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Note on Nuclear Disintegration Widths*

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Abstract

Symmetric expressions for fission and neutron (or other particle) emission widths Γ_f, Γ_n etc. are derived by applying uniformly the canonical version of the transition-state method to both cases. The resulting expression for the ratio Γ_n/Γ_f etc. is somewhat different from conventional formulae.

1. Introduction

In connection with conventional discussions of the competition between decay of an excited compound nucleus by fission or neutron (or other particle) emission, one may ask to what extent is it justified or necessary to use formally asymmetric expressions for the partial widths Γ_f and Γ_n, when both widths can be derived by a general transition-state method that ought to yield symmetric expressions.

The usual lack of symmetry is evident, for example, in the formulae (31) and (33) for Γ_f and Γ_n in the Bohr and Wheeler (1939) paper on nuclear fission (see also e.g. Vandenbosch and Huizenga 1973, p. 228). Thus, in the formula for Γ_n, there is an extra power of the kinetic energy in the integrand and a different factor in front of the integral.

The reason for the lack of symmetry may be traced to the different treatments of the transition state in the two cases. In the case of fission, the transition state or 'activated complex' is defined in the canonical way as the system consisting of all the degrees of freedom near the saddle point except *one*, the fission or disintegration degree of freedom. In the case of neutron emission, all *three* (translational) degrees of freedom of the neutron (as well as its spin) are singled out for special treatment. But is this asymmetry justified? Why not treat neutron (or proton, alpha, etc.) emission according to the simplest version of the transition-state method, in which only the normal (radial or separation) degree of freedom is singled out as the disintegration mode, and the activated complex is defined as the residual nucleus *plus* the two transverse degrees of freedom *plus* the spin degree of freedom of the neutron? Or conversely, if one believes that it is worth the trouble to single out also the transverse and spin modes for special treatment, why not then single out analogously the transverse and spin modes in the case of fission? [A transverse displacement of a

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neutron about to leave the surface of a nucleus may be regarded as a rotation of the system consisting of the residual nucleus and the neutron, accompanied by a slight (second-order) increase in the separation coordinate. The corresponding degree of freedom of a saddle-point shape in fission would be a slight rotation of this shape, accompanied by a second-order deformation in the fission degree of freedom. (In both cases, angular momentum may be preserved by an intrinsic counter-rotation of the constituents in the residual nucleus or in the saddle shape.)]

The singling out for special treatment of various degrees of freedom (such as collective rotations or vibrations) may, indeed, be justified when an improved treatment, going beyond the simplest statistical approach, is desired and believed possible. However, it is probably true to say that the use of asymmetric formulae for Γ_f and Γ_n is often not the result of a desire to go beyond the approximation provided by a uniform statistical treatment of all degrees of freedom, but is due to the apparent unavailability in the literature of the relevant symmetric formulae. The purpose of the present paper is to derive these symmetric formulae by applying the canonical version of the transition-state method uniformly to fission and particle emission. [Moretto (1975) provided a discussion of decay widths for the emission of fragments of any size—from neutrons to fission fragments—but the emphasis was on the form of the fragment kinetic energy spectrum at infinity. The problem of energy spectra, involving as it does assumptions concerning how the fragments are accelerated beyond the saddle point by a combination of electric and nuclear forces, is more difficult than the discussion of decay rates (which are determined by conditions in the vicinity of the saddle point alone). Thus, the estimates of the energy spectra reported by Moretto (1975) do not bear directly on the question raised in the present paper.]

2. Canonical Transition-state Method

In this method (see e.g. Bohr and Wheeler 1939), a microcanonical ensemble of $\rho(E)\,\Delta E$ nuclei is considered, where $\rho(E)$ is the level density of the system at energy E, and a calculation is performed of the number of systems that disintegrate per unit time. The ratio

$$\frac{\text{Number of disintegrations per unit time}}{\text{Total number of systems } [=\rho(E)\,\Delta E]}$$

is equal to the reciprocal of the average lifetime t for the disintegration of a system, and $\hbar/t$ is the partial width Γ associated with the decay in question.

The number of disintegrations per unit time is calculated by focusing attention on the microcanonical ensemble near the saddle-point configuration. [In the case of neutron emission the saddle point is the configuration with the neutron (just) outside the range of interaction with the residual nucleus.] The total number of degrees of freedom (say N) is imagined to be split up into two sets: a single degree of freedom q in the disintegration direction and the remaining $N-1$ degrees of freedom, which constitute the 'activated complex'. If p is the momentum conjugate to q then the density of states in the two-dimensional phase space (p, q) is $1/h$, and the number of states of the total system with q in the interval $\mathrm{d}q$, p in the interval $\mathrm{d}p$, and energy in the interval ΔE is

$$h^{-1}\,\rho^*(X)\,\Delta E\,\mathrm{d}p\,\mathrm{d}q\,, \tag{1}$$

where $\rho^*(X)$ is the level density of the activated complex at excitation X. This excitation is given by

$$X = E-B-K, \tag{2}$$

where B is the potential energy of the saddle point (i.e. the potential energy barrier against disintegration) and K is the kinetic energy in the disintegration degree of freedom q. We note the relations

$$K = p^2/2m, \tag{3}$$

$$\mathrm{d}K = (p\,\mathrm{d}p)/m = v\,\mathrm{d}p, \tag{4}$$

where v is the velocity and m the effective mass associated with q. (The effective mass will not appear in the final rate formula.)

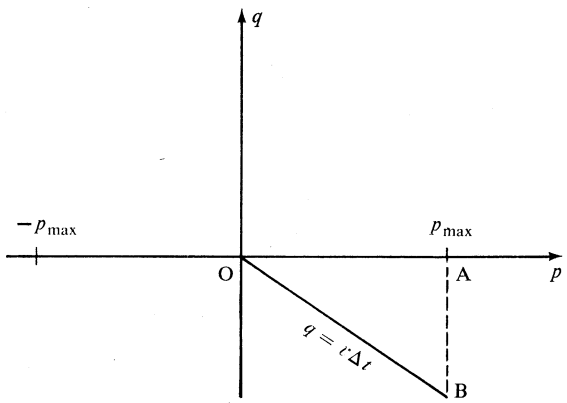

Fig. 1. Phase space of the disintegration degree of freedom q and its conjugate momentum p.

The subspace (p, q) is displayed in Fig. 1. The line $q = 0$ corresponds to the location of the saddle point. The possible values of p range from $-p_{max}$ to $+p_{max}$, where

$$p_{max}^2/2m = E-B. \tag{5}$$

Consider now a time interval Δt. The number of representative points in phase space that cross the saddle-point location $q = 0$ in the positive direction (and are, therefore, assumed to disintegrate in this time Δt) is given by summing the number of systems inside the triangle OAB in Fig. 1 defined by $0 < p < p_{max}$ and $0 < q < -v\Delta t$, i.e. by the integral

$$\int_{p=0}^{p_{max}} \int_{q=-v\Delta t}^{0} \frac{1}{h}\rho^*(X)\,\Delta E\,\mathrm{d}p\,\mathrm{d}q = \frac{\Delta t\,\Delta E}{h}\int_0^{p_{max}} \mathrm{d}p\; v\,\mathrm{d}\rho^*(X)$$

$$= \frac{\Delta t\,\Delta E}{h}\int_0^{E-B} \mathrm{d}K\;\rho^*(X). \tag{6}$$

The number of disintegrations per unit time follows by dividing by Δt, and the reciprocal of the lifetime t by further division by $\rho(E)\Delta E$. Hence we get the final canonical transition-state rate or lifetime formula

$$\frac{1}{t} = \frac{1}{h\rho(E)}\int_0^{E-B} \mathrm{d}K\, \rho^*(X) = \frac{1}{h\rho(E)}\int_0^{E-B} \mathrm{d}X\, \rho^*(X). \tag{7}$$

If we assume that quantal barrier penetration and reflection can be expressed by a penetrability or reflection factor $W(K)$, the generalized lifetime formula becomes

$$\frac{1}{t} = \frac{1}{h\rho(E)}\int_0^{E} \mathrm{d}X\, \rho^*(X)\, W(K). \tag{8}$$

[For a thick parabolic barrier, whose characteristic 'inverted frequency' $\hbar\omega$ is much less than B, we have

$$W(K) \approx \{1+\exp(-K/c)\}^{-1}, \tag{9}$$

where $c = \hbar\omega/2\pi$. When the barrier is not thick (such as in proton or alpha emission), a more sensible formula is actually the simpler expression

$$W(K) \approx \exp(K/c), \quad \text{for } K < 0; \tag{10a}$$

$$\approx 1, \quad \text{for } K > 0. \tag{10b}$$

For neutrons, $W(K)$ is identically zero when $K < 0$.]

3. The Γ_f/Γ_n Ratio

The ratio of a fission width to a neutron (or other particle) width is now given by the perfectly symmetric expression

$$\frac{\Gamma_f}{\Gamma_n} = \int_0^{E} \mathrm{d}X\, \rho_f^*(X)\, W_f(E-B_f-X) \Big/ \int_0^{E} \mathrm{d}X\, \rho_n^*(X)\, W_n(E-B_n-X), \tag{11}$$

or

$$\frac{\Gamma_f}{\Gamma_n} = \int_0^{E-B_f} \mathrm{d}X\, \rho_f^*(X) \Big/ \int_0^{E-B_n} \mathrm{d}X\, \rho_n^*(X), \tag{12}$$

if quantal reflection and penetrability are disregarded.

The level densities ρ_f^* and ρ_n^* refer to the respective activated complexes: namely, the saddle-point shape, or the residual nucleus with a neutron outside it, and with *one* degree of freedom (the disintegration mode) frozen out in *both* cases.

4. Estimates of $\rho(E)$, $\rho_f^*(X)$ and $\rho_n^*(X)$

As a first unrefined baseline estimate for all the above functions ρ, ρ_f^* and ρ_n^*, let us take the formula (2.5) given by Bohr and Mottelson (1975) for the level density of a (spherically symmetric) Fermi gas with fixed (in particular, zero) angular momentum and fixed parity:

$$\rho(A,E) = \frac{a^{\frac{1}{4}}}{24}\left(\frac{\hbar^2}{2\mathscr{I}_{rig}}\right)^{3/2} \frac{1}{E^2}\exp\{2(aE)^{\frac{1}{2}}\} = \frac{C}{E^2}\exp\{2(aE)^{\frac{1}{2}}\}. \tag{13}$$

Here A is the particle number (so the number of degrees of freedom is $4A$ if the particles have spin), E is the excitation energy and $\mathscr{I}_{\text{rig}}$ is the moment of inertia of a rigid body with the same density distribution as the Fermi gas. The level density parameter a is given by

$$a = \tfrac{1}{4}\pi^2 A/E_{\text{F}} + \text{shape-dependent corrections}, \tag{14}$$

where E_{F} is the Fermi energy. (The shape-dependent corrections may be estimated as in Toke and Swiatecki 1981.) Empirically one finds

$$a \approx A/(8{\cdot}5\ \text{MeV}).$$

[In applying these formulae we shall disregard the difference between the a appropriate to ρ (with $4A$ degrees of freedom) and that to ρ_{f}^* and ρ_{n}^* (with $4A-1$ degrees of freedom). To illustrate the main point of the present paper we shall also disregard shell effects and the shape dependences of C and a.]

The integrals over $\rho^*(X)$ appearing in the expressions for $1/t$ are now readily carried out, either exactly or by making the usual expansion about the upper limit of integration, where $X = E-B = X_0$, say. Thus we get

$$\begin{aligned}\int_0^{X_0} \frac{1}{X^2}\exp\{2(aX)^{\frac{1}{2}}\}\,\mathrm{d}X &= \int_0^{X_0} \exp\{-2\ln X + 2(aX)^{\frac{1}{2}}\}\,\mathrm{d}X \\ &= \int_0^{X_0} \exp\left[-2\ln X_0 + 2(aX_0)^{\frac{1}{2}} + \left\{-\frac{2}{X_0} + \left(\frac{a}{X_0}\right)^{\frac{1}{2}}\right\}(X-X_0) + \ldots\right]\mathrm{d}X \\ &\approx X_0^{-2}\exp\{2(aX_0)^{\frac{1}{2}}\}T[\exp\{(X-X_0)/T\}]_0^{X_0} \\ &= TX_0^{-2}\exp\{2(aX_0)^{\frac{1}{2}}\} + \text{small corrections}, \end{aligned} \tag{15}$$

where the temperature T is given by

$$\frac{1}{T} = \left(\frac{a}{X_0}\right)^{\frac{1}{2}} - \frac{2}{X_0}. \tag{16}$$

Using this result we find

$$\frac{1}{t_{\text{f}}} \approx \frac{1}{h}T_{\text{f}}\left(\frac{E}{E-B_{\text{f}}}\right)^2 \exp[2\{a(E-B_{\text{f}})\}^{\frac{1}{2}} - 2(aE)^{\frac{1}{2}}], \tag{17}$$

$$\frac{1}{t_{\text{n}}} \approx \frac{1}{h}T_{\text{n}}\left(\frac{E}{E-B_{\text{n}}}\right)^2 \exp[2\{a(E-B_{\text{n}})\}^{\frac{1}{2}} - 2(aE)^{\frac{1}{2}}], \tag{18}$$

so that

$$\frac{\Gamma_{\text{f}}}{\Gamma_{\text{n}}} = \frac{t_{\text{n}}}{t_{\text{f}}} \approx \frac{T_{\text{f}}}{T_{\text{n}}}\left(\frac{E-B_{\text{n}}}{E-B_{\text{f}}}\right)^2 \exp[2\{a(E-B_{\text{f}})\}^{\frac{1}{2}} - 2\{a(E-B_{\text{n}})\}^{\frac{1}{2}}]. \tag{19}$$

In the above equations, B_{n} is the neutron separation energy, B_{f} the fission barrier energy, and the transition-state temperatures are given by

$$\frac{1}{T_{\text{f}}} = \left(\frac{a}{E-B_{\text{f}}}\right)^{\frac{1}{2}} - \frac{2}{E-B_{\text{f}}}, \qquad \frac{1}{T_{\text{n}}} = \left(\frac{a}{E-B_{\text{n}}}\right)^{\frac{1}{2}} - \frac{2}{E-B_{\text{n}}}. \tag{20a, b}$$

Note that when $B_n = B_f$, equation (19) predicts Γ_f/Γ_n to be independent of energy and equal to one. On the other hand, using the conventional formula for Γ_n gives, for $B_n = B_f$, the ugly formula (see Vandenbosch and Huizenga 1973, p. 229, or Moretto 1975, p. 335)

$$\frac{\Gamma_n}{\Gamma_f} \approx \frac{2TA^{2/3}}{\hbar^2/2mr_0^2} \approx \frac{TA^{2/3}}{5}, \tag{21}$$

where m is the neutron mass and r_0 the nuclear radius constant. According to Fig. VII-7 of Vandenbosch and Huizenga (1973), the empirical trends for Γ_n/Γ_f in the case of a number of heavy nuclei at excitations where the temperature appears to be $T \approx 0{\cdot}4$ MeV show considerable scatter, but the average seems closer to 1 than to the value of ≈ 3 suggested by equation (21). The significance of this is far from obvious, since the use of equation (13) (with fixed C and a) for all the level densities is questionable, especially if shell effects are present. Thus, the approximate agreement with equation (19) might be spurious and due to the cancellation of errors associated with the neglect of shell and deformation effects.

5. Note on the Neutron Spectrum

The present use of the canonical transition-state method for calculating the average lifetime for neutron emission does *not* affect the estimate of the form of the energy spectrum of the emitted neutrons. Thus, the probability of finding the normal component of the emitted neutron momentum (denoted now by p_z) in the slot dp_z is, according to equation (6), proportional to

$$\rho^*(X)\,dp_z^2 \propto \rho^*(E-B-K_z)p_z\,dp_z, \tag{22}$$

where K_z is the z component of the neutron kinetic energy. The probabilities of finding the transverse components of neutron momentum p_x and p_y in the slots dp_x and dp_y are proportional respectively to $\rho^*(E-B-K_x)\,dp_x$ and $\rho^*(E-B-K_y)\,dp_y$, obtained directly from expressions analogous to equation (1), *before* the flux integration. (The extra power of p_z in the probability associated with the normal direction reflects the enrichment of the outgoing flux in particles whose original momenta were toward the escape direction.) The probability of finding the final momentum in the box $dp_x\,dp_y\,dp_z$ is now proportional to

$$\rho^*(E-B-K_x)\,\rho^*(E-B-K_y)\,\rho^*(E-B-K_z)\,dp_x\,dp_y p_z\,dp_z.$$

Using the standard approximation that $\rho^*(X)$ is proportional to $\exp(X/T)$, where T is the temperature, one finds the neutron spectrum to be proportional to

$$\exp\{-(K_x+K_y+K_z)/T\}\,dp_x\,dp_y\,p_z\,dp_z,$$

that is, to

$$K\exp(-K/T)\,dK, \tag{23}$$

where K is now the total kinetic energy $K_x+K_y+K_z$. This is the standard result as regards the single power of the energy K in front of the exponential.

What *is* affected by the present treatment is the absolute factor in the decay mode but not the *form* of the energy spectrum.

6. Conclusions

The discussion here is not meant to imply that the suggested expression for the neutron evaporation lifetime t_n or width Γ_n is more accurate than conventional formulae, derived by a nonstandard transition-state method that singles out the three translational degrees of freedom of the neutron and its spin (or by the equivalent method of detailed balance, underlying the evaporation model). What I do believe is that, especially in a discussion of the competition between fission and particle emission, the present treatment provides a conceptually cleaner set of baseline formulae, against which one may more readily ascertain the advantages of further refinements (such as the inclusion of shell effects, deformations, or the singling out for special treatment of selected degrees of freedom). The problem of disentangling these effects is quite complex and it may avoid confusion if at least a *standard procedure*—the canonical transition-state method—is used uniformly to discuss different types of nuclear disintegrations.

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