*Aust. J. Phys.*, 1984, **37**, 105–18

# Toroidal Flow in Axisymmetric Pulsar Magnetospheres

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*Abstract*

This paper deals with dissipation-free toroidal flow in steadily-rotating axisymmetric pulsar magnetospheres; for each species, relativistic inertia is balanced by the Lorentz force. A complete integral for such flows is obtained; precise corotation with the star corresponds to a singular solution. Except for the case of corotation, all quantities follow from a single scalar quantity, which is determined by the Stokes stream function of the magnetic field. A fundamental differential equation for the problem is obtained. Particular attention is paid to flows that tend toward corotation as the symmetry axis is approached, and implications of the results for model building are discussed.

## 1. Introduction

It is very likely that pulsar magnetospheres contain zones dominated by particles whose motion is close to one of corotation with the star—a flow enforced by rotation of the strong magnetic field. At any position, species of only a single charge-to-mass ratio can corotate (Burman 1981*a*; Holloway and Pryce 1981). If precise corotation is assumed, then such zones are confined within the light cylinder, on which the speed of corotation equals $c$, the vacuum speed of light. This assumption is usually made, no doubt for simplicity in analysis, but is unjustified: there is no reason to believe that zones dominated by toroidally flowing particles do not extend beyond the light cylinder, or that precise corotation holds anywhere.

So an obvious and necessary step in pulsar magnetosphere theory is to achieve an understanding of toroidal flows that are not corotating with the star. Remarkably little work has been done on this problem. Jackson (1980, 1981) treated these flows, but neglected inertia and all forces other than the Lorentz force, so that the equation of motion of a species was reduced to the equating of the Lorentz force to zero. A start on incorporating inertia has been attempted by Holloway and Pryce (1981), who concentrated on trying to treat small departures from corotation. This has been followed up by Mestel (1981), who suggested that a complete magnetosphere model may have a sub-rotating ion domain that can extend beyond the light cylinder.

In this paper, I shall develop the theory of purely toroidal flows in axisymmetric pulsar magnetospheres. The pulsar is taken to be steadily rotating and the flow is dissipation free, with the relativistic inertia of a species balanced by the Lorentz force on it. The analysis is exact within the steady-rotation and dissipation-free constraints.

0004-9506/84/010105$02.00

The result is, at least to me, quite unexpected: it is the full and exact solution for these flows. The flows are described by a complete integral, together with a singular integral representing corotation.

## 2. Basic Theory

Let $\tilde{\omega}$, $\phi$ and $z$ be cylindrical polar coordinates, with the $z$-axis as the rotation axis of the star. The system under consideration is steady in the rotating frame: the changes in time at points fixed in the inertial frame of the star result from the steady rotation of a structure at angular frequency $\Omega$. The dimensionless cylindrical radial coordinate $\Omega\tilde{\omega}/c$ is denoted by $x$ and the unit toroidal vector by $\boldsymbol{t}$.

It follows from Faraday's law and $\nabla . \boldsymbol{B} = 0$ that the electric and magnetic fields are connected by $\boldsymbol{E}+x\boldsymbol{t} \times \boldsymbol{B} = -\nabla\Phi$ (Mestel 1971; Westfold 1981) where the gauge-invariant potential $\Phi$ is defined in terms of the familiar scalar and vector potentials $\phi$ and $\boldsymbol{A}$ as $\phi - xA_\phi$ (Endean 1972*a*). Thus $\boldsymbol{E}$ is expressed as the sum of a part $x\boldsymbol{B} \times \boldsymbol{t}$, generated by the rotation of the magnetic field structure, and an irrotational part $-\nabla\Phi$.

Endean (1972*a*, 1972*b*) pointed out that, under the steady-rotation constraint, there exists a constant of the motion $\Psi_k$ for particles of species $k$:

$$\Psi_k \equiv \Phi + (\gamma_k m_k c^2/e_k)(1 - xv_{k\phi}/c), \tag{1}$$

where $\gamma_k$, $m_k$, $e_k$ and $v_{k\phi}$ denote the Lorentz factor, rest mass, charge and $\phi$ component of velocity of the particles of this species. If all particles of species $k$ are nonrelativistic in an arbitrarily thin neighbourhood of the stellar surface, then $\Psi_k$ is constant throughout all the space connected to the surface by flow lines of that species (Burman and Mestel 1978).

For a species represented as a cold dissipation-free fluid, the equation of motion, expressing the balance of the Lorentz force by relativistic inertia, can be written as (Burman and Mestel 1978)

$$c^{-1}\boldsymbol{u}_k \times \{\boldsymbol{B} + (m_k c/e_k)\nabla \times \gamma_k \boldsymbol{v}_k\} = \nabla\Psi_k, \tag{2}$$

where $\boldsymbol{u}_k$ denotes $\boldsymbol{v}_k - \Omega\tilde{\omega}\boldsymbol{t}$, the velocity reduced by the local velocity of corotation with the star. Inserting the toroidal flow restriction $\boldsymbol{v}_k = v_{k\phi}\boldsymbol{t}$ into equation (2) shows that $\Psi_k$ must be independent of azimuth, leaving the poloidal equation

$$(u_{k\phi}/c\tilde{\omega})\{\tilde{\omega}\boldsymbol{t} \times \boldsymbol{B} + (m_k c/e_k)\nabla(\tilde{\omega}\gamma_k v_{k\phi})\} = \nabla\Psi_k. \tag{3}$$

Attention will now be restricted to the axisymmetric case, in which the magnetic and rotation axes of the star are either parallel or antiparallel. The poloidal magnetic field (which, because of the absence of poloidal electric currents, is here the total magnetic field) is expressible in the form $\tilde{\omega}^{-1}\boldsymbol{t} \times \nabla P$, where $P \equiv -\tilde{\omega}A_\phi$. The equation of motion (3) reduces to

$$(u_{k\phi}/c\tilde{\omega})\nabla\{(m_k c/e_k)\tilde{\omega}\gamma_k v_{k\phi} - P\} = \nabla\Psi_k. \tag{4}$$

After inserting the Endean integral (1) for $\Psi_k$ and using the fact that, for toroidal flow, $c^2\nabla\gamma_k = \gamma_k^3 v_{k\phi}\nabla v_{k\phi}$, equation (4) becomes

$$\nabla\Phi = (m_k/e_k)\gamma_k v_{k\phi}^2 \boldsymbol{i}/\tilde{\omega} - (u_{k\phi}/c\tilde{\omega})\nabla P, \tag{5}$$

where $\boldsymbol{i}$ denotes the unit cylindrical radial vector. It follows that

$$\nabla^2 \Phi = \frac{m_k}{e_k}\frac{1}{\tilde{\omega}}\frac{\partial}{\partial \tilde{\omega}}(\gamma_k v_{k\phi}^2) - \frac{u_{k\phi}}{c\tilde{\omega}}\nabla^2 P - (\nabla P).\nabla\left(\frac{u_{k\phi}}{c\tilde{\omega}}\right), \tag{6}$$

which will be used in Section 4.

The physical significance of equation (5) is clear: it is a direct representation of the balance of the Lorentz force, which can be expressed as $-e_k \nabla\Phi + e_k c^{-1}\boldsymbol{u}_k \times \boldsymbol{B}$ (Burman and Mestel 1978), by the relativistic rotational inertia. For corotational flow, equation (2) shows that $\Psi_k$ is constant throughout the flow, while it follows from equation (5) that (Burman 1980) $e_k \Phi/m_k c^2 = 1-(1-x^2)^{\frac{1}{2}}$, satisfying the boundary condition $\Phi = 0$ on $x = 0$. Equation (5) shows that departure from corotation introduces a $z$ dependence into $\Phi$.

Taking the curl of the force-balance equation (5) shows that the velocity's dependences on $\tilde{\omega}$ and $z$ are linked identically by the relation

$$\{B_z + (m_k c/e_k)(\gamma_k^3 + \gamma_k)\alpha_k\}\partial\alpha_k/\partial z = -B_{\tilde{\omega}}\,\partial\alpha_k/\partial\tilde{\omega}, \tag{7}$$

where $\alpha_k$ denotes $v_{k\phi}/\tilde{\omega}$, the angular speed of the flow. Note that if $v_{k\phi}$ is taken to be independent of $z$, then it cannot be other than proportional to $\tilde{\omega}$—corresponding to exact corotation if that is imposed as a boundary condition as $\tilde{\omega} \to 0$. More generally, since $B_z$ is positive in an electron zone and negative in a positive ion zone, the $\tilde{\omega}$ derivative of $\alpha_k$ vanishes on the surfaces

$$|\,\omega_{Bkz}\,| = (\gamma_k^3 + \gamma_k)\alpha_k, \tag{8}$$

except, perhaps, where $B_z = 0$; here $\omega_{Bkz}$ denotes $e_k B_z/m_k c$, the $z$ component of the nonrelativistic vector angular gyrofrequency of species $k$. The angular speed of the flow reaches a maximum, for each $z$, on one of these surfaces.

## 3. Admissible Toroidal Flows

For convenience, the subscript $k$ labelling the species will now be dropped and the dimensionless variable $\tilde{P} \equiv (e/mc^2)\Omega P/c$ will be used; also, $z$ will be taken as normalized to one at distance $c/\Omega$ above or below the equatorial plane.

The identity (7) takes the essentially dimensionless form

$$\left(\frac{\partial\tilde{P}}{\partial x} - (\gamma^3+\gamma)\frac{v_\phi}{c}\right)\frac{\partial\alpha}{\partial z} = \frac{\partial\tilde{P}}{\partial z}\frac{\partial\alpha}{\partial x}. \tag{9}$$

This shows that, when inertia is neglected, $\alpha$ is a function of $\tilde{P}$ only, which just expresses magnetic domination of the flow. The full relation implies, because of the factor $\partial\tilde{P}/\partial z$ on the right-hand side, that $\alpha$ varies with $z$ only through a dependence on $\tilde{P}$; that is, $\alpha$ can be regarded as a function $\alpha(x, \tilde{P})$ of $x$ (explicitly) and $\tilde{P}$. Therefore, on transforming to $x$ and $\tilde{P}$ as independent variables, remembering that $\partial\alpha/\partial x$ in (9) is at constant $z$, the identity becomes

$$\left(\frac{\partial\alpha}{\partial x}\right)_{\tilde{P}} + (\gamma^3+\gamma)\frac{v_\phi}{c}\left(\frac{\partial\alpha}{\partial\tilde{P}}\right)_x = 0, \tag{10}$$

except perhaps where $B_{\tilde{\omega}} = 0$.

Consider, in particular, a small departure from corotation: $\alpha = \Omega - \varepsilon$ where $|\varepsilon| \ll \Omega$. The relation (10) becomes, to first order in $\varepsilon$,

$$\left(\frac{\partial \varepsilon}{\partial x}\right)_{\tilde{P}} + \frac{2-x^2}{(1-x^2)^{3/2}} x \left(\frac{\partial \varepsilon}{\partial \tilde{P}}\right)_x \approx 0. \tag{11}$$

So, in a small, nearly toroidal departure from corotation, $\varepsilon$ must be a function of a single variable:

$$\varepsilon \approx \varepsilon(\tilde{P} - \gamma + 1/\gamma), \tag{12}$$

with $\gamma = (1-x^2)^{-\frac{1}{2}}$; this is just the statement that $\varepsilon$ is constant on the characteristics of the linear first-order partial differential equation (11) that it satisfies. Even these flows will not be purely toroidal, since the functional form (12) will not, in general, satisfy precisely the full toroidal flow identity (9).

In particular, for small, nearly toroidal departures from corotation where $x^2 \ll 1$ [the problem studied by Holloway and Pryce (1981)], the functional form (12) reduces to $\varepsilon \approx \varepsilon(\tilde{P} - x^2)$. In the dipole approximation, we have $-\tilde{P}/x^2 = (\omega_B/\Omega)(1+3\cos^2\theta)^{-\frac{1}{2}}$, where $\omega_B \equiv eB/mc$ and $\theta$ is the angle from the dipole axis. So, for $x^2 \ll 1$, $\varepsilon$ must be close to being a function of $\tilde{P}$ only, which is the negligible inertia result once again.

In terms of $\beta(x, \tilde{P})$, denoting $v_\phi/c$, as the independent variable, the toroidal flow identity (10) becomes

$$(\partial\beta/\partial x)_{\tilde{P}} + (\gamma^3 + \gamma)\beta(\partial\beta/\partial\tilde{P})_x = \beta/x, \tag{13}$$

with $\gamma \equiv (1-\beta^2)^{-\frac{1}{2}}$. This quasilinear first-order partial differential equation has the singular solution $\beta = x$, corresponding to corotation, and a complete integral

$$\gamma\beta + b/\beta = (\tilde{P}+a)/x \equiv Q, \tag{14}$$

where $a$ and $b$ are independent of $x$ and $\tilde{P}$. Although $z$ is being treated as a parameter at this stage, $a$ and $b$ cannot depend on $z$ since $\beta$ can depend on $z$ only through $\tilde{P}$; thus $a$ and $b$ are constants. Another form of this complete integral is

$$\gamma - 1/\gamma + b = \beta Q. \tag{14'}$$

The complete integral shows that $\beta$ can, in fact, be regarded as a function of the single variable $Q$; pure corotation, being a singular solution, is an exception. For toroidal flow in axisymmetric pulsar magnetospheres, excluding precise corotation, all quantities are determined by a single scalar quantity: they follow from the magnetic field through its Stokes stream function $P$.

Eliminating either $\gamma$ or $\beta$ from the complete integral, using either the forms (14) or (14') respectively, yields quartic equations for $\beta$ and $\gamma$:

$$\beta^4 = (1-\beta^2)(Q\beta - b)^2, \tag{15}$$

$$\gamma(\gamma - 1/\gamma + b)^2 = (\gamma - 1/\gamma)Q^2. \tag{16}$$

These give $\beta$ and $\gamma$ in terms of $Q$.

Taking the gradient of the complete integral (14) results in

$$(\gamma^3 - b/\beta^2)\nabla\beta = \nabla Q. \tag{17}$$

Since $\beta$ is a function of $Q$ only, this is, in effect, an equation for $\mathrm{d}\beta/\mathrm{d}Q$; it will be found useful in calculations. Note that

$$\partial Q/\partial x = -\omega_{Bz}/\Omega - (b+\gamma\beta^2)/\beta x \quad \text{and} \quad \partial Q/\partial z = \omega_{B\tilde{\omega}}/\Omega . \tag{18a, b}$$

It is readily checked, using equations (17) and (18a), that $\partial\alpha/\partial x$ vanishes on the surfaces defined by equation (8).

Let $\tilde{\Psi}$ and $\tilde{\Phi}$ denote the dimensionless variables $e\Psi/mc^2$ and $e\Phi/mc^2$. The equation of motion (4) for toroidal flows in axisymmetric magnetospheres can be rewritten in the present notation as

$$(\beta/x - 1)\nabla(x\gamma\beta - \tilde{P}) = \nabla\tilde{\Psi} . \tag{19}$$

After using the complete integral (14) to substitute for $x\gamma\beta - \tilde{P}$, equation (19) integrates to

$$\tilde{\Psi} = 1 + b\{x/\beta - \ln(x/\beta) - C\}, \tag{20}$$

where $C$ is a constant. Comparison with the Endean integral (1) shows that

$$\tilde{\Phi} = 1 - \gamma(1 - x\beta) + b\{x/\beta - \ln(x/\beta) - C\} . \tag{21}$$

By taking the gradient of this equation and making use of the complete integral again, it is readily verified that the toroidal flow condition (5) is satisfied. Equations (20) and (21) present the variables $\tilde{\Psi}$ and $\tilde{\Phi}$ for flow corresponding to the complete integral (14).

The relativistic vorticity $\nabla \times \gamma \boldsymbol{v}$ of flow corresponding to the complete integral can readily be calculated by using the integral in the form (14), taking $\beta = \beta(Q)$, obtaining $\mathrm{d}\beta/\mathrm{d}Q$ from equation (17) and using equations (18) for $\nabla Q$:

$$\nabla \times \gamma\boldsymbol{v} = \frac{\gamma^3}{b - \gamma^3\beta^2}\left(\beta^2\boldsymbol{\omega}_B + b\frac{\beta}{x}(2-\beta^2)\Omega\boldsymbol{k}\right), \tag{22}$$

where $\boldsymbol{\omega}_B \equiv e\boldsymbol{B}/mc$ and $\boldsymbol{k}$ is a unit vector parallel to the symmetry axis.

Note that flows with $b = 0$ are characterized by having zero generalized vorticity, $\boldsymbol{\omega}_B + \nabla \times \gamma\boldsymbol{v}$; in other words, their magnetoidal field $\boldsymbol{B} + (mc/e)\nabla \times \gamma\boldsymbol{v}$ vanishes. Equation (20) shows that $\tilde{\Psi} = 1$ for such flows, as is the case for pure corotation when the boundary condition $\Phi = 0$ is imposed at the star. These are the only toroidal flows for which $\Psi$ is constant. The integral (14) shows that flow with zero generalized vorticity cannot extend to the star, since $Q$ diverges there.

## 4. Magnetic Field and Charge Density

Forms taken by the Gauss and Ampère laws under the constraints of steady rotation and axisymmetry have been given by Mestel *et al.* (1979). The former is

$$\nabla^2(\Phi - \Omega P/c) = -4\pi\rho^{\mathrm{e}}, \tag{23}$$

where $\rho^{\mathrm{e}}$ denotes the electric charge density. If only a single species is present in a region, or if all species present have the same azimuthal velocity component $v_\phi$, so that the $\phi$ component of electric current density in the region is $\rho^e v_\phi$, then the azimuthal component of the Ampère law is

$$\nabla^2 P + 2B_z = (4\pi/c)\tilde{\omega}\rho^e v_\phi. \tag{24}$$

In the absence, as here, of any poloidal electric current, $B_\phi = 0$ satisfies the remainder of Ampère's law.

For toroidal flow, $\nabla^2\Phi$ is related to $\nabla^2 P$ and $\nabla P$ by equation (6). Eliminating $\Phi$ and $\nabla^2 P$ among equations (6), (23) and (24) results in

$$\rho^e = -\gamma^2\frac{(x\beta)'}{2x}\frac{\Omega B_z}{2\pi c} + \gamma^2\frac{B_{\tilde{\omega}}}{4\pi}\frac{\partial\beta}{\partial z} - \frac{m\Omega^2}{4\pi e}\gamma^2(\gamma^3+\gamma)\frac{\beta\beta'}{x}, \tag{25}$$

where the prime denotes partial differentiation with respect to $x$ with $z$ constant.

Suppose that the flow velocity $v_\phi$ can be specified throughout an ion or electron zone. Equation (24) together with (25) for $\rho^e$ and appropriate boundary conditions on $P$—for example, that it takes dipolar form near the star—determine the magnetic field throughout that zone.

Once the magnetic field in the zone has been found, equation (25) determines the charge density there. The first two terms on the right-hand side are a direct generalization of the Goldreich–Julian (1969) charge density, valid for corotation, to general toroidal flow; these two terms are necessarily connected by the identity (7). The third contribution to $\rho^e$ has resulted from allowing for inertia, as indicated by the presence of the rest mass in its coefficient.

In the case of corotation, equation (25) for the charge density reduces to

$$\rho^e = -\gamma^2(\Omega/2\pi c)\{B_z + \tfrac{1}{2}\Omega(mc/e)(\gamma^3+\gamma)\},$$

which is the same as equation (3.9) of Mestel *et al.* (1979). The magnetic contribution is the Goldreich–Julian charge density, and is positive where $B_z < 0$ and negative where $B_z > 0$, vanishing with $B_z$: when inertia is neglected, positively and negatively charged zones of corotation are separated by the $B_z = 0$ surfaces.

The inertial contribution to $\rho^e$ is negligible in the bulk of a corotating zone, but determines an outer boundary to the zone (Wang 1978; Mestel *et al.* 1979): Since its sign is opposite to that of the magnetic term, the inertial term reduces $\rho^e$, and hence the particle number density, to zero where $|\omega_{Bkz}|/\Omega = \tfrac{1}{2}(\gamma^3+\gamma)$. So ion and electron zones are, if corotating, separated by thin neighbourhoods of the $B_z = 0$ surfaces, and terminate inside the light cylinder where $\gamma_k \approx (2|\omega_{Bkz}|/\Omega)^{1/3}$ (Wang 1978). The non-corotational part $-\nabla\Phi$ of the electric field, which provides the centripetal force on the corotating particles, produces the inertial term in $\rho^e$ through its contribution to $\nabla\cdot\boldsymbol{E}$; it hence leads to the vanishing of $\rho^e$ when the Wang condition is satisfied (Mestel *et al.* 1979).

Since $\beta$ is a function of $Q$ only, equation (25) for the charge density in terms of the magnetic field can be rewritten as

$$\frac{4\pi c}{\Omega}\frac{\rho^e}{\gamma^2} = -\frac{\beta}{x}B_z - \left\{\left(B_z + \Omega\frac{mc}{e}(\gamma^3+\gamma)\frac{\beta}{x}\right)\frac{\partial Q}{\partial x} - B_{\tilde{\omega}}\frac{\partial Q}{\partial z}\right\}\frac{\mathrm{d}\beta}{\mathrm{d}Q}. \tag{25'}$$

After using equations (18) for $\nabla Q$, equation (17) for $\mathrm{d}\beta/\mathrm{d}Q$ and the complete integral (14), it follows that

$$\rho^{\mathrm{e}} = -\frac{\Omega}{2\pi c}\frac{b+\gamma\beta^2}{b-\gamma^3\beta^2}\gamma^2\frac{\beta}{x}\left\{B_z+\frac{\Omega mc}{2e}\frac{\beta}{x}\left(\gamma^3+\gamma+\frac{(x\omega_B/\Omega)^2}{b+\gamma\beta^2}\right)\right\}. \tag{26}$$

For $\rho^{\mathrm{e}}$ to have a finite zero-inertia limit, the formula (26) shows that $b$ must diverge at least as fast as $1/m$ as $m \to 0$, giving

$$\rho^{\mathrm{e}} \approx -\frac{\Omega}{2\pi c}\gamma^2\frac{\beta}{x}\left(B_z+\frac{e}{2mc}\frac{\beta x}{\Omega b}B^2\right). \tag{27}$$

Equation (26) expresses the charge density in terms of the magnetic field and the flow velocity, both of which are determined by the variable $Q$. An alternative procedure, which leads to a particularly simple result, is to eliminate the magnetic field from equation (25′) in favour of $Q$ by using equations (18) relating $\boldsymbol{B}$ to $\nabla Q$ and $Q$. After using equation (17) for $\mathrm{d}\beta/\mathrm{d}Q$, and the complete integral (14) as well, there results, since $\mathrm{d}\gamma = \gamma^3\beta\,\mathrm{d}\beta$,

$$\frac{4\pi e}{\Omega^2 m}\rho^{\mathrm{e}} = \frac{1}{\beta}\left((\gamma^2-1)\frac{Q}{x^2}+\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}Q}(\tilde{\nabla}Q)^2\right), \tag{28}$$

where $\tilde{\nabla}$ denotes $(c/\Omega)\nabla$. With only a single species of number density $n$ present at any point, this is an equation for $\omega_{\mathrm{p}}^2/\Omega^2$, where $\omega_{\mathrm{p}}$ is the angular plasma frequency $(4\pi e^2 n/m)^{\frac{1}{2}}$.

In principle, equation (16) enables $\gamma$, and hence $\beta$, to be eliminated from (28), giving an expression for $\rho^{\mathrm{e}}$ as a function of $Q$ and $x$. Equation (16) is a quartic for $\gamma$ in terms of $Q$, but gives $Q$ directly in terms of $\gamma$:

$$Q = (\gamma^2-1)^{-\frac{1}{2}}(\gamma^2-1+b\gamma), \tag{29}$$

which is just another form of the complete integral; it follows that

$$\mathrm{d}Q/\mathrm{d}\gamma = (\gamma^2-1)^{-3/2}(\gamma^3-\gamma-b). \tag{30}$$

Hence it is much easier to eliminate $Q$ than $\gamma$ from equation (28), yielding

$$\frac{4\pi e}{\Omega^2 m}\rho^{\mathrm{e}} = \gamma\frac{\gamma^2-1+b\gamma}{x^2}+\frac{\gamma^3-\gamma-b}{(\gamma^2-1)^2}(\tilde{\nabla}\gamma)^2, \tag{31}$$

giving $\rho^{\mathrm{e}}$ in terms of $\gamma$, $\tilde{\nabla}\gamma$ and $x$.

Equation (24) for the magnetic field has the dimensionless form

$$\tilde{\nabla}^2\tilde{P}+2\omega_{Bz}/\Omega = (4\pi e/\Omega^2 m)x\beta\rho^{\mathrm{e}}. \tag{32}$$

On taking $Q$ as the independent variable, using equations (18) relating the magnetic field to $\tilde{\nabla}Q$, equation (32) becomes

$$\tilde{\nabla}^2 Q - Q/x^2 = (4\pi e/\Omega^2 m)\beta\rho^{\mathrm{e}}. \tag{33}$$

Any of the above equations for $\rho^{\mathrm{e}}$, namely (26), (28) or (31), can be substituted into equation (33) to provide the fundamental differential equation of the problem.

After inserting $\rho^e$ from (28), equation (33) can be written in the form

$$\tilde{\nabla}.(\gamma^{-1}\tilde{\nabla}Q) = \gamma Q/x^2. \tag{34}$$

On applying equations (29) and (30), giving $Q$ and $\mathrm{d}Q/\mathrm{d}\gamma$ in terms of $\gamma$, equation (34) becomes

$$\tilde{\nabla}.\{\gamma^{-1}f'(\gamma)\,\tilde{\nabla}\gamma\} = f(\gamma)\,\gamma/x^2, \tag{35}$$

with $f(\gamma)$ and $f'(\gamma)$ defined by the right-hand sides of equations (29) and (30). So the fundamental equation of the problem is now a nonlinear second-order partial differential equation for the Lorentz factor.

## 5. Inside the Light Cylinder

I shall now study toroidal flows that approach corotation on the axis of symmetry: $\beta \to x$ as $x \to 0$. Since $B_{\tilde{\omega}}$ must vanish on the axis, $\tilde{P}$ must approach zero faster than $x$ as $x \to 0$, except perhaps on the equatorial plane; in the dipole approximation, $\tilde{P}$ varies as $x^2$ as $x \to 0$, except on $z = 0$ where it varies as $1/x$. Also $\tilde{\Phi} \to 0$ and $\tilde{\Psi} \to 1$ as $x \to 0$. Hence, for flows in which $\beta \to x$ as $x \to 0$, the complete integral (14) shows that $b = a$, while equation (20) for $\tilde{\Psi}$ shows that $C = 1$.

The integral describing the flow, namely equation (14) with $b = a$, can be written in the forms

$$\frac{\beta}{x} = \frac{1+\gamma\beta^2/a}{1+\tilde{P}/a} \quad \text{and} \quad \varepsilon = \frac{\Omega}{a}\frac{\tilde{P}-\gamma+1/\gamma}{1+\tilde{P}/a}. \tag{36a, b}$$

The integral (36a) shows that the flow is close to corotation so long as the terms $\gamma\beta^2/a$ in the numerator and $\tilde{P}/a$ in the denominator can be neglected. In the dipole approximation, we have $\tilde{P} \approx -x^2\omega_B/\Omega$. Thus the flow is close to corotation so long as

$$x^2 \ll \Omega|\,a/\omega_B\,|. \tag{37}$$

Hence, for the flow to remain close to corotation for a sensible distance from the star, the constant $a$ must be huge, at least of order $\omega_{Bs}/\Omega$; here, $\omega_{Bs} \equiv eB_s/mc$ with $B_s$ denoting the magnetic field strength on the surface of the star at its poles. If $a$ is written as $d\omega_{Bs}/\Omega$, where $d$ is a constant, then $\tilde{P}/a = -(x^2/2d)(R/r)^3$ in the dipole approximation; $r$ is the distance from the centre of the star and $R$ is the star's radius.

The condition $\gamma \ll |\,a\,|$ will be very easily satisfied, implying that the integral (14) with $b = a$ reduces to $\beta \approx a/Q$; that is

$$\beta/x \approx (1+\tilde{P}/a)^{-1}. \tag{38}$$

The flow is sub-rotating or super-rotating according to whether $\tilde{P}/a$ is positive or negative respectively. Since $|\,\tilde{P}/a\,|$ is everywhere small, equation (38) can be further approximated to

$$\beta/x \approx 1-\tilde{P}/a \quad \text{or} \quad \varepsilon \approx \Omega\tilde{P}/a. \tag{39a, b}$$

The condition $\gamma \ll |\,a\,|$ cannot be a sufficient one for equation (38) to be valid: somewhere, extremely close to the light cylinder, that equation will begin to violate the relativity restriction $\beta < 1$. Thus, the condition

$$x < 1+\tilde{P}/a \tag{40}$$

is required. Equations (38) and (39) should be applicable from the rotation axis to slightly beyond the light cylinder or to slightly inside it according to whether $\tilde{P}/a$ is positive or negative respectively.

Use of equation (38) for $\beta$ in equations (20) and (21) for $\tilde{\Psi}$ and $\tilde{\Phi}$, with $b = a$ and $C = 1$, shows that

$$\tilde{\Psi} \approx 1+a\{\tilde{P}/a-\ln(1+\tilde{P}/a)\} \approx 1+\tilde{P}^2/2a, \tag{41a, b}$$

giving

$$\tilde{\nabla}\tilde{\Psi} \approx \tilde{P}(a+\tilde{P})^{-1}\tilde{\nabla}\tilde{P} \approx (\tilde{P}/a)\tilde{\nabla}\tilde{P}, \tag{42a, b}$$

$$\tilde{\Phi} \approx \tilde{\Psi}-\{(1+\tilde{P}/a)^2-x^2\}^{-\frac{1}{2}}(1+\tilde{P}/a-x^2). \tag{43}$$

Use of equation (38) in the force-balance equation (5) shows that

$$\tilde{\nabla}\tilde{\Phi} \approx (1+\tilde{P}/a)^{-1}[\{(1+\tilde{P}/a)^2-x^2\}^{-\frac{1}{2}}x\boldsymbol{i}+(\tilde{P}/a)\tilde{\nabla}\tilde{P}] \tag{44a}$$

$$\approx \{(1+\tilde{P}/a)^2-x^2\}^{-\frac{1}{2}}x\boldsymbol{i}+(\tilde{P}/a)x\Omega^{-1}\boldsymbol{\omega}_B\times\boldsymbol{t}. \tag{44b}$$

The small size of $\tilde{P}/a$ has been used in obtaining equations (41b), (42b) and (44b).

So long as $x$ is not extremely close to one, equations (43) and (44b) reduce to

$$\tilde{\Phi} \approx 1-(1-x^2)^{\frac{1}{2}}+\tilde{P}^2/2a, \tag{45}$$

$$\tilde{\nabla}\tilde{\Phi} \approx (1-x^2)^{-\frac{1}{2}}x\boldsymbol{i}+(\tilde{P}/a)x\Omega^{-1}\boldsymbol{\omega}_B\times\boldsymbol{t}. \tag{46}$$

When the terms involving $\tilde{P}$ in equations (41)–(46) are neglected, the results corresponding to pure corotation are recovered.

These flows require further study, but the preliminary investigation of this section is sufficient for certain inferences to be drawn.

## 6. Implications for Model Building

The standard Goldreich–Julian (1969) model of the pulsar magnetosphere features zones of corotating electrons and positive ions, in which particle inertia and all forces other than the Lorentz force are neglected, implying that $\boldsymbol{E}$ is orthogonal to $\boldsymbol{B}$. The electron and ion zones are separated by the $B_z = 0$ surfaces, corresponding to $\cos^2\theta = \frac{1}{3}$ in the dipole approximation. Jackson (1978) pointed out that this configuration is unstable to charge mixing. He studied the infinitesimal additional electrical fields introduced by infinitesimal perturbations to this configuration. The resulting component $E_\parallel$ of the electric field along $\boldsymbol{B}$ is, in the neighbourhood of the $B_z = 0$ surfaces, directed so as to force the electrons into the ion zone and the ions into the electron zones. Jackson argued that recombination would occur in those regions; the resulting neutral particles would be removed by gravity and could not be replaced from the star.

When particle inertia is allowed for, with the assumption of corotation retained, the function $\Phi$ must be introduced in order to support the motion. The resulting $\Phi$ is proportional to the mass-to-charge ratio of the species: it is negative and comparatively small in the electron zones and positive and very much larger in the ion zone. Thus, there is a big jump in $\Phi$ between the zones, and the resulting electric field is directed so as to accelerate electrons into the ion zone and ions into the electron zones (Burman 1981*a*, 1981*b*).

So far, this is consistent with Jackson's (1978) work. But, whereas a stability analysis of the kind he used is intrinsically unable to give the magnitude of the effect, other work (Burman 1981*a*, 1981*b*) showed that the effect of inertia is very large: beyond $x \approx 1/30$, the jump in $\Phi$ is sufficient to accelerate the electrons to relativistic speeds. The implications of these results for model building are quite different from those advocated by Jackson.

Because of the rapid particle acceleration across the thin regions between the electron and ion zones, very little recombination is to be expected—rather, processes of acceleration and mirroring of particles occur, with electrons penetrating the ion zone and ions penetrating the electron zones. The electron-dominated and ion-dominated zones are separated, not by gaps, but by inertial boundary layers (Burman 1981*c*) of reduced density because of the increased poloidal velocities. The mirroring provides the means to keep the electron and ion zones replenished.

The mirroring is not by the usual mechanism of magnetic mirroring in regions of converging magnetic field lines—rather, the electrons undergo electrical mirroring, caused by the $E_{\|}$ fields, in the northern and southern boundary layers, oscillating between the two, while the ions oscillate between electrical mirroring in the boundary layers and centrifugal mirroring closer to the star. Both mirroring mechanisms owe their origin to inertia: while the centrifugal mirroring is directly inertial, the electrical mirroring is indirectly so, since the $E_{\|}$ fields causing it have been set up because of the differing inertia of the toroidally flowing electrons and ions.

But what is the result of removing the constraint of corotation—which, after all, is now seen to represent but a singular solution—while retaining that of toroidal flow? The equations of the last section provide the answer: the fundamental problem of mismatch of $\Phi$ and its derivatives between electron and ion zones remains. The form of $\Phi$ is not so simple as in the corotational case, and is not just proportional to the mass-to-charge ratio. But in the expanded forms of $\Phi$ and $\nabla\Phi$, some of the terms do have that dependence; again, $\Phi$ and its gradient cannot be matched between electron and ion zones.

With $a$ written as $d\omega_{Bs}/\Omega$, equations (45) and (46) take the forms

$$\Phi \approx (mc^2/e)\{1-(1-x^2)^{\frac{1}{2}}\}+(\Omega/2cdB_s)(\Omega P/c)^2\,, \tag{47}$$

$$\nabla\Phi \approx (\Omega mc/e)(1-x^2)^{-\frac{1}{2}}x\boldsymbol{i}+(x/d)(\Omega^2 P/c^2)\boldsymbol{B}\times\boldsymbol{t}/B_s\,. \tag{48}$$

If $d$ is independent of the species, then the jump in $\Phi$, and the corresponding $E_{\|}$, between electron and ion zones are much the same as with precisely corotating zones: relaxing the assumption of corotation does not alter the physical implications of my earlier analysis (Burman 1981*a*, 1981*b*).

Consider the component $F_{\|}$ of the Lorentz force parallel to the magnetic field acting on a stray electron in the ion zone or on a stray positive ion in an electron zone. Equation (48) shows that, in both cases, $F_{\|}$ has the sign of $B_{\tilde{\omega}}$: it is parallel to $\boldsymbol{B}$ in the northern (magnetic) hemisphere and antiparallel in the southern. The centrifugal effect experienced by the electron will be negligible in comparison with the electric force, which has arisen through the need to support the centrifugal effect on the ions. The centrifugal effect on the stray ion will act in support of $F_{\|}$.

Once in the ion zone, the electrons experience a Lorentz force component $F_{\|}$ directed so as to accelerate them deeper into the ion zone, until they cross into the opposite hemisphere where $F_{\|}$ will decelerate them. They will travel approximately along magnetic field lines, mirroring between the northern and southern boundaries separating the electron and ion zones.

Once they have penetrated into an electron zone, the ions will be decelerated by a combination of the Lorentz force component $F_{\parallel}$ and the centrifugal effect, mainly the latter. They will travel approximately along magnetic field lines, mirroring between some point above the stellar surface and the boundary separating the electron and ion zones.

It is easily seen that the usual idea of introducing a vacuum gap between the electron and ion zones cannot be used to overcome the matching difficulty. In an axisymmetric magnetosphere, we have $\Phi = \phi + \Omega P/c$. In a vacuum region, the scalar potential $\phi$ satisfies Laplace's equation. So, in a vacuum gap stretching indefinitely away from the star, $\phi$ can be represented as a series in inverse powers of $r$, beginning with the quadrupole term varying as $1/r^3$, but with a $1/r$ term if there is a net charge. The magnetic stream function has a similar series expansion, beginning with the dipole term varying as $1/r$. Thus, the behaviour of the potential $\Phi$ as a function of $r$ is nothing like that described by equation (47) for $\Phi$ in a toroidally flowing zone near the star: matching between such a zone and a vacuum gap is clearly impossible.

It might be feasible to circumvent this argument by making the vacuum gap terminate at some finite distance from the star, so enabling terms involving positive powers of $r$ to be introduced into $\Phi$ in the gap. But, in any case, introduction of a vacuum gap is unphysical. The forces in the boundary regions between zones are in the wrong direction, acting to produce charge mixing instead of charge separation, and they accelerate the particles too powerfully to allow much recombination to occur.

Proposed models of the axisymmetric pulsar magnetosphere invariably have zones occupied solely by toroidally flowing particles. There are almost always, in addition, zones containing outflowing or poloidally circulating particles emitted from the star's polar caps. A few authors (e.g. Michel 1980; Michel and Pellat 1981) have advocated purely rotational models, in which there is no poloidal flow at all.

But it is clear from the above discussion that there can be no zones occupied solely by toroidally flowing particles: although the toroidal flow condition can be satisfied locally, global considerations imply that poloidal flow is endemic. In particular, the purely rotational models do not work.

It is interesting that no explicit use of any dynamical boundary condition at large distances from the star has been required in order to deduce the physical picture discussed here. Solutions satisfying the boundary conditions on the star form only a single-parameter family, and the qualitative behaviour is determined.

## 7. Comparison with Other Work

Jackson (1980, 1981) studied purely toroidal flows using the approximation in which the component $E_{\parallel}$ of the electric field parallel to the magnetic field vanishes in the plasma. This approximation arises when inertia, and all forces other than the Lorentz force, are neglected, so that the equation of motion of a species reduces to the statement that the Lorentz force on it is zero. Thus, the differential equation of motion is approximated by a simple algebraic equation. Jackson dealt with both finite and infinite axisymmetric magnetospheres, and took $E_{\parallel}$ to vanish throughout the plasma.

As a result of these studies, Jackson (1981) claimed that the angular speed of the flow must go to zero on the symmetry axis, a notion which is in conflict with the behaviour described by the complete integral above. I shall now discuss the reason for this disagreement.

The precise result established by Jackson can be described as follows. Consider an axisymmetric pulsar magnetosphere of infinite extent, in which the flow is purely toroidal, of angular speed $\alpha$, and the magnetic field, with stream function $P$, has dipolar form at large distances from the star. Application of the $E_{\parallel} = 0$ approximation in which $\alpha$ is a function of $P$ only, throughout the magnetosphere, together with the requirement that the flow speed must remain below $c$ at large distances from the star, implies that (Jackson 1981, Section III)

$$\lim_{P \to 0} \{\alpha(P)/P\} \leqslant M < \infty, \tag{49}$$

where $M$ is a constant. It appears from this that $\alpha(P)$ must tend to zero at least as fast as $P$ as the surface $P = 0$ is approached; this surface consists of the symmetry axis together with a surface at infinity.

The essential point is that the neglect of inertia in the $E_{\parallel} = 0$ approximation means that the flow speeds are not automatically kept below $c$, so some step must be taken in order to satisfy this requirement. In the Goldreich–Julian (1969) model, which is based on the same approximation, it is the need to keep the speeds outside the light cylinder below $c$ that leads to the existence of a stellar wind, as Mestel *et al.* (1979) emphasized. Goldreich and Julian imposed the boundary condition of perfect conductivity on the stellar surface; as a result, the $E_{\parallel} = 0$ approximation yields the isorotation law $\boldsymbol{u} \approx \kappa \boldsymbol{B}$ for each species, where $\kappa$ is a scalar: the lines of the reduced flow velocity coincide with those of the magnetic field. The toroidal part of this relation shows that, beyond the light cylinder, $\kappa$ must be nonzero in order to keep $v_{\phi}$ below $c$. Hence, the poloidal part of the isorotation law shows that there is necessarily a poloidal flow beyond the light cylinder. Jackson did not impose the boundary condition of perfect conductivity on the stellar surface and was led, not to a poloidal flow, but to the limit (49).

It should perhaps be emphasized that the Jackson limit (49) arose from the *global* imposition of the $E_{\parallel} = 0$ approximation. It would not arise from neglecting inertia just where the flow speed is small.

In my work, since relativistic inertia is fully incorporated, the flow speeds are automatically kept below $c$, except for the singular integral corresponding to corotation, a solution which must be restricted to the region inside the light cylinder. Yet my complete integral does not show $\alpha$ to have the behaviour that Jackson deduced for it. What is the explanation?

The reason lies in taking the apparent freedom of choice of the functional form $\alpha(P)$ in the $E_{\parallel} = 0$ approximation at its face value. That apparent freedom is illusory: it is not that $\alpha(P)$ was left free in Jackson's work but that it was undetermined. The correct interpretation is that there does exist a specific functional form of $\alpha(P)$ appropriate to the $E_{\parallel} = 0$ limit, but the equations of that limit are too weak to determine it.

In contrast, the work set out in the present paper, with relativistic inertia fully incorporated, fully determines the functional form of $\alpha$, leaving only two constants free to be chosen to satisfy boundary conditions.

To study the zero-inertia limit of the rigorous theory, one can regard $\gamma$ in the complete integral (14) as a parameter labelling the inertial term and let it go to zero, giving

$$\alpha \approx \Omega b/(\tilde{P} + a). \tag{50}$$

Whereas the equations of the $E_{\parallel} = 0$ approximation leave $\alpha$ an undetermined function of $P$, approaching that approximation as a limit of the rigorous theory determines the functional form of $\alpha$ appropriate to that approximation. Equation (50) gives the required functional form. It shows that $\alpha(P)/P$ diverges as $P \to 0$: the Jackson limit (49) is violated. Equation (50) shows that, if the plasma extends to the axis, then the constant $a$ must be nonzero so that $\alpha$ will remain finite there. It follows that $\alpha$ must take the finite nonzero value $\Omega b/a$ on the axis.

## 8. Concluding Remarks

The purpose of this paper has been to report a study of toroidal flows that are not restricted to corotation with the star. The general analysis has been exact within the constraints of steady rotation, axisymmetry and no dissipation, representing, for each species, a balance between the Lorentz force and relativistic inertia. I have shown that the flows are described by a complete integral, together with a singular solution corresponding to precise corotation with the star. This makes the full range of such flows now available for study.

The complete integral and singular solution follow from a quasilinear first-order partial differential equation that toroidal flows must satisfy, so demonstrating the existence of an underlying quasilinear structure to this highly nonlinear problem.

I have used the complete integral to show that there can be no zones occupied solely by toroidally flowing particles, though there may well be zones dominated by particles whose motion is largely toroidal. It is not surprising that the qualitative implications of this analysis of toroidal flow are just the same as in my earlier work based on corotation: it was always intuitively clear that the electric forces were simply too great for any reasonable toroidal departure from corotation to have a major effect. But it is, perhaps, more compelling to see the conclusion emerge from a rigorous analysis of general toroidal flow.

The real problem in understanding the pulsar magnetosphere is not the Goldreich–Julian one of satisfying the boundary conditions on the star: it is that of matching regions dominated by different species.

The complete integral extends by one the list of known integrals of the motion (Westfold 1981). These will undoubtedly feature prominently in the process of obtaining a self-consistent pulsar magnetosphere model. Recent work of mine has shown that knowledge of the complete integral for purely toroidal flow, invoked as a limiting case, is very helpful, perhaps essential, in developing the full flow equations for axisymmetric magnetospheres.

## Acknowledgment

I thank the referees for their comments, which have led to a considerable improvement in the presentation of this paper.

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Manuscript received 21 December 1982, accepted 23 August 1983