

Formal Scattering Theory for Energy-dependent Potentials*

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Abstract

We show how the usual results of formal scattering theory should be modified when the two particle potential is energy dependent.

1. Introduction

The use of energy-dependent potentials (which are often also complex) in scattering theory has a long history in atomic, nuclear and particle physics. The best known examples are the 'optical potentials' which arise when many channel problems are reduced to equivalent one channel problems (see e.g. Goldberger and Watson 1964 or Mott and Massey 1965). Other examples are the nucleon-nucleon potentials which are now regarded as energy dependent (Epstein and McKellar 1972, 1974; Cottingham *et al.* 1973). It is therefore surprising that, as far as we are aware, the formal theory of scattering by such energy-dependent potentials has not been developed. We rectify this deficiency in this paper.

Of course this is not to suggest that these potentials have not been correctly used to derive scattering observables. In general they have in that, when phase shifts are obtained by solution of the Schrödinger equation for an energy-dependent potential and the results are inserted in the standard formulae, the correct results are obtained. However, some applications, particularly those which use either the formula $S_{fi} = \langle \Psi_f^{(-)} | \Psi_i^{(+)} \rangle$ or the two potential formula (Mott and Massey 1933, p. 258), are formally in error and need to be reformulated. We plan to discuss some examples in subsequent papers.

The major difficulties which arise in constructing the formal scattering theory for energy-dependent potentials $V(E)$ are

- (1) the eigenstates $\Psi_k^{(\pm)}$ of $H(E_k) = T + V(E_k)$ do not form an orthonormal set;
- (2) the action of $H(E)$ on arbitrary states in the Hilbert space is not defined.

These circumstances are not unrelated. The first can be avoided by introducing the states $\tilde{\Psi}_k^{(\pm)}$ which are biorthogonal to $\Psi_k^{(\pm)}$ in the sense that

$$\langle \tilde{\Psi}_k^{(\pm)} | \Psi_{k'}^{(\pm)} \rangle = \delta(k - k'), \quad (1)$$

* Dedicated to the memory of Professor S. T. Butler who died on 15 May 1982.

and, if $\{\Psi_k^{(\pm)}\}$ is a complete set*,

$$\int dk |\Psi_k^{(\pm)}\rangle \langle \tilde{\Psi}_k^{(\pm)}| = \int dk |\tilde{\Psi}_k^{(\pm)}\rangle \langle \Psi_k^{(\pm)}| = 1. \quad (2)$$

The second difficulty is then surmounted by introducing the generalized Hamiltonian operator

$$\mathcal{H} = \int dk E_k |\Psi_k^{(\pm)}\rangle \langle \tilde{\Psi}_k^{(\pm)}|, \quad (3)$$

which is energy independent, has the eigenstates $\Psi_k^{(\pm)}$ with eigenvalues E_k , and may be regarded as an extension of $H(E)$ to the whole Hilbert space. The operator $\mathcal{H}$ is not Hermitian, even if $V(E)$ is real, and the eigenstates of $\mathcal{H}^+$ are $\tilde{\Psi}_k^{(\pm)}$.

We can state our results in terms of $\tilde{\Psi}_k^{(\pm)}$, and $\mathcal{H}$, and do so immediately for the convenience of the reader.

S Matrix

The S matrix elements are given by

$$S_{k',k} = \langle \tilde{\Psi}_{k'}^{(-)} | \Psi_k^{(+)} \rangle. \quad (4)$$

The usual formula is modified by the introduction of $\langle \tilde{\Psi}_{k'}^{(-)} |$ in place of $\langle \Psi_{k'}^{(-)} |$. This modification is typical of our results.

T Matrix

The T matrix elements are given by

$$T_{k',k}^{(+)} = \langle \Psi_{k'} | V(E_k) | \Psi_k^{(+)} \rangle \quad (\text{prior form}), \quad (5)$$

$$T_{k',k}^{(-)} = \langle \tilde{\Psi}_{k'}^{(-)} | \mathcal{H} - E_k | \chi_k \rangle \quad (\text{post form}). \quad (6)$$

On the energy shell the prior and post forms become equal. The prior form remains unaltered, but the post form of the T matrix is changed.

Full Green Function

This is given by

$$G(z) = \frac{1}{z - \mathcal{H}}, \quad (7)$$

which is the expected form, but with the generalized Hamiltonian $\mathcal{H}$ replacing the naive form $H(E)$.

Two Potential Formula

We suppose that $V(E) = V_1(E) + V_2(E)$, that $\mathcal{H}_1$ is the generalized Hamiltonian associated with $V_1(E)$, and that $T_{k',k}^{(1)}$ and $\tilde{\phi}_k^{(-)}$ are the related T matrix elements

* We assume that there are no bound states, for convenience. As long as $V(E)$ is well defined for negative E the extension of our results to include the case of bound states presents no additional problems.

and biorthogonal states. Then the T matrix elements $T_{k',k}$ of the complete potential $V(E)$ are given by

$$T_{k',k} = T_{k',k}^{(1)} + \langle \tilde{\phi}_k^{(-)} | \mathcal{H} - \mathcal{H}_1 | \Psi_k^{(+)} \rangle. \quad (8)$$

The two potential formula is modified in such a way that the distorted wave term is changed by the introduction of the biorthogonal states $\tilde{\phi}_k^{(-)}$ and by the replacement of V_2 by the difference of generalized Hamiltonians $\mathcal{H} - \mathcal{H}_1$.

Unitarity

In spite of the non-Hermitian nature of the generalized Hamiltonian $\mathcal{H}$, we find that the S matrix is unitary unless the potential $V(E)$ has a nonzero imaginary part. If however $V(E)$ does have a nonzero imaginary part, we are able to derive relationships for T matrix elements which resemble the usual optical theorem, but which relate T matrix elements of $\mathcal{H}$ to those of $\mathcal{H}^+$.

We devote the next section to an outline of the derivation of our results and to some discussion of their significance.

2. Derivation of Results

Construction of Biorthogonal States

The key step in the derivation of the formal scattering theory of energy-dependent potentials is the construction of the set of $\tilde{\Psi}_k^{(\pm)}$, which are biorthogonal to the eigenstates $\Psi_k^{(\pm)}$ of the Hamiltonian $H(E)$.

We will assume that the eigenstates $\Psi_k^{(\pm)}$ of $H(E)$ form a complete set of states in the Hilbert space. It is useful to characterize this property of the eigenstates by two lemmas, which are 'proved' in the Appendix.

Lemma 1: A set of states $\{\Psi_k\}$ in a Hilbert space is complete if and only if the operator $P = \int dk |\Psi_k\rangle\langle\Psi_k|$ has an inverse P^{-1} .

Lemma 2: A set of states $\{\Psi_k\}$ in a Hilbert space is complete if and only if the operator D , whose matrix elements are $\langle k' | D | k \rangle = \langle \Psi_{k'} | \Psi_k \rangle$, has an inverse D^{-1} .

Either P^{-1} or D^{-1} , whose existence follows from our hypothesis of completeness of the eigenstates, may be used to construct the biorthogonal states $\tilde{\Psi}_k$ through

$$\tilde{\Psi}_k = P^{-1} \Psi_k, \quad (9)$$

$$\tilde{\Psi}_k = \int dk' \Psi_{k'} \langle k' | D^{-1} | k \rangle. \quad (10)$$

It is readily shown that (9) and (10) are equivalent. For example, act on (10) with the operator P to show that $P\tilde{\Psi}_k = \Psi_k$, whence (9) follows. Equally, it is readily demonstrated that the definitions (9) and (10) give a set of states $\tilde{\Psi}_k$ which satisfy the desired biorthogonality relations

$$\langle \tilde{\Psi}_{k'} | \Psi_k \rangle = \delta(k' - k), \quad (11)$$

$$\int dk |\Psi_k\rangle\langle\tilde{\Psi}_k| = 1. \quad (12)$$

It seems to us that equation (10) leads to the most straightforward practical method for the construction of the biorthogonal states $\tilde{\Psi}_k$. We introduce in place of $\langle k' | D | k \rangle$ the deviation $d(k', k)$ of these matrix elements from a delta function, i.e.

$$\langle k' | D | k \rangle = \delta(k' - k) - d(k', k). \quad (13)$$

If the energy dependence of $V(E)$ is weak we may expect $d(k', k)$ to be small. The definition of D^{-1} , substituted into (13), leads to the integral equation

$$\langle k' | D^{-1} | k \rangle = \delta(k' - k) + \int dk'' \langle k' | D^{-1} | k'' \rangle d(k'', k) \quad (14)$$

for the matrix elements of D^{-1} . Using equation (14) in (10) we obtain a Fredholm integral equation of the second kind for $\tilde{\Psi}_k$:

$$\tilde{\Psi}_k = \Psi_k + \int dk' \tilde{\Psi}_{k'} d(k', k). \quad (15)$$

This integral equation may be solved in a number of ways. If the energy dependence of $V(E)$ is sufficiently weak we would expect the Neumann series to converge rapidly, giving $\tilde{\Psi}_k$ perturbatively in terms of Ψ_k .

Time-dependent Approach to Scattering by Complex Energy-dependent Potentials

In the usual scattering theory in the time-dependent framework (Goldberger and Watson 1964) the time evolution operator $U(t, t')$ or $\exp\{-iH(t-t')\}$ is the basic element. To construct such an operator in the case of energy-dependent potentials we must extend $H(E)$, whose action is defined only on its eigenstates, to the full Hilbert space. Using the decomposition of the unit operator provided by (12) we see that the natural extension of $H(E)$ is the generalized Hamiltonian operator

$$\mathcal{H} = \int dk E_k |\Psi_k\rangle \langle \tilde{\Psi}_k|. \quad (16)$$

This operator has the property of being energy independent, but it has the same eigenvalues and eigenstates as $H(E)$. The energy independence of $\mathcal{H}$ allows us to use it in much the same way as the normal Hamiltonian of scattering theory, as long as care is taken to allow for the fact that $\mathcal{H}$ is not Hermitian, although its eigenvalues are real. The adjoint $\mathcal{H}^+$ has the same eigenvalues E_k , but its eigenstates are $\tilde{\Psi}_k$.

In particular, we may write $U(t, t')$ formally as $\exp\{-i\mathcal{H}(t-t')\}$ and take the Fourier transform in the usual way to obtain

$$\Psi_k^{(\pm)} \equiv U(0, \mp\infty)\chi_k = \lim_{\eta \rightarrow 0+} \frac{\pm i\eta}{E_k - \mathcal{H} \pm i\eta} \chi_k, \quad (17)$$

$$\tilde{\Psi}_k^{(\mp)} \equiv U^+(\pm\infty, 0)\chi_k = \lim_{\eta \rightarrow 0+} \frac{\mp i\eta}{E_k - \mathcal{H}^+ \mp i\eta} \chi_k. \quad (18)$$

In fact the state $\tilde{\Psi}_k^{(-)}$ is none other than $\tilde{\Psi}_k^{(-)}$. To see this, equations (17) and (18) can be manipulated as in Goldberger and Watson (1964, p. 190) to show that

$$\langle \Psi_k^{(-)} | \tilde{\Psi}_k^{(-)} \rangle = \delta(k' - k), \quad (19)$$

whence $P\tilde{\Psi}_k^{(-)} = \Psi_k^{(-)}$ and, as P^{-1} is uniquely defined by Lemma 1,

$$\tilde{\tilde{\Psi}}_k^{(-)} = \tilde{\Psi}_k^{(-)}. \quad (20)$$

Our result (4) then follows from the identification of the S operator with $U(\infty, -\infty)$:

$$S_{k',k} = \langle \chi_{k'} | U(\infty, -\infty) | \chi_k \rangle = \langle \chi_{k'} | U(\infty, 0) U(0, -\infty) | \chi_k \rangle, \quad (21)$$

that is, equation (4).

We might have expected the result (4) on the basis of the following simple heuristic argument: $S_{k',k}$ may be regarded as the probability amplitude for finding the state $\Psi_{k'}^{(-)}$ in the state $\Psi_k^{(+)}$. We may use the decomposition of the identity (2) to write

$$| \Psi_k^{(+)} \rangle = \int dk' | \Psi_{k'}^{(-)} \rangle \langle \tilde{\Psi}_{k'}^{(-)} | \Psi_k^{(+)} \rangle, \quad (22)$$

and the result (4) follows.

Lippmann-Schwinger Equations and Prior Form of the T Matrix Elements

Equation (17) may be rewritten as

$$\Psi_k^{(\pm)} = \chi_k + \frac{1}{E_k - \mathcal{H} \pm i\eta} (\mathcal{H} - E_k) \chi_k. \quad (23)$$

Here we may replace $\mathcal{H} - E_k$ by $\mathcal{H} - T$ but, as this operator does not act on an eigenstate of $\mathcal{H}$, we cannot make the further replacement of $\mathcal{H} - T$ by $V(E_k)$ to obtain the usual form of the equation for $\Psi_k^{(\pm)}$ in terms of χ_k . The identity

$$\frac{1}{z - \mathcal{H}} (\mathcal{H} - T) = \frac{1}{z - T} (\mathcal{H} - T) \left(1 + \frac{1}{z - \mathcal{H}} (\mathcal{H} - T) \right) \quad (24)$$

may be used to convert (23) into an integral equation for $\Psi_k^{(\pm)}$ which is

$$\Psi_k^{(\pm)} = \chi_k + \frac{1}{E_k - T \pm i\eta} (\mathcal{H} - T) \Psi_k^{(\pm)}. \quad (25)$$

Now $(\mathcal{H} - T) \Psi_k^{(\pm)} = \{H(E_k) - T\} \Psi_k^{(\pm)} = V(E_k) \Psi_k^{(\pm)}$ and we may write (25) in the canonical form for the Lippmann-Schwinger equation:

$$\Psi_k^{(\pm)} = \chi_k + \frac{1}{E_k - T \pm i\eta} V(E_k) \Psi_k^{(\pm)}. \quad (26)$$

If we write (26) in a momentum space representation we get

$$\Psi_k^{(+)}(\mathbf{p}) = \delta(\mathbf{k} - \mathbf{p}) + \frac{1}{E_k - E_p + i\eta} \langle \chi_p | V(E_k) | \Psi_k^{(+)} \rangle. \quad (27)$$

We see that the on-shell T matrix element is given by

$$T_{k',k}^{(+)} = \langle \chi_{k'} | V(E_k) | \Psi_k^{(+)} \rangle$$

(cf. Dirac 1958, Section 50). Thus the prior form of the T matrix elements has the usual structure, as promised.

Post Form of the T Matrix Elements

We can write down the analogue of equation (25) which is satisfied by $\tilde{\Psi}_k^{(\pm)}$, namely

$$\tilde{\Psi}_k^{(\pm)} = \chi_k + \frac{1}{E_k - T \pm i\eta} (\mathcal{H}^+ - T) \tilde{\Psi}_k^{(\pm)}, \quad (28)$$

and substitute it into the expression (4) for the S matrix to obtain the prior form of the T matrix elements on the energy shell ($E_k = E_{k'}$):

$$T_{k',k}^{(-)} = \langle \tilde{\Psi}_{k'}^{(-)} | \mathcal{H} - E_k | \chi_k \rangle. \quad (29)$$

This time however we cannot reduce (29) to the usual form in which $V(E_k)$ appears as the operator, because $\tilde{\Psi}_{k'}^{(-)}$ is an eigenstate of $\mathcal{H}^+$, but *not* an eigenstate of $T + V^+(E_{k'})$.

Green Function

From equation (23) it is obvious that the full Green function at $z = E \pm i\eta$ is (see equation 7) $G(z) = (z - \mathcal{H})^{-1}$ and that this then applies by analytic continuation to all z in the cut plane.

Two Potential Formula

Let us first review the two potential formula in the theory of scattering by real energy-independent potentials (Mott and Massey 1933, 1965; Goldberger and Watson 1964). If $V = V_1 + V_2$, and the scattering states for scattering by V_1 alone are $\Phi_k^{(\pm)}$, then the T matrix for scattering by V can be written as

$$T_{k',k}^{(+)} = \langle \chi_{k'} | V_1 | \Phi_k^{(+)} \rangle + \langle \Phi_{k'}^{(-)} | V_2 | \Psi_k^{(+)} \rangle. \quad (30)$$

The first term will be recognized as the matrix element for scattering by V_1 alone, and the second term is the familiar distorted wave scattering matrix.

To obtain the equivalent two potential formula for energy-dependent (and possibly complex) potentials, we introduce the biorthogonal states $\tilde{\Phi}_k^{(\pm)}$ and the effective Hamiltonian $\mathcal{H}_1$ associated with the potential $V_1(E_k)$. We may repeat the standard manipulations to obtain

$$\Psi_k^{(\pm)} = \Phi_k^{(\pm)} + \frac{1}{E_k - \mathcal{H}_1 \pm i\eta} (\mathcal{H} - \mathcal{H}_1) \Psi_k^{(\pm)}, \quad (31)$$

or equivalently

$$\Psi_k^{(\pm)} = \Phi_k^{(\pm)} + \frac{1}{E_k - \mathcal{H} \pm i\eta} (\mathcal{H} - \mathcal{H}_1) \Phi_k^{(\pm)}. \quad (32)$$

The biorthogonal functions $\tilde{\Psi}_k^{(\pm)}$ satisfy similar equations in terms of $\tilde{\Phi}_k^{(\pm)}$, but with $\mathcal{H}$ and $\mathcal{H}_1$ replaced by their adjoints $\mathcal{H}^+$ and $\mathcal{H}_1^+$.

We may then follow the method of Goldberger and Watson (1964, pp. 202 ff) to obtain as the analogue of (30) (see equation 8)

$$T_{k',k}^{(+)} = \langle \chi_{k'} | V_1(E_k) | \Phi_k^{(+)} \rangle + \langle \tilde{\Phi}_{k'}^{(-)} | \mathcal{H} - \mathcal{H}_1 | \Psi_k^{(+)} \rangle.$$

The first term is again just $T_{1k',k}^{(+)}$, but the distorted wave term is modified from its usual form.

The post form of the two potential formula becomes

$$T_{k',k}^{(-)} = \langle \tilde{\Phi}_{k'}^{(-)} | \mathcal{H}_1 - E_k | \chi_k \rangle + \langle \tilde{\Psi}_{k'}^{(-)} | \mathcal{H} - \mathcal{H}_1 | \Phi_k^{(+)} \rangle, \quad (33)$$

and again the first term is simply related to scattering by the first potential. It is just $T_{1k',k}^{(-)}$, the post form of the T matrix for scattering by V_1 .

Equations (8) and (33) have the simplest general form. There are two special cases in which we have been able to simplify the distorted wave term:

(1) If V_1 is energy independent (but not necessarily real), the distorted wave term in the prior T matrix $T^{(+)}$ takes the almost familiar form $\langle \tilde{\Phi}_{k'}^{(-)} | V_2(E_{k'}) | \Psi_k^{(+)} \rangle$.

(2) If $V_1 + V_2$ is energy independent and $E_{k'} = E_k = E$ (i.e. we are on the energy shell), the distorted wave terms may be written as $\langle \Psi_{k'}^{(-)} | V_2(E) | \Phi_k^{(+)} \rangle$.

Unitarity Relations and the Optical Theorem

Intuitively, since the S matrix conserves energy when acting on any state, we would not expect the energy dependence of the potential to influence the unitarity of S . That this is indeed the case may be seen by following the derivation of the optical theorem (and hence the unitarity of S) given by Mott and Massey (1965, pp. 51, 52). The validity of the optical theorem, and the unitary nature of S , depend only on the reality of the phase shifts, and hence on the reality of the potential. These results remain valid for an energy-dependent potential, as long as it is real (McKay 1979).

In the event that the potential is complex, the optical theorem, which we may write as

$$i(T_{kk}^* - T_{kk}) = 2\pi \int d\mathbf{k}' T_{kk'}^+ T_{k'k} \delta(k'^2 - k^2), \quad (34)$$

is easily seen to be modified so that the left side exceeds the right if the potential is absorptive, i.e. the imaginary part of V is negative (see Mott and Massey 1965, pp. 184 ff).

Now let us introduce, as well as $S_{k'k} = \langle \tilde{\Psi}_{k'}^{(-)} | \Psi_k^{(+)} \rangle$, the 'conjugate' S matrix for the 'potential' $\mathcal{H}^+ - T$. Calling this the $\tilde{S}$ matrix, we have

$$\tilde{S}_{k'k} = \langle \Psi_{k'}^{(-)} | \tilde{\Psi}_k^{(+)} \rangle. \quad (35)$$

It is immediate that

$$[S\tilde{S}^+]_{k'k} = [\tilde{S}^+S]_{k'k} = \delta(k' - k). \quad (36)$$

The steps by which we derive the optical theorem (34) from the unitarity of S may now be applied to obtain

$$i\{(\tilde{T}_{k'k}^{(+)})^* - T_{k'k}^{(+)}\} = 2\pi \int dE_{k''} \delta(E_{k''} - E_{k'}) T_{k'k''}^{(+)} (T_{k''k}^{(+)})^*, \quad (37)$$

on the energy shell. This equation may also be readily obtained directly.

3. Discussion

We find it surprising that the inapplicability of the standard techniques of the scattering formalism to scattering by energy-dependent potentials is not more widely

appreciated. This problem can be appreciated by a moment's consideration of the most common energy-dependent potential—the optical potential, for which we use the Feshbach formalism (1958). The optical potential is that potential which, when used in a one channel problem, reproduces the ground state component of the wavefunction of a many channel problem. The difficulties we have discussed which arise in applying the standard approach to the optical model potential simply reflect the existence of the additional channels.

The many channel system is described in terms of a projectile p , with kinetic energy operator T_p , which interacts through a potential V_{pA} with a target A , whose internal Hamiltonian is H_A , with eigenstates Φ_i and eigenenergies ε_i . (We assume that all the Φ_i are bound states and ignore the complications arising from breakup.) The ground state of the target has the wavefunction Φ_0 and energy ε_0 .

The unperturbed Hamiltonian operator for the complete system is $T = T_p + H_A$, and the complete Hamiltonian is $H + T + V_{pA}$. The eigenstates of the projectile plus target system are solutions of the many channel Lippmann–Schwinger equation

$$\Psi_{ik}^{(\pm)} = \chi_k \Phi_i + \frac{1}{E_k - T \pm i\eta} V_{pA} \Psi_{ik}^{(\pm)}, \quad (38)$$

which are generated by asymptotic states in which the projectile is incident in a plane wave state χ_k on the target in the state Φ_i .

The optical potential $\mathcal{V}_{\text{opt}}(E)$ is defined so that the scattering states $\psi_k^{(\pm)}$, which are solutions of the single channel Lippmann–Schwinger equation

$$\psi_k^{(\pm)} = \chi_k + \frac{1}{E_k - T_p \pm i\eta} \mathcal{V}_{\text{opt}}^{(\pm)}(E_k) \psi_k^{(\pm)}, \quad (39)$$

are the projections of $\Psi_{0k}^{(\pm)}$ onto the ground state channel, with the projection operator $P = |\Phi_0\rangle\langle\Phi_0|$, i.e.

$$\psi_k^{(\pm)} = P \Psi_{0k}^{(\pm)}. \quad (40)$$

Feshbach (1958) has shown that

$$\mathcal{V}_{\text{opt}}^{(\pm)}(E) = P V_{pA} P + P V_{pA} Q \frac{1}{E - H \pm i\eta} Q V_{pA} P, \quad (41)$$

where $Q = 1 - P$. The energy dependence and non-Hermitian nature of $\mathcal{V}_{\text{opt}}^{(\pm)}(E)$ is immediately apparent. It is immediately obvious that the $\psi_k^{(\pm)}$ do not form an orthonormal set. Straightforward calculation gives

$$\langle \psi_k^{(\pm)} | \psi_{k'}^{(\pm)} \rangle = \delta(k' - k) - \langle \Psi_{0k'}^{(\pm)} | Q | \Psi_{0k}^{(\pm)} \rangle, \quad (42)$$

$$\int dk |\psi_k^{(\pm)}\rangle\langle\psi_k^{(\pm)}| = 1 - \sum_{i \neq 0} \int dk P |\Psi_{ik}^{(\pm)}\rangle\langle\Psi_{ik}^{(\pm)}| P. \quad (43)$$

The existence of the terms with $i \neq 0$ on the right of (43) was earlier emphasized by one of us (McKellar 1970). In each of (42) and (43) the non-orthonormal nature of the optical model wavefunctions is directly linked with the presence of additional channels in the model, and it is just these additional channels which produce the

energy dependence and non-Hermiticity of $\mathcal{V}_{\text{opt}}(E)$. The kernel $d(\mathbf{k}', \mathbf{k})$ of equation (13) which can be used to construct the biorthogonal states is readily identified in (42):

$$d(\mathbf{k}', \mathbf{k}) = \langle \Psi_{0\mathbf{k}'}^{(\pm)} | Q | \Psi_{0\mathbf{k}}^{(\pm)} \rangle, \quad (44)$$

again showing explicitly that the non-orthogonality of the optical model wavefunctions is a consequence of other channels.

The optical potential $\mathcal{V}_{\text{opt}}^{(\pm)}(E)$ is constructed so that the T matrix elements in the ground state channel are given by $\langle \chi_{\mathbf{k}'} | \mathcal{V}_{\text{opt}}^{(+)}(E_{\mathbf{k}}) | \psi_{\mathbf{k}}^{(+)} \rangle$ on the energy shell. Thus we have

$$\begin{aligned} S_{\mathbf{k}', \mathbf{k}} &= S_{0\mathbf{k}', 0\mathbf{k}} \\ &= \delta(\mathbf{k}' - \mathbf{k}) - 2\pi i \delta(E_{\mathbf{k}'} - E_{\mathbf{k}}) \langle \chi_{\mathbf{k}'} | \mathcal{V}_{\text{opt}}^{(+)}(E_{\mathbf{k}}) | \psi_{\mathbf{k}}^{(+)} \rangle. \end{aligned} \quad (45)$$

It is clear that we may write

$$S_{\mathbf{k}', \mathbf{k}} = \langle \Psi_{0\mathbf{k}'}^{(-)} | \Psi_{0\mathbf{k}}^{(+)} \rangle = \langle \psi_{\mathbf{k}'}^{(-)} | \psi_{\mathbf{k}}^{(+)} \rangle + \langle \Psi_{0\mathbf{k}'}^{(-)} | Q | \Psi_{0\mathbf{k}}^{(+)} \rangle, \quad (46)$$

where $\psi_{\mathbf{k}}^{(-)}$, defined (see equation 40) as $P\Psi_{0\mathbf{k}}^{(-)}$, is the solution of the Lippmann-Schwinger equation for $\mathcal{V}_{\text{opt}}^{(-)}(E)$ with the boundary condition of incoming spherical waves. As $\mathcal{V}_{\text{opt}}^{(-)}(E) \neq \mathcal{V}_{\text{opt}}^{(+)}(E)$ (in fact $\mathcal{V}_{\text{opt}}^{(-)}(E) = [\mathcal{V}_{\text{opt}}^{(+)}(E)]^+$), $\psi_{\mathbf{k}}^{(-)}$ is not the $(-)$ type of solution for the usual optical potential. In any case, from (46) even $\langle \psi_{\mathbf{k}'}^{(-)} | \psi_{\mathbf{k}}^{(+)} \rangle$ is not the S matrix, showing that $T_p + \mathcal{V}_{\text{opt}}^{(-)}(E)$ cannot be identified as the restriction of $\mathcal{H}^+$ to its eigenstates.

We have been unable to continue the analysis of the general optical model further, but the separable potential case discussed by McKellar (1975) has been shown to provide a model in which the validity of our general results can be illustrated by explicit calculation. We will report the details of these results elsewhere.

Finally it is appropriate to remark that our work, while formal, does have practical implications. Applications of the two potential formalism in the case of energy-dependent complex potentials, such as the theory of stripping reactions (Butler 1950, 1951, 1952), requires some reformulation. So do more sophisticated applications of scattering theory; for example, to two step processes. It is not yet clear how much practical results will be altered.

More ambitiously we note that a consistent three body (or many body) theory involving energy-dependent interactions has not yet been achieved. We would hope that our achievement of such a formulation of the two body problem will encourage attempts to resolve the more difficult many body problem.

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Stuart Butler introduced one of us (B.McK.) to scattering theory in particular and to theoretical physics in general. This paper is gratefully and respectfully dedicated to his memory.

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Appendix

We outline heuristic proofs of Lemmas 1 and 2 introduced in Section 2.

Lemma 1. A set of states $\{\Psi_k\}$ in a Hilbert space is complete if and only if the operator $P = \int dk |\Psi_k\rangle\langle\Psi_k|$ has an inverse P^{-1} .

Proof: If P^{-1} exists, then if Φ is any state in the Hilbert space we have $P\Phi \neq 0$. In other words, $\int dk |\Psi_k\rangle\langle\Psi_k|\Phi\rangle \neq 0$, so for k in some set of finite measure, $\langle\Psi_k|\Phi\rangle \neq 0$. Thus any Φ has a nonzero projection onto the space spanned by the $\{\Psi_k\}$, which is therefore a complete set.

The steps in this proof may be reversed to obtain the 'only if' part of the lemma.

Lemma 2. A set of states $\{\Psi_k\}$ in a Hilbert space is complete if and only if the operator D , whose matrix elements are $\langle k'|D|k\rangle = \langle\Psi_{k'}|\Psi_k\rangle$, has an inverse D^{-1} .

Proof:

(1) Necessity of the condition D^{-1} exists: Suppose $\{\Psi_k\}$ is not complete, then by Lemma 1 P^{-1} does not exist, and there is a nonzero state Φ for which $P\Phi = 0$, i.e.

$$\int dk' \langle\Psi_k|\Psi_{k'}\rangle\langle\Psi_{k'}|\Phi\rangle = 0 \quad \text{for all } k, \quad \text{or}$$

$$\int dk' \langle k|D|k'\rangle\langle k'|\Psi\rangle = 0 \quad \text{for all } k,$$

where $\langle k'|\Psi\rangle = \langle\Psi_{k'}|\Phi\rangle$. Thus $D\Psi = 0$, where $\Psi = \int dk' |k'\rangle\langle k'|\Psi\rangle$ is nonzero, and D^{-1} does not exist.

(2) Sufficiency of the condition D^{-1} exists: Suppose the contrary, that D^{-1} does not exist. Then there is a state Φ such that $D\Phi = 0$ or, for any k ,

$$0 = \int dk' \langle k|D|k'\rangle\langle k'|\Phi\rangle = \int dk' \langle\Psi_k|\Psi_{k'}\rangle\langle k'|\Phi\rangle.$$

Thus, for any k , $\langle\Psi_k|\Psi\rangle = 0$ where $|\Psi\rangle = \int dk' |\Psi_{k'}\rangle\langle k'|\Phi\rangle \neq 0$. Therefore $\{\Psi_k\}$ is not a complete set in the Hilbert space.