the corresponding reduction to the Schrödinger equation with a Yukawa potential has never been accomplished by a generally accepted procedure (Moravcsik and Noyes 1961). We believe the difference is due to the fact that for QED the nonrelativistic limit leads to a potential defined in classical physics, which is scale invariant, whereas the range $\hbar/mc$ of the Yukawa potential is not scale invariant and hence intrinsically nonclassical. This fact has frustrated attempts to construct generally accepted unique models for nonrelativistic nuclear physics. In this communication we demonstrate that by starting from covariant Faddeev equations for two particles and one massive quantum we can derive integral equations defining covariant and unitary amplitudes describing single quantum exchange and production. The production channel can be closed without destroying unitarity, leading to fully covariant equations for elastic scattering which reduce to the Lippmann–Schwinger equation for the scattering by a Yukawa potential in the nonrelativistic kinematic region, but which are valid at any energy. The extension to sectors with higher particle and quantum number, the connection to field theory, and some possible applications are briefly discussed.

Fully covariant Faddeev equations driven by separable two-particle amplitudes define unitary and time-reversal invariant three-particle amplitudes (Freedman *et al.* 1966; Brayshaw 1978). For the minimal case discussed by Lindesay (1981), they reproduce the Efimov effect in the appropriate limit in quantitative agreement with nonrelativistic calculations. For this communication we restrict ourselves to two scalar particles of masses m_1, m_2 and a scalar quantum of mass m_Q. The quantum is distinguished from the particles by our postulate that there is no direct particle–particle

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scattering, which reduces the number of Faddeev amplitudes from nine to four. Our second assumption is that the quantum-particle scattering input amplitudes describing $m_i + m_Q \rightarrow m_i + m_Q$ are a single s-channel state of mass m_i which is physically indistinguishable from the particle m_i ($i \in 1, 2$). Following the usual convention of labelling this amplitude by the spectator index j ($\neq i$) and writing it as a function of the initial and final spectator momenta $\boldsymbol{k}_j^{(0)}$ and $\boldsymbol{k}_j$ in the three-particle zero momentum system and of the invariant c.m. energy M, we have for this invariant amplitude

$$t_j(\boldsymbol{k}_j, \boldsymbol{k}_j^{(0)}; M) = s^{\frac{1}{2}}(2\pi)^{-2}\varepsilon_j\,\delta^3(\boldsymbol{k}_j - \boldsymbol{k}_j^{(0)})[\{-q^2(m_i^2)\}^{\frac{1}{2}} - \{-q^2(s)\}^{\frac{1}{2}}]^{-1}.$$

If in the coordinate system in which m_i and m_Q have zero total momentum we allow the spectator m_j to have any momentum between zero and infinity, these limits in the three-particle zero momentum system transform covariantly (Brayshaw 1978) to $0 \leqslant k_j \leqslant (M^2 - m_j^2)/2M$, where M is the invariant four momentum. Consequently in this system we have

$$s = M^2 + m_j^2 - 2M\varepsilon_j, \qquad \text{with} \quad \varepsilon_j = (m_j^2 + k_j^2)^{\frac{1}{2}};$$

$$q^2(s) = \{s - (m_i + m_Q)^2\}|\, s - (m_i - m_Q)^2\,|/4s.$$

This model differs from the minimal model previously discussed in that the 3 (i.e. Q) channel is closed and that we have taken $\mu_j \equiv \mu_{iQ} \equiv m_i$.

By inserting this driving term in the relativistic Faddeev equation (Freedman *et al.* 1966; Brayshaw 1978; Lindesay 1981) for the three-particle amplitudes M_{ij}, defining $t_j = \tau_j \varepsilon_j \delta^3$ and $M_{ij} = t_i \delta_{ij} + \tau_i Z_{ij} \tau_j$, and iterating once, we find that the Z_{ij} satisfy the coupled equations

$$Z_{ij} = -\bar{\delta}_{ij} R - \int \bar{\delta}_{ik} R \tau_k Z_{kj} = -\bar{\delta}_{ij} R - \int Z_{ik} \tau_k R \bar{\delta}_{kj},$$

where $\bar{\delta}_{ij} = 1 - \delta_{ij}$, R is the three-particle propagator and the variable content is defined below. To isolate the elastic scattering and rearrangement amplitudes we rationalize the denominator in τ_j and separate the pole by defining

$$\tau_j = \Gamma_i^2(s_j - m_i^2)^{-1} + \hat{\tau}_j.$$

Following equation (IV.7) of Osborn and Bollé (1973), we isolate the primary singularities in M_{ij} and define the physical amplitudes whose squares are directly related to cross sections by

$$M_{ij} = t_i \delta_{ij} + F_{ij} + G_{ij}\Gamma_j(s_j - m_i^2)^{-1} + \Gamma_j(s_i - m_j^2)^{-1}\tilde{G}_{ij}$$
$$+ \Gamma_i(s_i - m_j^2)^{-1} K_{ij}(s_j - m_i^2)^{-1}\Gamma_j. \tag{1}$$

From this definition it follows immediately that F_{ij} (the 3–3 amplitude) is equal to $\hat{\tau}_i Z_{ij} \hat{\tau}_j$, that the amplitudes needed to compute breakup and coalescence [cf. equation (I.2) of Osborn and Bollé (1973)] are $G_{ij} = \hat{\tau}_i Z_{ij} \Gamma_j$ and $\tilde{G}_{ij} = \Gamma_i Z_{ij} \hat{\tau}_j$, while the elastic scattering and rearrangement amplitudes are $K_{ij} = \Gamma_i Z_{ij} \Gamma_j$.

Since we are primarily concerned here with two-particle elastic scattering, we note that the quantum production channel can be closed simply by taking $\hat{\tau} = 0$. Noting that the three-particle propagator

$$R(\boldsymbol{k}_i, \boldsymbol{k}_j; M) = \varepsilon_{ij}^{-1}(\varepsilon_{ij} + \varepsilon_i + \varepsilon_j - M - \mathrm{i}0^+)^{-1}$$

with $\varepsilon_{ij} = \{m_Q^2+(\boldsymbol{k}_i+\boldsymbol{k}_j)^2\}^{\frac{1}{2}}$, and that since $M = \varepsilon_i^{(0)}+\varepsilon_j^{(0)}$, when we start from a two-body channel,

$$s_j-m_i^2 = -2M(\varepsilon_j-\varepsilon_j^{(0)}-\mathrm{i}\,0^+) \equiv -P(k_j,k_j^{(0)})^{-1},$$

we find that the equations for the physical amplitudes are

$$K_{ij}(\boldsymbol{k}_i,\boldsymbol{k}_j^{(0)};M)+\bar{\delta}_{ij}\Gamma_i R(\boldsymbol{k}_i,\boldsymbol{k}_j^{(0)};M)\Gamma_j$$

$$= \bar{\delta}_{ik}\int_0^{(M^2-m_k^2)/2M} \mathrm{d}^3k_k\,\varepsilon_k^{-1}\,\Gamma_i R(\boldsymbol{k}_i,\boldsymbol{k}_k;M)\Gamma_k P(\boldsymbol{k}_k,\boldsymbol{k}_k^{(0)})\,K_{kj}(\boldsymbol{k}_k,\boldsymbol{k}_j^{(0)};M)$$

$$= \int K_{ik}P_k\Gamma_k R\Gamma_j\bar{\delta}_{kj}. \tag{2}$$

If the bound states m_{iQ} are physically distinguishable from the particles m_i all four of these amplitudes would describe different observable processes. Actually, so far as observation goes we have only elastic scattering, and since we are in the zero momentum system of $\boldsymbol{k}_j = -\boldsymbol{k}_i$, everything can be described in terms of one vector variable. Taking this vector to be $\boldsymbol{k}$, the momentum of m_1 as a spectator, and noting that we can tell, relative to this direction, whether it is m_1 or m_2 that had initial momentum $\boldsymbol{k}'$, the physical amplitude whose square gives the elastic scattering cross section is

$$T(\boldsymbol{k},\boldsymbol{k}';M) = K_{11}(\boldsymbol{k},\boldsymbol{k}';M)+K_{21}(-\boldsymbol{k},\boldsymbol{k}';M)$$

$$= K_{22}(-\boldsymbol{k},-\boldsymbol{k}';M)+K_{12}(\boldsymbol{k},-\boldsymbol{k}';M);$$

the second form expresses the time-reversal invariance guaranteed by the two forms of equation (2).

We note that equation (2) is a coupled channels relativistic Lippmann–Schwinger equation with the exchange potential $V_{ij} = -\bar{\delta}_{ij}\Gamma_i R\Gamma_j$. Since the form of the equation automatically guarantees two-particle unitarity independent of the (finite) value of the product $\Gamma_i\Gamma_j$, we can treat the strength of this potential as arbitrary and call it $g_i g_j$. Since the Γ are the *asymptotic* normalizations of the 'bound state' wavefunction, this amounts to saying that the 'zero range' or pole form of the wavefunction does not hold down to an infinitesimal distance but, other than the reflection of this fact in the 'reduced width' $g_i^2 \neq \Gamma_i^2$, we need not specify this behaviour, a point we will return to below. Further, since on-shell where in the zero momentum system (i.e. $\boldsymbol{k}_i+\boldsymbol{k}_j = 0 = \boldsymbol{k}_i'+\boldsymbol{k}_j'$) we have

$$t = (\varepsilon_i-\varepsilon_i')^2-(\boldsymbol{k}_i-\boldsymbol{k}_i')^2 = -(\boldsymbol{k}_i+\boldsymbol{k}_j')^2, \qquad M = \varepsilon_i+\varepsilon_j,$$

or in the nonrelativistic kinematic region, our 'potential' is

$$-g_1g_2\{m_Q^2+(\boldsymbol{k}-\boldsymbol{k}')^2\}^{-1} = g_1g_2/(t-m_Q^2),$$

and hence can be interpreted as either a nonrelativistic Yukawa potential or as the lowest order field theory result for single quantum exchange. Furthermore, in the nonrelativistic kinematic region, P_k^{-1} reduces to the nonrelativistic propagator $(k^2-k'^2-\mathrm{i}\,0^+)^{-1}$ and we can add the two equations to obtain the usual Lippmann–

Schwinger equation for the amplitude T due to a Yukawa potential. Thus our equation for single quantum exchange, although fully unitary and covariant, has an unambiguous nonrelativistic limit.

The generalization of our treatment to a first approximation for the nuclear force problem is immediate. Instead of scalar particle functions we can use spinors, and since our driving term in the three-particle space with which we start is, kinematically, simply the s-channel absorption and re-emission of the quantum, we know how to put in the vertex operators for pseudoscalar, vector or pseudovector quanta; they are the same as lowest order field theory. Thus we can write three-particle coupled channels equations, and by isolating the pole terms as before obtain a fully covariant and unitary 'one-boson-exchange' model for nucleon–nucleon scattering. Solving these equations then gives us directly the fully off-shell amplitude $T_{\mathrm{NN}}(\boldsymbol{k}, \boldsymbol{k}'; M)$ which could be used directly to compute three-nucleon observables from relativistic Faddeev equations, or N-nucleon observables from relativistic Faddeev–Yakubovsky equations. Noyes (1982) has shown that the Faddeev–Yakubovsky equations for $N = 4$ can easily be derived using our 'zero range' approach. Simply by comparing the results with the same equations using nonrelativistic kinematics we can find out quantitatively how important relativistic 'recoil corrections' are for nuclear physics. But we can go further; by using Faddeev–Yakubovsky equations for N nucleons plus one meson and comparing them with the (relativistic) equations for N nucleons, we can isolate (within our model) the effect of 'three-body forces' from the effect of 'two-body off-shell' behaviour. A still simpler way to test the adequacy of the static potential concept for nuclear physics is to use our fully off-shell T_{NN} to compute the potential, starting from the Low equation (Noyes 1968). Explicitly, since the nonrelativistic energy parameter z is related to M by $M = z + m_1 + m_2$, we have

$$V(\boldsymbol{k}, \boldsymbol{k}') = T(\boldsymbol{k}, \boldsymbol{k}'; z + m_1 + m_2) - \left(\sum + \int_0^\infty\right) \frac{q^2\, \mathrm{d}q\, T(\boldsymbol{k}, \boldsymbol{q}; \tilde{q}^2 + m_1 + m_2)\, T^*(\boldsymbol{q}, \boldsymbol{k}'; \tilde{q}^2 + m_1 + m_2)}{\tilde{q}^2 - z}, \tag{3}$$

where $\tilde{q}^2 = q^2/2\mu$, with $\mu = m_1 m_2/(m_1 + m_2)$, and the summation is included with the integral to remind us to include any bound state pole terms predicted by our interaction. Thus we can determine up to what energy and to what accuracy the V so computed is indeed independent of z, and hence can be used in a nonrelativistic Schrödinger equation for nuclear physics.

Returning to our covariant equation, we note that it is not the ladder approximation to the Salpeter–Bethe (1951) equation, because it is a single time equation, and it is not the Blankenbecler–Sugar (1966) equation, because it has no spurious singularities. Since we have shown above that it has an unambiguous and reasonable limit in non-relativistic scattering theory, we claim to have obtained the correct equation for single quantum exchange. Note that if we make one of the particles a spinor, let m_{Q} go to zero and the mass of the second particle go to infinity, we obtain the momentum space Dirac equation for a Coulomb potential. Clearly, if we do not take these limits, we have a correct relativistic equation with full 'recoil' and, as noted above, can introduce spin for the quantum (or quanta) as easily as for the particles. If we have $m_1 = m_2$ and treat the two particles as identical, then we must, as usual, symmetrize or antisymmetrize the amplitude depending on whether the particles are bosons or fermions. As can be seen from the expressions discussed above, this will give us t–u 'crossing'. Clearly we cannot have s–t or s–u 'crossing' in a finite particle number theory since our ladder would imply an infinite number of particles in the intermediate states when crossed. We can, however, introduce antiparticles in a straightforward way, and compute unitary amplitudes for particle–antiparticle processes with appropriate symmetries, as we will discuss elsewhere.

To extend our theory to higher particle number is, as already noted, straightforward. Since our equations will always give finite results, the test of whether our theory can be generalized in a way consistent with known physics will come when we compute four-particle (or more specifically two-particle and two-quantum) processes and compare with renormalized perturbation theory in the weak coupling limit. So far as we can see, we are including the same physics as quantum field theories with Yukawa-type couplings at that level, and should anticipate the same results. If we fail, this will show that even though, once renormalized, perturbation theory to order g^4 seems only to refer to a finite number of real particles, some trace of the infinite renormalization was left behind; this would also be interesting.

It remains to extend our three-particle theory to quantum production, by restoring the elastic scattering amplitude $\hat{\tau}$ to our two-particle input. This leads to coupled equations for the K_{ij} and G_{ij} which are easy to write down. The three-particle amplitudes calculated from them are clearly unitary so long as we retain the residues Γ_i^2 and Γ_j^2 at the poles which come from a unitary two-particle amplitude, since it is easy to show (Freedman *et al.* 1966; Noyes 1982) that this plus the Faddeev form of the equations guarantees unitarity. Whether we can in this case take g_i^2 to be arbitrary depends to some extent on interpretation.

Again, if we introduce a 'form factor' so that the normalization of the bound state wavefunction corresponds to precisely two particles, we have simply gone back to the more general model discussed by Freedman *et al.* (1966) and Brayshaw (1978), and there is no problem with unitarity; however, the form factors then enter the equations and change our fundamental theory to phenomenology, which we wish to avoid doing. But if we retain the simple pole form for the bound state wavefunction with an arbitrary residue, we are in some sense saying that the bound state is partly elementary and partly composite. With this interpretation flux conservation is preserved, as can be seen from the way it is achieved by Osborn and Bollé (1973). In the 'nonrelativistic field theory' for n–d scattering by Aaron *et al.* (1965) a similar argument has been used in treating the n–d vertex constant as a free parameter, and we do not see why we do not have the same freedom in our context. A fuller discussion of this point, and an approach in which we use the density matrix to describe physical states which are partly 'bare' and partly 'composite' will be presented elsewhere.

We conclude that we have achieved a unitary and covariant description of single quantum exchange with immediate application in nuclear physics, and with possible interesting extensions to a much broader class of problems.

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Stuart Butler was the office mate of one of us (H.P.N.) in Birmingham at the time when Stuart was completing his thesis on deuteron stripping reactions. As is well known, Stuart's fundamental contribution became one of the primary tools for making progress in the analysis of nuclear reactions and nuclear structure, neatly avoiding the ambiguities discussed in the present paper. At the time H.P.N. was struggling

with the problems discussed here but, in contrast to Stuart's success, it has taken until now to achieve what might be a satisfactory resolution of them. We can only wish that we could still profit from his comments on this work.

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