

Correlations, Long Range Order and Dimensionality*

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Abstract

After a retrospective look at the nature of correlations in the case of the helium λ transition, we consider long range correlations and their information content for critical phenomena in general. Separation of the statistical from the geometric information leads to the resolution of a long-standing dilemma concerning the dimensionality dependence of the hyper-scaling relations. The hyper-scaling inequalities are the result of the significant correlations suffering a re-arrangement when a regular contribution dominates the singular part of the free energy. The inequalities can each be replaced by a pair of equalities. This resolution has a bearing on the special place of four dimensions as the origin for the ϵ -expansion technique of the renormalization group approach.

1. Introduction

A requirement of any 'good' theory is not so much that it is 'right' but that it leads to experimentally testable consequences. This truism is exemplified nowhere more perhaps than in the field of phase transitions and critical phenomena. Amongst the well-known good theories concerning the nature of phase transitions which preceded recent developments are those which carry the names of Van der Waals (1873), Ornstein and Zernike (1914), Ehrenfest (1933), Landau (1937) and Onsager (1944). It is remarkable that, of these, only the latter is 'right', and it involved the solution of a well-defined mathematical problem, the Ising model in two dimensions, and so was a calculation rather than a 'theory'. Its great significance as a conceptual advance lay in the nature of the results it revealed rather than in the calculation itself, brilliant and universally admired as that was.

The other theories are not right in any general sense, although each can be correct under certain circumstances. The Ehrenfest (1933) classification of phase transitions into first order, second order etc., was never satisfactory except for the first order; the more interesting critical transitions are quite outside the scope of the classification scheme and it is to these that each of the other theories mentioned is relevant. The Landau (1937) theory embodies elements of both the Ornstein–Zernike (OZ) (1914) theory of the correlation function and the Van der Waals (1873) theory of the critical transition. The latter two can each be correct, but only under certain special circumstances and it is significant that in general these circumstances are mutually incon-

* Dedicated to the memory of Professor S. T. Butler who died on 15 May 1982.

sistent. Furthermore, each is inconsistent also with the exact results discovered by Onsager (1944) for the Ising model in two dimensions.

In the present paper it is not our purpose to recall particularly the case of two dimensions, but rather to examine the role of dimensionality itself and its relation to statistical correlations and to critical phenomena. We briefly review some aspects of equilibrium correlations, particularly in superfluid and critical systems, and look back at the theory of limited range momentum correlations put forward in the mid-1950s by Stuart Butler and his colleagues at the University of Sydney (Butler and Blatt 1955; Butler *et al.* 1956). This theory shares with those of Van der Waals, Landau and OZ, the essential property of a good theory, in that it made experimentally testable predictions and stimulated significant advance; like those earlier theories also, however, its predictions failed when put to the test.

We pay particular attention to the nature of the information carried by two quite different kinds of long range correlation and are led to express the correlation of fluctuations in terms of *size* rather than *distance* dependence. This enables us to write the hyper-scaling relations for the critical exponents in what might be called a 'value-free' form, independent of dimensionality. We can then effect a resolution of a long-standing dilemma concerning the status of the hyper-scaling equations. Under certain circumstances the usual homogeneity properties break down, but there emerge in their place two distinct hyper-scaling *equalities*, which together embody the well-established hyper-scaling *inequalities*. For ordinary critical transitions this occurs at the unique special circumstance which restores consistency between the Van der Waals and OZ theories. This is at dimensionality $d = 4$, which marks the coalescence of the renormalization group non-trivial fixed point with the trivial gaussian fixed point and the interchange of their stability. This resolution between Van der Waals and OZ is necessary for the Landau theory to be 'right'. It is also essential for the validity of one of the first triumphs of the renormalization group approach, the ε -expansion technique, since deviations from both the Van der Waals thermodynamic exponent values and the OZ correlation exponents must both vanish together as $\varepsilon \rightarrow 0$.

2. Order in Superfluids

After interruption by the Second World War, developments in the understanding of cooperative phenomena resumed, stimulated particularly by a growing awareness of the dramatic but little understood properties displayed by superconductors and liquid helium. There was as yet no microscopic theory, but powerful and elegant phenomenological theories of these new low-temperature phases of matter had been developed and embodied in the two volume masterpiece 'Superfluids' by London (1950, 1954). London's theory emphasized that the superfluid phases were macroscopic quantum states and involved long range order in momentum space, rather than in ordinary coordinate space as in more familiar ordered phases. He also underlined the connection with the condensation phenomenon occurring in the ideal Bose-Einstein gas. A basic implication of the London theory was that at low speeds the superfluid motion is irrotational, i.e. $\text{curl } \mathbf{p}_s = 0$, where $\mathbf{p}_s$ is the momentum field of the superfluid. This gives the well-known irrotational hydrodynamics of superfluid helium and the famous London equation in the case of superconductors.

For a system in thermodynamic equilibrium to behave in this way is highly singular. In fact, whether or not it is a solid, a system in equilibrium with a container slowly rotating with angular velocity Ω can be shown (see Landau and Lifshitz 1958) to possess 'solid body' rotation; that is, a velocity field $\mathbf{v} = \Omega \times \mathbf{r}$ such that $\text{curl } \mathbf{v} = 2\Omega$, which is by no means irrotational. The proof of this result actually depends on the tacit assumption of the absence of long range momentum correlations. It is just such correlation, extending indeed over long range, that forms the basis of London's theory. The singular nature of the equilibrium response function of a system with such properties is best seen in terms of the Fourier transforms (see Schafroth 1951; Schafroth and Blatt 1956). Thus, if M_k is the equilibrium diamagnetic magnetization produced by the k th Fourier component B_k of an imposed magnetic field $\mathbf{B}(\mathbf{r})$, the linear response coefficient χ_k is just the ratio M_k/B_k . For London's equation this susceptibility ratio at long wavelengths diverges as k^{-2} . In contrast the diamagnetic susceptibility of, for example, a normal metal would correspond to a coefficient which approaches a small constant as $k \rightarrow 0$. Furthermore, in classical mechanics the equipartition theorem shows that momentum correlations vanish in equilibrium and that the diamagnetic coefficient, far from diverging, is not even a finite constant; it is identically zero.

Thus London's theory for both superconductors and superfluid helium requires, as a purely quantum effect, a correlation of the momentum of particles over arbitrarily long range. Casimir (1955) put the matter very clearly: '... if London's explanation of persisting currents is valid, we have to consider such stable wave functions extending over a mile or so of dirty lead wire. This idea of macroscopic quantum states has also been considered in connection with liquid helium. If London's equation on the other hand would only hold in cells of the order of for instance 10^{-4} cm, then we would still find [the diamagnetic properties] with sufficient accuracy, but the existence of persisting currents would remain unexplained.'

3. Limited Range Order

A theory, based on the idea of a large but finite correlation length for momentum correlations, was advanced by Butler *et al.* (1956). It suggested that the diamagnetic response of superconductors should be characterized by a function $\{k(k+\mu)\}^{-1}$, a form less singular than the London k^{-2} and containing a constant $\mu = \Lambda^{-1}$, determined by the finite correlation length Λ . London's theory would be regained if the correlation length were taken as infinite. Schafroth (1954) had shown earlier that the ideal Bose-Einstein gas of charged (but non-Coulomb interacting) particles effectively obeys the London equation, so that in 1955 the cases of known behaviour of the equilibrium diamagnetic response function χ_k in the $k \rightarrow 0$ limit were the following: In classical mechanics, χ_k vanishes; in 'normal' quantum mechanics, χ_k approaches a small constant; in the unphysical ideal Bose-Einstein gas $\chi_k \rightarrow k^{-2}$. The finite correlation length theory assumed the intermediate behaviour $\chi_k \rightarrow k^{-1}$.

In the case of liquid helium the angular momentum response to equilibrium in a rotating container plays the role of the diamagnetic response in superconductors, with similar consequences. As Casimir (1955) implied in the statement quoted above, persistent currents and superfluid flow must in this theory be metastable not equilibrium properties, and this aspect was emphasized in the work 'Nonequilibrium Nature of the Superfluid State' by Butler and Blatt (1955). Of course the equilibrium properties

were also implied by the theory. Their evaluation, however, required extensive computation and had to await completion of the SILLIAC computer.*

4. Helium λ Transition

The results of the computations on the λ transition in liquid helium by Butler *et al.* (1956) presented the specific heat and superfluid density as functions of temperature. Calculated essentially from first principles but incorporating the assumption of a finite correlation length $\Lambda \sim 10^{-5}$ cm, the results involved calculations for clusters of up to 10^7 or more atoms. The agreement with the then existing experimental results was excellent and far better than that obtained by any previous theoretical approximation.

Because of the limited correlation length, a singularity in the thermodynamic properties and thus a true transition was not predicted by the theory; instead, a smoothed out 'quasi-transition' resulted. In this the value of the specific heat, although a continuous and differentiable function of temperature, nevertheless does change extremely suddenly, falling from a maximum value by a factor of two within a millidegree at the λ point; and this occurs after the value has risen steadily, for two thousand times that temperature interval, as the quasi-transition is approached from below. These conclusions were soon to be subjected to experimental test.†

The first experimental report, which described a specific heat singularity sharp to nearly a microdegree, was presented in April 1957 (Fairbank *et al.* 1957). This made it quite clear that the quasi-transition concept and the finite correlation length theory were untenable. This unequivocal conclusion from the preliminary findings was confirmed in more detail as measurements were further refined in the next few years (Fairbank *et al.* 1958; Buckingham and Fairbank 1961; Fairbank and Kellers 1966). The results were consistent with the specific heat diverging as the logarithm of the temperature interval ΔT , where $\Delta T = T - T_\lambda$, with the same coefficient above and below the transition temperature T_λ , but with a constant subtracted on the high temperature side. (This constant difference accounts for the asymmetry of the cusp-shaped curve, making it look somewhat like the Greek letter λ , for which the transition is named.) In fact, although it was this transition that Ehrenfest (1933) had in mind when devising his classification, it is not the step but the logarithmic divergence which dominates the asymptotic nature of the singularity. Ehrenfest's second order type of transition would have been able to describe a step and his first

* It is interesting to recall now that this machine, the first fully engineered electronic computer to be built in Australia, was the Sydney version of the ILLIAC machine at the University of Illinois. When commissioned, SILLIAC was one of the fastest machines in the world and could carry out, on twelve decimal digits, half a million operations *per minute*! To do so it used 2800 electronic valves and, for its fast access memory, 40 cathode ray tubes. It consumed 35 kW and, of course, had to be programmed in machine language. The first scientific calculation carried out on SILLIAC, on 5 July 1956, was in fact that for the equilibrium properties of liquid helium.

† It happened that I was visiting the Sydney department at just this time, on leave from Duke University, where I had been doing theoretical research in close collaboration with the experimental group led by W. M. Fairbank. Immediately on my return to North Carolina in late September, Fairbank and I decided to measure the specific heat near the λ point with high resolution and to try and obtain the qualitative result by the end of the year. Butler was at Princeton for the last part of 1956 and visited the laboratory at Duke. Soon after, he became one of the first to know that our initial results, obtained in 1956, showed that the transition was in fact very sharp and that the quasi-transition theory could not be correct.

order a δ function, but a logarithm or a diverging power $|\Delta T|^{-\alpha}$ cannot be described at all.

As far as the precise form of the singularity is concerned, it was subjected during the 1970s to intense experimental probing, principally by Ahlers and his co-workers. [This work has been reviewed by Ahlers (1980) and here we will only recall some results.] By accurate measurements, not only at the vapour pressure but also under higher pressure and with admixtures of the isotope ^3He , Ahlers could convincingly demonstrate that singular terms additional to the dominant ones were numerically important in practice. By allowing for confluent singularities in the form predicted theoretically (Wegner 1972), the conclusion reached was that the value of the specific heat exponent for the helium λ transition is actually less than zero, the final result being $\alpha = -0.026 \pm 0.004$, with the ratio of the amplitude for the high and low temperature branches given by $A/A' = 1.112 \pm 0.022$. For comparison, the most accurate calculation, using the renormalization group approach, is by Le Guillou and Zinn-Justin (1980) who very effectively employed Borel transformation methods to obtain the result $\alpha = -0.007 \pm 0.006$. One striking result from Ahler's work is the empirical confirmation of the hyper-scaling equation $dv = 2 - \alpha$ to one part in 10^3 .

5. Statistical Order and Broken Symmetry

The fact that the λ transition of liquid helium is neither an Ehrenfest nor a quasi-transition means that it is a 'mainstream' cooperative transition; it is like other critical phenomena such as those that arise in a ferromagnetic or gas-liquid system.

In general, a thermodynamic system undergoes a transition when, as its temperature is reduced or some other general property changes, it cannot both remain stable and retain its symmetry. It must abandon the latter and stability is then preserved at the expense of a broken symmetry, which is manifest in a long range ordering of the property whose stability was threatened. Thus, in the case of a ferromagnet in a zero magnetic field, the total magnetization is zero by symmetry, but the homogeneous locally mean-zero state existing at high temperature becomes unstable for temperatures below the Curie temperature T_c . Then each macroscopic region has a spontaneous magnetization, in which the total can be cancelled if necessary by an opposite magnetization in another such region. These macroscopic regions may be of arbitrarily large size. They thus reflect the long range of the order as well as the broken symmetry; maintenance of the zero total, meanwhile, becomes only a question of determining the fraction of each phase present in a two (or more) phase state.

In the case of liquid helium, as elsewhere, the long range ordering is a purely statistical effect. It is merely what can become ordered that depends on the specifics of the mechanics, not the phenomenon of ordering itself. In the quantum mechanics of a system that remains a fluid, it is the macroscopic occupancy of a particular quantum state that can be the analogue of the magnetization; then a complex wavefunction is the order parameter, its phase being coherent and long range ordered, if necessary over 'a mile of dirty lead wire'. The gradient of the scalar phase function gives the momentum field, which is therefore irrotational, just as foreseen by London (1950, 1954).

6. Critical Dimensions and Universality

The order parameter in the case of the superfluids can be written as $\Psi = |\Psi| \exp(i\chi)$, a complex function with phase angle χ . The free energy depends only on the magnitude

$|\Psi|$, which is a function of temperature and equivalent to the magnitude of the spontaneous magnetization. The symmetry broken in the transition is a gauge symmetry reflected in the phase. A particular value for χ can be regarded as defining a direction in the complex plane, or as a vector to the surface of a unit-radius hypersphere in an abstract n -dimensional vector space, with $n = 2$. A mathematical model of a magnetic system in which each atomic magnet is confined to a plane, say the xy plane, has given its name to the 'XY' or 'planar model'. Here also the order parameter can be represented, like that of the superfluids, as a vector in a certain $n = 2$ dimensional space. On the other hand the Ising model, solved by Onsager (1944) for the case of geometrical space dimension $d = 2$, represents atomic magnets with only two possible orientations, somewhat similar to a spin $\frac{1}{2}$. Thus the possible values for the order parameter in this case are those of an $n = 1$ dimensional unit vector, namely only the two values ± 1 , instead of the continuous infinity of values for $n = 2$ or for larger values of n . Although these last examples are only theoretical models, real physical systems do have an identifiable order parameter dimension n and a spatial dimensionality d , which of course is usually $d = 3$.

The pair of dimensionality numbers n and d are the fundamental symmetry characters for a phase transition and determine the *universality class* of the transition. Thus, the two-dimensional Ising model is class $n = 1, d = 2$. A given system can display more than one class; for example, liquid helium has its λ transition in universality class $n = 2, d = 3$, while its critical transition is in class $n = 1, d = 3$ like that of any other fluid.

The idea of a universal aspect to critical phenomena is a long-standing one (Landau and Lifshitz 1958) that became clarified on the basis of scaling (Griffiths 1970; Kadanoff 1971) and was at last given full theoretical support by the renormalization group theory (Wilson 1971). The assertion of the *concept of universality* is that the specific form of a thermodynamic singularity is determined solely by the universality class to which it belongs.

An example of universality is provided by the ideal Bose-Einstein gas which has a long history as a valuable and suggestive, although unphysical, model for the superfluid transition. It can be looked at from the point of view of a critical transition (Gunton and Buckingham 1968b) and its order parameter can be expressed as a 'wavefunction' $\Psi_{\text{BE}}(\mathbf{r})$, but because of the absence of interaction between the atoms Ψ_{BE} is, in effect, merely a temperature dependent coefficient multiplying a particular function of spatial position, namely the normalized single particle ground state wavefunction $\psi_0(\mathbf{r})$. It is thus qualitatively quite different from the superfluid order parameter $\Psi(\mathbf{r})$ discussed at the beginning of this section. The $\psi_0(\mathbf{r})$ can be taken as a real function of position determined by the boundary conditions. This order parameter is therefore a 'vector' in an infinite-dimensional function space, and so $n = \infty$. Thus, the d -dimensional ideal Bose-Einstein gas at constant density is in universality class $n = \infty, d$. This is the same class as that shown by Stanley (1968, 1971) to represent the spherical model of ferromagnetism, first introduced by Berlin and Kac (1952). The concept of universality therefore implies that the critical exponents are the same for the spherical model and the ideal Bose gas at constant density, and indeed this is known to be the case (Gunton and Buckingham 1968b).

7. Long Range Information

The long range correlations so far discussed are associated with a symmetry-broken two (or more) phase state. While the associated ordering is only fully developed at

absolute zero temperature, its *character* is already established at the transition, when the symmetry is only just broken. In a gas-liquid system at its critical density for example, the system is homogeneous at temperatures T greater than the critical temperature T_c . This symmetry is lost for $T < T_c$, in that some regions are at the liquid density ρ_l , while the remainder are at the gas density ρ_g . As $T \rightarrow T_c^-$, when $\rho_l \rightarrow \rho_g$, the density distribution still has a long range ordered structure, so that every macroscopic region is decidedly either liquid or gas. The long range order carries *precisely the information for this decision*. In some volume element the deviation of the actual density from ρ_l or ρ_g corresponds to the equilibrium fluctuation. In ordinary two phase states such fluctuations can be described by narrow gaussian distribution functions with the fluctuations of different volume elements occurring at random, just as they do in ordinary one phase states.

There is another quite different type of correlation effect that can also become long ranged. It is again a purely statistical effect and is associated with the *onset* of symmetry breaking at the critical transition. Near this transition the fluctuations from the local mean no longer occur independently at random in different volume elements. Their distribution is not narrow, nor gaussian, nor indeed uncorrelated—even at long range. This new structure of the fluctuations and their long range correlations now contain information which implies the *precise asymptotic nature of the critical singularity*. This is in contrast to the two phase long range order discussed above, in which the information carried over long range concerns merely the broken symmetry itself.

The correlation function $C(r)$ can be introduced to describe the correlation of the fluctuations of the order parameter in two volume elements, separated by a distance r . In general $C(r)$ decays exponentially to zero with a correlation length R_c , where the value of R_c depends on the thermodynamic state. In ordinary one or two phase states R_c is only a few times the interparticle spacing. Near the critical point, however, it ceases to be the case that R_c is microscopic; it must become infinite in such a way that, as the temperature approaches T_c , it diverges as $(T - T_c)^{-\nu}$, where ν is the *critical exponent for the correlation length*. The space integral of $C(r)$ is just the compressibility and, while this is ordinarily finite, it becomes infinite at the critical point, so that the integral and hence R_c also become infinite. Since the magnitude of $C(r)$ is bounded, the integral can only diverge because of the contributions from greater and greater distances as the critical point is approached. At the critical point itself $C(r)$ decreases, say as r^{-m} , where m is the *critical point correlation function exponent*. An alternative to m is the conventional exponent η which is defined by $\eta = m - d + 2$. The information carried in the structure of the fluctuations and determining the critical singularity is summarized in the two exponents ν and m (or ν and η). If R_c is the only length characterizing all fluctuations, not only of the order parameter, the system will satisfy the scaling conditions and then the exponents ν and m are all that is needed to specify completely the critical singularity. In Section 11 we will find it useful to transform this information into a form not dependent on the geometry of space.

8. Critical Exponents

Without attempting to review the details of critical phenomena here, we merely recall the four principal thermodynamic exponents α , β , γ and δ . [We refer the reader for background to the comprehensive volumes by Domb and Green (1972-7), to the

review by Fisher (1967) for an excellent account of the theoretical situation prior to the impact of the renormalization group method, to Fisher (1974a) for an account of that impact and, for a discussion of its wider implications, to the book by Pfeuty and Toulouse (1977).]

At the critical density ρ_c , the specific heat of a fluid diverges as $|T - T_c|^{-\alpha}$, while the inverse compressibility vanishes as $|T - T_c|^\gamma$; the latter vanishes on the critical isotherm $T = T_c$ as $|\rho - \rho_c|^{\delta-1}$, as the density ρ approaches ρ_c . In the two phase state the density difference $\rho_1 - \rho_g$, which is the order parameter for the present case, vanishes as $(T_c - T)^\beta$, as $T \rightarrow T_c^-$. Thermodynamic stability demands certain inequalities, such as $\beta(\delta+1) \geq 2-\alpha$, between the exponents but, within the universality classes we mainly wish to consider, the scaling relations apply so that only two exponents are independent; then

$$2-\alpha = \beta(\delta+1) = 2\beta+\gamma. \quad (1)$$

9. Van der Waals and Classical Critical Points

The remarkable equation of state put forward by Van der Waals (1873) began the theoretical description of the gas-liquid critical phenomenon discovered by Andrews (1869). The germ of universality came already in 1880 with the Van der Waals 'law of corresponding states'. This played a major role at the turn of the century in searches for means to liquefy gases and the emergence particularly in the Netherlands, of low-temperature physics. The Weiss theory of magnetism and eventually the Bragg-Williams theory of order-disorder transformations in binary alloys share with the Van der Waals theory the same simplification and crucial weakness; they all describe the critical singularity in terms of thermodynamic functions which possess Taylor expansions at the critical point. These are examples also of a type of universality, all having 'classical' critical points, and are characterized by the set of *classical exponents*

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3. \quad (2)$$

10. Ornstein-Zernike, Landau and Classical Correlations

The Ornstein-Zernike (OZ) (1914) theory (see Fisher 1964, 1967) involves the simple and appealing idea of representing the correlation function by an integral equation in terms of a supposedly simpler 'direct' correlation function. The theory recognizes that the influence at some point caused by a fluctuation at the origin, a distance r away, can be seen as the sum of a direct contribution from the origin plus the integral over all space of an indirect one from other regions, which themselves are affected by the fluctuation at the origin. The space integral of the direct correlation function converges, even at the critical point, and the OZ theory effectively assumes that its Fourier transform, in terms of wavenumber k , is analytic and possesses a Taylor expansion in k^2 at the long wavelength limit ($k \rightarrow 0$). This leads to the result that the correlation exponents m and ν take the OZ values

$$m = d-2, \quad \nu = \frac{1}{2}\gamma. \quad (3)$$

The results of light scattering and other experiments have long ago shown (see Heller 1967) that the value of m at the critical point of fluids is very close to unity,

just the value given by the OZ theory for dimension $d = 3$. Because of this success and no doubt because the theory was advanced as long ago as 1914, the OZ theory became referred to as the classical theory of the correlation function. In fact, although we will make use of the exponent m to describe the decrease with distance of the correlation function at the critical point, the conventional exponent is η (see Section 7 above), defined as the departure from the 'classical' value, i.e. $\eta = m - (d - 2)$. This choice however, was not a very happy one: It effectively incorporates, merely because of the choice of a convention, an apparent dimensionality dependence into what is a statistical and not a geometric concept. On the other hand, when the relationship between statistical mechanics and quantum field theory is under discussion, the geometry of space must enter and then η does indeed become an appropriate parameter as a measure of field interaction effects; for non-interacting fields, $\eta = 0$.

The most succinct version of the classical theory of critical phenomena is the Landau theory (Landau 1937; Landau and Lifshitz 1958). This phenomenological theory involves an expansion of the free energy in powers of a mean order-parameter field, augmented in the Ginzburg form (Ginzburg and Landau 1950) by a term proportional to the square of the field gradient, to give a type of hydrodynamic approximation. This gradient contribution invokes a geometrical element which provides a new assumed link between the thermodynamics and the correlations. In effect the OZ result is enforced, since an expansion in powers of the square of the gradient is essentially the same as an expansion, tacitly assumed to be convergent, of the Fourier transform in powers of k^2 for long wavelengths. Thus the Landau theory results in the full set of exponents given by equations (2) and (3) together. However, the thermodynamic properties themselves determine, independently of the geometry, the important structure of the correlations. Thus, as the renormalization group approach reveals clearly (Wilson 1974) from another point of view, it cannot in general be self-consistent to impose an additional assumption which effectively also determines the structure of the correlations.

11. Thermodynamics and Geometry

Amongst the features which determine the actual thermodynamic properties of a particular system, its dimensionality d is, of course, one of the most crucial. Even the monatomic perfect gas in d dimensions, for example, has its specific heat at constant volume given by $C_V/R = \frac{1}{2}d$ per mole, where R is the gas constant. Also crucial, at least near a critical point, is the other dimensionality number n , discussed in Section 6.

As opposed to the actual properties possessed by particular systems however, the structure of thermodynamics itself is indifferent to the geometry of space or the tensor character of any parameter. So too are the relations of statistical mechanics, including the general expressions for thermodynamic coefficients in terms of equilibrium fluctuations. It is nevertheless customary for these fluctuations to be described in terms of a function of distance, or displacement, namely the correlation function $C(r)$. Now the integral of $C(r)$ over all d -dimensional space gives the relevant coefficient just as described in Section 7. In this, aspects of geometry necessarily enter the discussion and, seemingly, the results, even though the general conclusions must be expressible without such reference, in particular to the dimensionality d .

It is a simple task to transform the fluctuation theorems into a *size* rather than *distance* dependent form and thus ensure their integrity. As a familiar and elementary example, let us write $\langle \Delta E^2 \rangle_N$ for the mean square deviation from the mean of the energy of a system with fixed boundaries at a temperature T and containing N parts or moles or just 'units'; we use N here as an effective measure of 'size', but we could use any *extensive* variable, such as volume. The value of $\langle \Delta E^2 \rangle_N$ equals the total heat capacity times $k_B T^2$, where k_B is Boltzmann's constant. This relation, which can be written as

$$\langle \Delta E^2 \rangle_N / N = k_B T^2 \langle C_b \rangle_N, \quad (4)$$

is *exact* for a system of finite N , where $\langle C_b \rangle_N$ stands for the heat capacity per 'unit' for that particular finite system with its own particular boundary conditions, indicated by the subscript b . In the limit $N \rightarrow \infty$, $\langle C_b \rangle_N$ becomes independent of N and equals the true thermodynamic specific heat coefficient, no longer dependent on finite size effects or on *details* of boundary conditions, other than the overall thermodynamic constraints (such as, say, constant volume for the specific heat C_V of a fluid).

Another example, involving order parameter instead of energy fluctuations, is provided by a fluid whose isothermal compressibility χ (times $k_B T$) is V^{-1} times the mean square deviation from the mean of the volume V occupied by a fixed number N of atoms subject to a fixed pressure. We write this, again in the form used in (4), as

$$\langle \Delta V^2 \rangle_N / V = k_B T \langle \chi_b \rangle_N. \quad (5)$$

The fluctuations described in equations such as (4) and (5) can also easily be expressed, again *exactly* if we are careful, in terms of sums of correlations. Thus, if we write E as $\sum_{i=1}^N E_i$ and V as $\sum_{i=1}^N V_i$ (i.e. as sums over suitably defined contributions from each of N parts), we have

$$\langle \Delta E^2 \rangle_N = \langle (\sum_i \Delta E_i)^2 \rangle_N = \sum_i \sum_j \langle \Delta E_i \Delta E_j \rangle_{N,b}, \quad (6a)$$

$$\langle \Delta V^2 \rangle_N = \sum_i \sum_j \langle \Delta V_i \Delta V_j \rangle_{N,b}. \quad (6b)$$

Here the subscripts N and b are to remind us that the average is for the i th and j th parts embedded together in a possibly correlated system of total size N , with its boundary conditions b (still of course imposed at the boundaries of the total system). Expressions identical in appearance to (4)–(6), but different in precise value, will also represent the fluctuations in a part of size N , well inside a much larger (even, in the limit, an infinite) system. This would merely require a suitable meaning for the subscript b , which we continue to use to indicate any fully specified conditions of equilibrium. We note now that, except in the critical state, if the parts are large enough correlations between the different parts will vanish, so that for example $\langle \Delta E_i \Delta E_j \rangle = \langle \Delta E_i^2 \rangle \delta_{ij}$. Then the double sum in (6a) reduces to the single $\sum_i \langle \Delta E_i^2 \rangle$ and, if the parts are effectively the same as each other, this sum is $N \langle \Delta E^2 \rangle_1$, where the subscript 1 means the fluctuations are those appropriate for a system of unit size. Under these conditions the denominators N on the left in (4) and (5) cancel and we are left with a fluctuation expression, similar to (4), but now for the smaller size of only one part or unit and valid as long as this size is large enough compared with the size of the correlations.

This simplification enables us to identify the condition necessary to achieve the N independence for expressions such as the right-hand sides of (4) and (5) and mentioned earlier in this section in connection with the thermodynamic specific heat coefficient. This condition is just that the size is large compared with a 'correlation size' N_c (or volume V_c), itself determined by the spread of the correlations and depending on the thermodynamic state. (For a system in d -dimensional space with correlation length R_c , this size would be proportional to R_c^d .) It is this dependence that embodies information determining critical exponent values, but we postpone such a discussion until Section 14. In a state near, but not at, the critical point the correlation sums in (6) could only be reduced to a 'unit' size of order N_c , before the N dependence would cause (4) and (5) to depart from their limiting thermodynamic values which, while finite, are possibly very large. When condition $N > N_c$ does not apply, there is a dependence on N which will be taken to be a power law, at least for $N \ll N_c$. This case must include the critical state itself, for which N_c is infinite.

12. Correlation Exponents

At a critical point, a finite limit as $N \rightarrow \infty$ may not exist for $\langle C_b \rangle_N$ in (4) or for its temperature (or other) derivatives and certainly none exists for the order parameter 'susceptibility' coefficient $\langle \chi_b \rangle_N$. In this circumstance there is no size for which we could invoke a vanishing of the correlations, but we can nevertheless still extract what we need from the information in the correlation of the fluctuations by examining their *size dependence*. We therefore define a pair of correlation exponents g_1 and g_2 by taking the N dependence in (4) and (5) to be given *in the critical state*, as $N \rightarrow \infty$, by

$$\langle C_b \rangle_N \sim N^{g_1}, \quad (7)$$

$$\langle \chi_b \rangle_N \sim N^{g_2}, \quad (8)$$

where both exponents must lie between 0 and 1. We can find expressions relating the values of g_1 and g_2 to those of α and δ introduced in Section 8 by evaluating the fluctuations on the left in equations (4) and (5) applied to the critical state itself. In the following section g_1 is extended to negative values.

A characteristic of any critical state, relevant to the incipient instability, is the fact that the free energy change, associated with a slight change of the order parameter from its critical value, is anomalously small. Instead of being, as in a normal state, quadratic in this change, it is proportional to the $(1+\delta)$ th power (by definition of the exponent δ). The condition of being quadratic (i.e. $\delta = 1$), however, is what is necessary for the distribution function of the fluctuations to be a normal gaussian distribution and for the width of the distribution to be proportional to $N^{-\frac{1}{2}}$. For the $(1+\delta)$ th power the distribution width is easily seen to be proportional instead to $N^{-1/(1+\delta)}$. Thus, referring to equation (5) divided by V , we see that the square of the distribution width $\langle \Delta V^2 \rangle_N / V^2$ is just $N^{-2/(1+\delta)}$. From (5) this must be the same as $\langle \chi_b \rangle_N / N$, which in turn is N^{g_2-1} from the definition (8) of g_2 . Thus, we must have

$$g_2 = (\delta - 1)/(\delta + 1). \quad (9)$$

We observe that no geometry has entered the definition of g_2 nor the calculation of the important result (9). We obtain a known result for a system in d dimensions with correlation function exponent m , which would in fact have

$$g_2 = 1 - m/d, \quad (10)$$

since a correlation function $C(r)$ decreasing like r^{-m} at large r would have an integral over a d -dimensional volume V_d proportional to $V_d^{1-m/d}$; the result (10) then follows from (8), as found for example by Fisher (1971, 1974b). It is also easy to see the result if we apply the argument leading to (9) to a non-critical state, for which the free energy of the fluctuations would be quadratic, so that δ would be unity and g_2 identically zero. In (5), χ_N would then be independent of N (for N large enough) and would correspond to the normal and finite thermodynamic coefficient.

We now turn from the order parameter to the energy. The fluctuations of the latter in the critical state determine the N dependence of $\langle C_V \rangle_N$ defined by equation (7). By analogy with the argument leading to the expression for g_2 and using the exact equation (4), we would have in this case, instead of (9), the exponent $(a-1)/(a+1)$, where $1+a$ is the power (not quadratic in general, and analogous to $1+\delta$ for the order parameter) characterizing the free energy change associated with small *energy* changes. In terms of the conventional exponent α it is easily shown that when $\alpha > 0$, $1+a = (2-\alpha)/(1-\alpha)$. Thus at the critical point the N dependence indicated in equation (4) is actually given by equation (7) with $g_1 = (a-1)/(a+1)$ or

$$g_1 = \alpha/(2-\alpha). \quad (11)$$

This size dependence for the C_V coefficient from equations (7) and (11) will be shared by other constant-volume or adiabatic coefficients, such as the adiabatic compressibility; this is because the same power $1+a$ characterizes the free energy change for a fluctuation of any extensive variable, or any linear combination, excepting only the order parameter itself which has the power $1+\delta$ (see Buckingham 1972). This in turn reflects the fact that the same long range correlation of fluctuations causes anomalies in all these coefficients.

13. Non-dominant Exponents

The argument used to determine g_1 only applies when $a > 1$ (i.e. $\alpha > 0$), so that g_1 is positive. This is the case in which the singular part of the free energy change dominates any regular contribution near the critical point. When this does not apply, C_V remains finite and the relationship between a and α changes; two distinct α exponents can then be distinguished, the larger α_D 'sticking' at the value zero and describing the dominant regular contribution, the other α_S becoming negative and describing the singular part which contributes the higher derivatives of C_V that do become infinite. Thus the l th derivative with respect to T diverges with the $l+\alpha_S$ power of $|\Delta T|^{-1}$. (The lowest derivative to diverge is the smallest integer l for which $l+\alpha_S > 0$; if C_V itself diverges then $l = 0$ and α_S is not distinct from α_D or α , now positive.) Without going into further details here, we note that when the dominance passes to a regular contribution in the free energy, the other exponents also bifurcate. One branch continues and identifies the exponents for the singular part and the other 'sticks' at the values for a dominant regular part (with integer values for a and δ ; for the present case of the simple critical point $a = 1$, $\delta = 3$, while for a tricritical point $a = 2$, $\delta = 5$).

The significance of 'dominance' can be quite subtle, particularly for the correlation function. Whereas the dominant exponents α_D and δ_D and both the correlation exponents $g_{1,D}$ and $g_{2,D}$ are never less than the corresponding singular exponents α_S etc., the opposite is the case for the exponents ν and η . Thus $\nu_S \geq \nu_D$ and the range of the

correlations, while becoming infinite for both, is greater for the 'singular' than the 'dominant' and increasingly so as the critical point is more closely approached. The 'singular' not the 'dominant' part is what actually determines the longest distance correlations. These are then modulated strongly at shorter distance by the 'dominant' term which, for this reason, numerically determines the correlation integral. For sufficiently high moments of the integral, however, the singular part would in fact dominate, as it does for sufficiently high derivatives of the thermodynamic coefficients.

When dominance passes to a regular contribution and the exponents bifurcate, a case particularly relevant for systems with dimensionality $d > 4$, our notation requires some generalization. Thus (7) is inadequate since $\langle C_b \rangle$ may remain finite. We must generalize the definition of g_1 to include negative values, and therefore write

$$\langle C_b \rangle_N \propto (N^{g_1} - 1)/g_1 \quad (7')$$

as equivalent to (7) when $g_1 > 0$ and $\langle C_b \rangle_\infty$ is infinite, but capable of describing the asymptotic N dependence when $g_1 < 0$ and $\langle C_b \rangle_\infty$ is finite. With this generalization it is not hard to show that, if α_s is the generalized specific heat exponent defined above, equation (11) still applies, but with α_s for α and $g_{1,s}$ for g_1 . Thus in all cases, positive or negative,

$$g_{1,s} = \alpha_s/(2 - \alpha_s). \quad (11')$$

Another circumstance arises when α and g_1 vanish. Without going into details, it is easy to see that logarithms can replace the power laws in this case. Although we have not needed to identify it, in the equation analogous to (10) for g_1 instead of g_2 there is an exponent equivalent to m and it equals d when $g_1 = 0$; in this case the integral of the correlations over a volume of size V_d would still diverge, not as the usual power $V_d^{g_1}$ but as $\log V_d$. This case, in fact, applies to the two-dimensional Ising model, for which $g_1 = 0$, $g_2 = \frac{7}{8}$ and, as Onsager (1944) showed, the specific heat for the finite system diverges logarithmically. This situation is quite different from that in the non-critical 'normal' state, mentioned briefly following equation (10), with g_2 'identically' zero. In that state g_1 also is 'identically' zero, and the integral of $C(r)$ and all its moments converge so that no analogue of m really exists.

14. Statistical Exponents

The information in the correlation of the fluctuations in the critical state has enabled us to define exponents g_1 and g_2 and hence α and δ . We now briefly extend these considerations to the neighbourhood of the critical state to identify other exponents.

Away from the critical point, say at a temperature different by ΔT , the coefficients C_v and χ are proportional to $\Delta T^{-\alpha}$ and $\Delta T^{-\gamma}$, and using (7) and (8) we can equate these to $N_c^{g_1}$ and $N_c^{g_2}$ respectively. Thus we must have both $\Delta T^{-\gamma/g_2}$ and $\Delta T^{-\alpha/g_1}$ representing the divergence of what is taken to be the same size N_c . Equating these exponents and using (11) we obtain

$$\gamma/g_2 = \alpha/g_1 = 2 - \alpha, \quad (12)$$

$$\gamma = 2g_2/(1 + g_1). \quad (13)$$

Using (9) and (11), equation (13), in more familiar terms, is $\gamma = (2-\alpha)(\delta-1)/(\delta+1)$. Recalling that the exponents g_1 and g_2 are determined by the properties of the fluctuation only of the energy and of the order parameter respectively, we note that the relation (13) for γ involves both. While further relations for these and other exponents, such as that for the divergence at $T = T_c$ of C_V as a function of $\rho - \rho_c$, could readily be obtained, we note that the information for the coexistence exponent β is not contained directly in the correlations of the fluctuations but, as discussed in Section 7, is an aspect of the long range ordered state. This exponent is not specifically concerned with the fluctuations but with the order itself. Of course, if scaling is invoked, we can write β as we could any exponent in terms only of g_1 and g_2 , thus we set $\beta = (1-g_2)/(1+g_1)$. The other results obtained above can for convenience be collected and re-arranged in the form

$$\alpha = 2g_1/(1+g_1), \quad \gamma = 2g_2/(1+g_1), \quad \delta = (1+g_2)/(1-g_2). \quad (14)$$

These relations illustrate the way the properties of the correlations in the critical state, distilled into the fluctuation-size dependence exponents g_1 and g_2 , contain all the information necessary to describe the asymptotic character of the thermodynamic singularity.

In the development so far we have not included any information other than the statistical; the relations (14) are valid regardless, for example, of the dimensionality of the systems concerned. To invoke this dimensionality and the conventional exponents ν and η , we need to recall equation (10) and from Section 11 the fact that in d dimensions the critical size N_c is proportional to R_c^d and to $\Delta T^{-d\nu}$.

15. Weak-scaling and Hyper-scaling

Once we specify the dimensionality d , we can equate $d\nu$ to the exponent combinations in equation (12) and write

$$d\nu = 2/(1+g_1) = 2-\alpha. \quad (15)$$

Invoking the relation $g_2 = 1 - m/d$ from equation (10) and the definition of η , we obtain

$$m/d = 1 - (2-\eta)/d = 1-g_2 = 2/(\delta+1). \quad (16)$$

After they were first derived the two relations (15) (Josephson 1967) and (16) (Gunton and Buckingham 1968a; Buckingham and Gunton 1969) became known as the 'hyper-scaling' relations. They were put forward in the form of inequalities, the left-hand side having been shown in each case to be greater than or equal to the term farthest on the right. The non-equality is by no means a mere academic possibility; it can arise and indeed does so. This is in spite of the fact that the statistical exponent relations from the previous sections, used to derive (15) and (16), can be claimed to stand as *equalities* in all cases, that is, in all cases in the present context of thermodynamic or 'weak' scaling, in which no difference is allowed for between the correlation sizes characterizing the energy and the order-parameter fluctuations.

The hyper-scaling relations have always been something of an enigma. They are obeyed as equalities in some exactly known cases; for example, the ideal Bose-

Einstein gas at constant density (for which $d = 3$, $\alpha = -1$, $\gamma = 5$, $\delta = 2$, $\nu = 1$ and $\eta = 0$) and the two-dimensional Ising model ($d = 2$, $\alpha = 0$, $\delta = 15$, $\gamma = \frac{7}{4}$, $\nu = 1$ and $\eta = \frac{1}{4}$). The renormalization group requires the equalities, except under special conditions (Fisher 1973; Pfeuty and Toulouse 1977). Even the extreme case of first order transitions, which can be brought (Nienhuis and Nauenberg 1975) within the ambit of the renormalization group approach and scaling theory, provides another example, with a suitably generalized interpretation (Fisher and Berker 1982) of the exponents (any d , $\alpha = 1$, $\delta = \infty$, $\gamma = 1$, $\nu = 1/d$ and $\eta = 2-d$; $m = 0$ and $g_1 = g_2 = 1$). Nevertheless, the equalities can also fail; for example, in the limit of a large dimension d , where the mean-field theory applies, with the consequence that the exponents take the classical values given in equations (2) and (3) ($d \rightarrow \infty$, $\alpha = 0$, $\delta = 3$, $\gamma = 1$, $\nu = \frac{1}{2}$ and $\eta = 0$).

The hyper-scaling relations establish the important link between the thermodynamic and the correlation exponents but, because of failures of the equality and poor agreement with some early numerical estimates, their status has been much questioned and attempts have been made to find generalizations (Stell 1968, 1970a, 1970b, 1971; Domb 1968; Fisher 1973, 1974b). It is the explicit appearance of d that has provoked suspicion, rather than the linking relations themselves. Thus Fisher (1974b) has introduced the exponent ω^* (the 'anomalous dimension of the vacuum') to generalize (15), defining it by $d - \omega^* = (2 - \alpha)/\nu$, and it has been suggested that, for large d , $\omega^* = d - 4$. That the relations (15) and (16) are not necessarily deficient is indicated by the result of eliminating d between them. Thus, subtracting (16) from unity and multiplying by (15) gives, with (13), a result that still links the thermodynamic and correlation exponents:

$$(2 - \eta)\nu = \gamma. \quad (17)$$

This relation originally due to Fisher (1964, 1969) is apparently obeyed in all cases when weak-scaling applies.

16. Dominance Dilemma

The information determining all details of the thermodynamics of any equilibrium state of a system can be described, in the parameter space ('Gibbs space') of its extensive variables, in terms of the relationship of the entropy surface to its tangent plane at the point representing that state. In particular, the highly anomalous relationship at the critical state reflects the details of the singularity structure (Buckingham 1972). This information includes that determining the size dependence of fluctuations. Our relations (9) and (11) arise from the fact that the width of the distribution function is $N^{-1/(1+\delta)}$ and $N^{-1/(1+a)}$ respectively, instead of the $N^{-\frac{1}{2}}$ dependence in a 'normal' non-critical state with a gaussian distribution. This compelling result is applied, with little more than appropriate definitions, to the case of d dimensions to yield the hyper-scaling relations (15) and (16). Yet the latter are apparently not always correct as *equalities*.

This dilemma can be resolved if we take more care with the significance of 'dominance'. As in Section 13, when a regular part of the free energy changes dominates the singular part (in particular when $\alpha < 0$), we must identify two branches with distinct exponent sets. These we identified with subscript D or S to indicate their origin from the dominant, if distinct, or the singular term respectively. The 'dominant'

is never less than the 'singular' $g_{1,D} \geq g_{1,S}$, and similarly for g_2 , α and δ etc. However, we saw that the opposite inequality applies for ν and η , i.e. $\nu_S \geq \nu_D$. While this again arises merely from a particular definition, it does in fact reflect the functionally dominant quality as far as the correlation function is concerned. What has most significance, if more than one finite range term is involved, is that *term with the longer range*, i.e. with the larger ν . While this term may not, and indeed does not, yield the larger contribution to the convergent integral (since η is also larger) for the thermodynamic coefficient, it must do so for the higher moments of the integral. This reflects the fact that sufficiently high derivatives of the coefficients will be dominated (numerically) by the 'singular' rather than the 'dominant' contribution.

Thus, if we were to revert to not identifying the origin of the exponents and to the usual notation, we must put $\nu = \nu_S$ and $\eta = \eta_S$, but $\alpha = \alpha_D$ and $\delta = \delta_D$ etc. Then we have the originally proved hyper-scaling inequalities resulting now from the set of relations

$$d\nu = d\nu_S \geq d\nu_D = 2 - \alpha_D = 2 - \alpha, \quad (18)$$

$$d - 2 + \eta = d - 2 + \eta_S \geq d - 2 + \eta_D = 2d/(\delta_D + 1) = 2d/(\delta + 1). \quad (19)$$

It is more informative however to retain the precise relations which can be written in the combined form

$$d\nu_Q = 2 - \alpha_Q; \quad (2 - \eta_Q)/d = (\delta_Q - 1)/(\delta_Q + 1), \quad (20)$$

where Q stands for S or D in each equation to give both pairs of equalities.

It should be noted here that when the singular is distinct from the dominant term, the usual homogeneity property no longer applies. The free energy would be the sum of two terms, each by itself homogeneous but with its own set of exponents, the 'dominant' set with integer values of δ and $a = 1/(1 - \alpha)$ representing a regular contribution to the free energy. Although homogeneity fails, the condition of weak-scaling still applies, at least in the version we have employed, namely that the correlation size is the same for energy as for order-parameter fluctuations. Thus, contrary to conventional wisdom, even within the validity of weak-scaling the usual homogeneity property does not always hold, whereas appropriate hyper-scaling *equalities* do.

17. Classical Consistency and Conclusions

We have emphasized how the thermodynamic properties of the critical state and, in particular, the values of δ and a determine, independently of dimensionality d , the correlation exponents g_1 and g_2 (equations 9, 11 and 11'). Except when a regular term intervenes to dominate the singular thermodynamic coefficients and to re-arrange the dominance of exponents, the conventional correlation exponents are then determined by the hyper-scaling equalities (15) and (16). A system possessing the classical exponents $\alpha = 0$, $\gamma = 1$ and $\delta = 3$ must have $g_1 = 0$ and $g_2 = \frac{1}{2}$ and thus, unless dominated, $\nu = 2/d$ and $\eta = 2 - \frac{1}{2}d$. Except for the special case $d = 4$, the latter are not consistent with the OZ values $\nu = \frac{1}{2}$ and $\eta = 0$.

Even allowing for dominance it is impossible for both the Van der Waals and OZ theories to apply to any system in three dimensions, as pointed out by Gunton and Buckingham (1968a). The hyper-scaling *inequalities* (18) and (19) demand for a

Van der Waals system, with $d = 3$, that $\nu \geq \frac{2}{3}$ and $\eta \geq \frac{1}{2}$, completely inconsistent with the OZ values; this inconsistency applies for any dimension less than four. In the one case $d = 4$ in which the hyper-scaling equalities can apply directly, we nevertheless expect logarithmic terms to arise, in the manner mentioned in Section 13. For $d > 4$, inconsistency can be avoided by invoking the *inequalities*, which as we have seen imply that a regular contribution dominates the singular part of the free energy. This indeed happens according to renormalization group analysis. For $d > 4$ the singularity is that of the trivial 'gaussian' fixed point, which has precisely the expected dominant and singular contributions necessary to satisfy the generalized hyperscaling equalities (20). With the notation employed there, we can write the full sets of exponents, for this case $d > 4$, as follows:

$$\begin{aligned}\alpha_D &= 0, & \beta_D &= \frac{1}{2}, & \gamma_D &= 1, & \delta_D &= 3, \\ g_{1,D} &= 0, & g_{2,D} &= \frac{1}{2}, & \nu_D &= 2/d, & \eta_D &= -\frac{1}{2}(d-4); \\ \alpha_S &= -\frac{1}{2}(d-4), & \beta_S &= \frac{1}{2} + \frac{1}{4}(d-4), & \gamma_S &= 1, & \delta_S &= 3 - 2(d-4)/(d-2), \\ g_{1,S} &= -(d-4)/d, & g_{2,S} &= 2/d, & \nu_S &= \frac{1}{2}, & \eta_S &= 0.\end{aligned}$$

In this form it is easy to see explicitly how this 'trivial' fixed point and therefore the renormalization solution for $d > 4$ obeys the dominance rules embodied in equations (18) and (19). As foreshadowed, the two sets of exponents bifurcate from their identical values at $d = 4$ in such a way that the first four exponents of the 'dominant' D set and the last two of the 'singular' S set are the ones which actually dominate behaviour. Together they comprise the full 'classical' set and so, because of the subtle interplay of 'dominance' between statistics and geometry, reconciliation is finally brought about between the Van der Waals and Ornstein-Zernike theories.

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