Aust. J. Phys., 1984, **37**, 255–62

Nucleus–Nucleus Elastic Scattering at Ultra-high Energies

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Abstract

Nucleus–nucleus elastic scattering at ultra-high energy is studied within the scheme of Glauber multiple scattering and also within the geometrical picture of the Chou–Yang model. Both models offer a reasonable explanation of a recent CERN experiment on α–α elastic scattering and their differences are discussed. Results are presented for α–α elastic scattering at ultra-high energies, and for α–^{9}Be, α–^{12}C, α–^{16}O and α–^{24}Mg elastic scatterings at currently accessible energies. The A dependence of the total cross section is approximately A^n with $n \approx 0{\cdot}55$.

1. Introduction

The recent measurement of α–α elastic scattering at CERN (Faessler 1981, 1982; Bell *et al.* 1982; Ambrosio *et al.* 1982) indicates that the era of nucleus–nucleus scattering at ultra-high energies has dawned, and raises the interesting question of how to explain such scattering theoretically. The traditional Glauber (1959) multiple-scattering theory (see Li *et al.* 1981 and references therein) is quite capable of explaining p–d and p–α elastic scattering, and probably also all proton–nucleus (p–A) elastic scattering (Li and Lo 1982). In principle there should be no difficulty in extending the same technique to all nucleus–nucleus elastic scattering ($\mathrm{A+B \to A+B}$, where the nucleon number for both nuclei is $\geqslant 4$). However, in practice it is necessary to make a further approximation to facilitate easy calculation and this is done in Section 2. In Section 3 we suggest an alternative method: the geometrical model of Chou and Yang (Chou 1968). The Chou–Yang method is much simpler to use, and can explain all salient features of α–α elastic scattering as adequately as the Glauber (1959) multiple-scattering theory. We use it to calculate $\alpha+\mathrm{A}$ elastic scattering for nucleon number $A = 8, 12, 16, 24$.

2. Multiple Scattering

In the framework of the Glauber multiple-scattering theory, the amplitude of nucleus–nucleus scattering $\mathrm{A+B \to A^+ + B^+}$ is

$$F_{\mathrm{AB}}(\boldsymbol{q}) = \theta_{\mathrm{AB}}(\boldsymbol{q})\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2 b \exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{b})\langle \mathrm{f}\,|\,\Gamma(b;\,\boldsymbol{s}_1,\ldots \boldsymbol{s}_{\mathrm{A}};\,\boldsymbol{s}_1,\ldots \boldsymbol{s}_{\mathrm{B}})\,|\,\mathrm{i}\rangle\,, \tag{1}$$

where

$$\Gamma(b;\,\boldsymbol{s}_1,\ldots \boldsymbol{s}_{\mathrm{A}};\,\boldsymbol{s}_1,\ldots \boldsymbol{s}_{\mathrm{B}}) = 1 - \prod_{j=1}^{A}\prod_{k=1}^{B}\{1-\Gamma_{jk}(\boldsymbol{b}-\boldsymbol{s}_j^{\mathrm{A}}+\boldsymbol{s}_k^{\mathrm{B}})\}\,, \tag{2}$$

0004-9506/84/030255$02.00

and where Γ_{jk} is the profile function of two nucleons, $\boldsymbol{b}$ is the impact parameter, $\boldsymbol{s}_j^{\mathrm{A}}$ and $\boldsymbol{s}_k^{\mathrm{B}}$ are the projections of the coordinates of particles j and k onto a plane perpendicular to the incident direction and $\boldsymbol{q}$ is the momentum transfer. Particles j and k belong to nucleus A and nucleus B respectively. The initial and final states of the A+B system are denoted by $|\mathrm{i}\rangle$ and $|\mathrm{f}\rangle$ and $\theta_{\mathrm{AB}}(\boldsymbol{q})$ is the centre-of-mass correction factor. For elastic scattering, we make the approximation

$$\psi_{\mathrm{i}}^{*}\psi_{\mathrm{i}} = \prod_{j=1}^{A} \rho_{\mathrm{A}}(\boldsymbol{r}_j) \prod_{k=1}^{B} \rho_{\mathrm{B}}(\boldsymbol{r}_k),$$

where ψ_i is the wavefunction of A+B, and $\rho_{\mathrm{A}}(\boldsymbol{r}_j)$ and $\rho_{\mathrm{B}}(\boldsymbol{r}_k)$ are the nucleonic densities of nuclei A and B.

Let us discuss specifically α–α elastic scattering. We take the nucleonic density of the α from electron–α elastic-scattering experiments to be (Li and Lo 1982)

$$\rho(r) = N\{\exp(-k_1^2 r^2) - c\exp(-k_2^2 r^2)\}. \tag{3}$$

It is clear that some additional approximation is needed to carry out calculations with equation (2). We used the so-called rigid projectile approximation (RPA). We consider α–α elastic scattering to be composed of a sum of α–nucleon multiple elastic scatterings (Varma 1978). The α–nucleon scattering amplitude is taken to be (Li *et al.* 1981; Li and Lo 1982)

$$F_{\alpha\mathrm{p}}(\boldsymbol{q}) = \theta(\boldsymbol{q})\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2 b\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{b})[1-\{1-\Gamma_{\mathrm{NN}}(\boldsymbol{b})\}^4], \tag{4}$$

where the profile function is

$$\Gamma_{\mathrm{NN}}(\boldsymbol{b}) = \frac{1}{2\pi\mathrm{i}}\int \rho(\boldsymbol{r})\,\mathrm{d}^3 r\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{s})\int \mathrm{d}^2 q\exp(-\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{b})\,f_{\mathrm{NN}}(\boldsymbol{q}) \tag{5}$$

and the nucleon–nucleon elastic-scattering amplitude is chosen empirically as

$$f_{\mathrm{NN}}(\boldsymbol{q}) = \{\mathrm{i}\sigma(1-\mathrm{i}\rho_{\mathrm{i}})/4\pi\}\exp(-\tfrac{1}{2}\beta^2 q^2) = f(0)\exp(-\tfrac{1}{2}\beta^2 q^2). \tag{6}$$

Substituting equations (3), (5) and (6) into equation (4), one gets the analytic expansion

$$\begin{aligned} F_{\alpha\mathrm{p}}(\boldsymbol{q}) = {} & -\tfrac{1}{2}\mathrm{i}\exp(q^2/16k_1^2)\sum_{m=1}^{4}\sum_{n=0}^{m} C_m^4 C_n^m \\ & \times\left(\frac{\sigma(1-\mathrm{i}\rho)}{2\pi}\frac{k_1}{1+2k_1^2\beta^2}\right)^m A_1^n\left(A_2\frac{k_1^2(1+2k_1^2\beta^2)}{k_1^2(1+2k_2^2\beta^2)}\right)^{m-n} \\ & \times\left(\frac{nk_1^2}{1+2k_1^2\beta^2}+\frac{(m-n)k_2^2}{1+2k_2^2\beta^2}\right)^{-1} \\ & \times\exp\left\{-\tfrac{1}{4}q^2\left(\frac{nk_1^2}{1+2k_1^2\beta^2}+\frac{(m-n)k_2^2}{1+2k_2^2\beta^2}\right)^{-1}\right\}, \end{aligned} \tag{7}$$

where $A_1 = k_2^3/(k_2^3 - ck_1^3)$ and $A_2 = ck_1^3/(k_2^3 - ck_1^3)$. Then in the RPA the α–α scattering amplitude becomes

$$F_{\alpha\alpha}(\boldsymbol{q}) = \exp(q^2/16k_1^2)\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2 b\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{b})\,\Gamma(b), \tag{8}$$

where

$$\Gamma(b) = 1-\{1-\Gamma_{\alpha\mathrm{p}}(b)\}^4, \tag{9}$$

with

$$\Gamma_{\alpha p}(b) = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^2 q \exp(\mathrm{i}\boldsymbol{q}.\boldsymbol{b}) F_{\alpha p}(q)\, S(\boldsymbol{q}), \tag{10}$$

$$S(\boldsymbol{q}) = A_1 \exp(-q^2/4k_1^2) - A_2 \exp(-q^2/4k_2^2). \tag{11}$$

The formulae above look complicated but they are easier to use than equation (1) and actually contain no free parameters. All the parameters are either known from the shape of the α particle, or can be taken from the nucleon–nucleon elastic scattering parameters. They are

$$k_1^2 = 0{\cdot}0288\ \mathrm{GeV}^{-2}, \qquad k_2^2 = 0{\cdot}144\ \mathrm{GeV}^{-2}, \qquad c = 1{\cdot}0, \tag{12}$$

and for α–α scattering at $s^{\frac{1}{2}} = 126$ GeV, we take the nucleon–nucleon scattering parameters as

$$\sigma = 41\ \mathrm{mb}, \qquad \beta^2 = 14\,(\mathrm{GeV}/c)^{-2}, \qquad \rho_{\mathrm{i}} = -0{\cdot}2. \tag{13a, b, c}$$

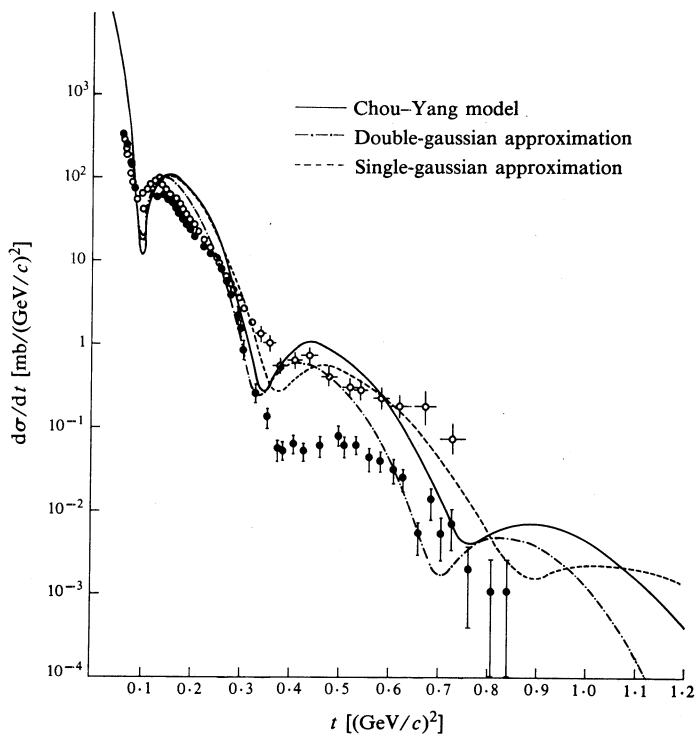

Fig. 1. Differential cross sections for α–α elastic scattering at $s^{\frac{1}{2}} = 126$ GeV. Data are from Faessler (1981) (solid circles) and from Ambrosio *et al.* (1982) (open circles). Curves are theoretical and show the prediction of the Chou–Yang model, and the Glauber model with double- and single-gaussian approximations.

The result is shown in Fig. 1. The first dip is correctly given at $|t| = 0{\cdot}1\ \mathrm{GeV}^2$ where t is the momentum transfer. The height of the second and third peaks are quite close to experimental values. We expect a third dip at $|t| = 0{\cdot}7$–$0{\cdot}9\ \mathrm{GeV}^2$. The data gathered at CERN ($s^{\frac{1}{2}} = 88$ and 126 GeV) almost cover this range; further experimental work may uncover this third dip. From the optical theorem we also obtained the total cross sections, which are compared with experiment in Table 1.

Table 1. Theoretical and experimental total cross sections for α–α scattering

σ_{NN} is the parameter which specifies energy

σ_{NN} (mb)	σ_T (mb) (Theor.)	σ_T (mb) (Exp.)
41	332	295 ± 40
60	400	
80	500	
120	520	
200	605	

Fig. 1 also shows the curve for a single-gaussian approximation of the density function ρ ($k_1^2 = 0{\cdot}0288\ \mathrm{GeV}^{-2}$, $c = 0$). It turns out that the difference between the calculations using single- and double-gaussian fits for the density ρ is minimal at small momentum transfer, and grows larger as the momentum transfer increases. Hence if one is only interested in the total cross section and the slope parameter of the first peak, the single-gaussian approximation is good enough. There is no need to be more sophisticated in the parametrization of the density function ρ. On the other hand, if one wants more accuracy at large momentum transfer, say $t \geqslant 1\ \mathrm{GeV}^2$, perhaps even a double-gaussian fit of ρ may not be sufficient. We suggest that the remaining small deviation of our fits from the data can be easily attributable to the

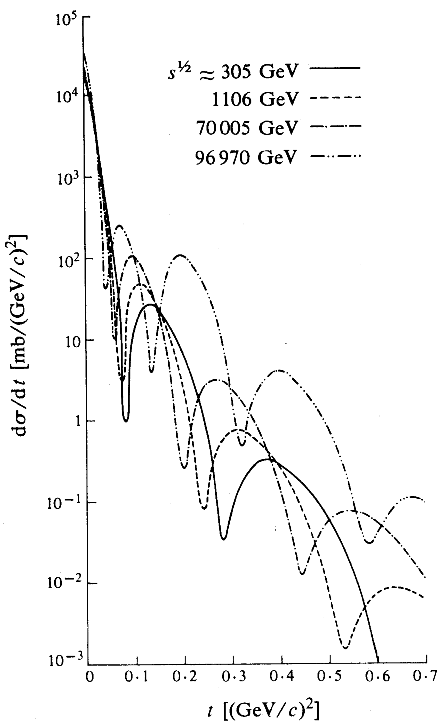

Fig. 2. Differential cross sections for α–α elastic scattering at ultra-high energies.

inability of equation (8) to represent adequately the nucleonic matter distribution of the α particle.

The above formulae are also valid for higher energies ($\geqslant 1$ TeV). The only energy-dependent parameter comes from the nucleon–nucleon elastic-scattering amplitude, whereas the shape of the nucleus and the mechanism of multiple scattering should be energy-independent. We calculated the differential cross section of α–α elastic scattering for higher energies, as shown in Fig. 2. The total cross sections for nucleon–nucleon scattering ($\sigma_{\mathrm{tot}} = 60, 80, 120, 200$ mb) and the corresponding slope parameters [$\beta^2 = 14{\cdot}73, 20{\cdot}33, 21{\cdot}79, 34{\cdot}67\ (\mathrm{GeV}/c)^{-2}$] serve as energy parameters, i.e. the larger the total cross section, the higher the energy. If one uses the formula

$$\sigma_{\mathrm{tot}}(\mathrm{pp}) = 38{\cdot}4 + 0{\cdot}49\ln^2(s/122),$$

the corresponding $s^{\frac{1}{2}}$ are 305, 1106, 70005 and 96970 GeV. As the energy increases, the differential cross section increases, the dip positions shift to smaller $|t|$ values and, for a given $|t|$ range, one sees more dips and peaks. We may see up to four dips at $s^{\frac{1}{2}} = 10^5$ GeV for the range $0 \leqslant |t| \leqslant 0{\cdot}7\ \mathrm{GeV}^2$. The slope parameters β^2 have been taken from the pp elastic-scattering calculations of Chou and Yang (1980) and Clarke and Lo (1974). In any case, these values seem a very reasonable and smooth extrapolation of present data.

3. Geometrical Model

For nucleus–nucleus elastic scattering, the larger the nucleon number A of the nucleus, the harder it is to do Glauber multiple-scattering calculations. As we showed in the previous section, even for $A = 4$ there must be some additional theoretical assumption before a numerical calculation can be made.

In this section we shall elaborate the geometrical or Chou–Yang model (Chou 1968; Lo and Lai 1977) to account for nucleus–nucleus elastic scattering. The physical picture is to consider the nucleus consisting of a continuous distribution of matter. The boundaries between nucleons are insignificant. As the energy increases, the Lorentz contraction makes the nuclei appear to each other as discs with a vanishing thickness. The interaction between the nuclei becomes the interaction between the nucleon matter in discs; there is virtually no distinction in principle between nucleus–nucleus scattering and nucleon–nucleon scattering. The practical differences are that the size of a nucleus is larger, and the interaction strength is A times that of a nucleon.

The scattering amplitude of $\mathrm{A}+\mathrm{B} \to \mathrm{A}+\mathrm{B}$ is therefore given by

$$a = \frac{1}{2\pi}\int \mathrm{d}^2 b \exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{b})\,[1-\exp\{-\Omega(b)\}], \tag{14}$$

where the opaqueness function is

$$\Omega(b) = \mu_{\mathrm{AB}}\int F_{\mathrm{A}}(q^2)\,F_{\mathrm{B}}(q^2)\exp(-\boldsymbol{q}\,.\,\boldsymbol{b})\,\mathrm{d}^2 q/2\pi. \tag{15}$$

The form factors $F_{\mathrm{A}}(q^2)$ and $F_{\mathrm{B}}(q^2)$ are those of the two scattering nuclei A and B. The interaction strength is

$$\mu_{\mathrm{AB}} = \mu_{\mathrm{NN}}\,AB, \tag{16}$$

where μ_{NN} is the interaction strength of nucleon–nucleon elastic scattering, and AB is the product of the nucleon numbers of the two colliding nuclei. To allow a real part to the scattering amplitude (14), one takes μ_{NN} to be complex:

$$\mu_{NN} \approx \sigma(1 - \mathrm{i}\rho_{\mathrm{i}})/8\pi^2, \tag{17}$$

where σ and ρ_{i} have the numerical values discussed in Section 2 (equations 13). For α–α scattering we have $A = B = 4$ and the form factor F_α is the Fourier transform of the density function ρ of equation (3), i.e. $S(\boldsymbol{q})$ of (11). The result is also shown in Fig. 1 (solid curve). It is clear that the curve is as good a fit as the one obtained from the Glauber multiple-scattering theory.

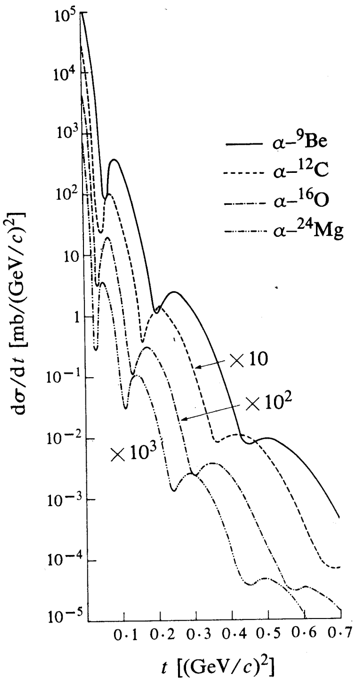

Fig. 3. Theoretical predictions for differential cross sections of α–nucleus elastic scattering.

It is extremely easy to extend our calculation to heavier nuclei. We chose a few examples which lend themselves more readily to experiment, namely

$$\alpha\text{–}^9\mathrm{Be}, \quad \alpha\text{–}^{12}\mathrm{C}, \quad \alpha\text{–}^{16}\mathrm{O}, \quad \alpha\text{–}^{24}\mathrm{Mg}.$$

The differential cross sections for elastic scattering for these cases are plotted in Fig. 3. We take the form factor as

$$F_{\mathrm{A}} = \exp(-q^2/4\alpha_{\mathrm{A}}^2), \tag{18}$$

where $\alpha_{\mathrm{A}}^2 = 0{\cdot}0102, 0{\cdot}0094, 0{\cdot}0079$ and $0{\cdot}0071$ (GeV/c)^2 for ^{9}Be, ^{12}C, ^{16}O and ^{24}Mg respectively and the α-particle form factor is given by $S(\boldsymbol{q})$ of equation (11). There is no adjustable parameter. The radii of ^{9}Be, ^{12}C, ^{16}O and ^{24}Mg are given by elastic electron–nucleus scattering (Landolt and Börnstein 1967) from which one obtains the values α_{A}^2.

The quantity μ_{NN} is chosen such that the nucleon–nucleon total cross section is 41 mb, as in equation (13a). Hence the scale shown in Fig. 3 is absolute. From the optical theorem we also obtained the total cross sections for these cases of α–nucleus scattering (see Table 2). The A-dependence of these total cross sections, $\sigma_T \sim A^n$, gives an index $n \approx 0 \cdot 55$. One notes that it is far from a linear dependence, and is in fact smaller than $A^{2/3}$. It would be most interesting to measure the A-dependence in these processes.

Table 2. Theoretical total cross sections for α–nucleus scattering

Parameters are $\sigma_{NN} = 41$ mb and $\beta^2 = 14\ (\mathrm{GeV}/c)^{-2}$

System	σ_T (mb)
α–^{9}Be	626
α–^{12}C	777
α–^{16}O	936
α–^{24}Mg	1140

Because of the simple form of equations (14) and (15), one easily observes that the elastic cross sections of two nucleus–nucleus scatterings,

$$A_1 + B_1 \to A_1 + B_1, \qquad A_2 + B_2 \to A_2 + B_2,$$

are equal if

$$A_1 B_1 = A_2 B_2, \tag{19}$$

$$F_{A_1}(q^2) F_{B_1}(q^2) = F_{A_2}(q^2) F_{B_2}(q^2), \tag{20}$$

where $A_{1,2}$ and $B_{1,2}$ are their nucleonic numbers respectively.

Table 3. Equalities among pairs of nucleus–nucleus scatterings

	^{12}C+^{12}C	^{11}B+^{13}C	^{12}C+^{16}O	^{11}B+^{18}O	^{12}C+^{20}Ne	^{13}C+^{18}O
AB	144	143	192	198	240	234
$\alpha_A^{-2} + \alpha_B^{-2}$ $(\mathrm{GeV}/c)^{-2}$ *	8·55	7·50	9·32	9·03	9·62	8·73

* Defined by equation (18) with values from Landolt and Börnstein (1967).

We found that the following pairs of nuclei seem to obey equation (19) to within approximately 20%:

$$^{12}\mathrm{C} + {}^{12}\mathrm{C} \quad \text{and} \quad {}^{11}\mathrm{B} + {}^{13}\mathrm{C},$$

$$^{12}\mathrm{C} + {}^{16}\mathrm{O} \quad \text{and} \quad {}^{11}\mathrm{B} + {}^{18}\mathrm{O},$$

$$^{12}\mathrm{C} + {}^{20}\mathrm{Ne} \quad \text{and} \quad {}^{13}\mathrm{C} + {}^{18}\mathrm{O}.$$

Values of AB and $\alpha_A^{-2} + \alpha_B^{-2}$ for these pairs are listed in Table 3. It would be most interesing to test these equalities in future experiments; they would be very difficult to understand within the framework of the Glauber multiple-scattering theory. And, in the black-disc approximation, there would be no need for the strength of the interaction ($A_1 B_1 = A_2 B_2$) to come into play. The nucleus–nucleus scatterings would have equal differential cross sections whenever the sums of their radii squared are equal, which is the content of equation (20). Hence equations (19) and (20) are unique to the Chou–Yang model.

4. Conclusions

Our calculations suggest that nucleus–nucleus scattering at ultra-high energies could show considerable structure. As the energy is increased, there would be more dips and peaks at a given t interval, and the dips would move to smaller $|t|$ values while the peaks would grow larger. We have also shown that, in the case of larger nuclei, the Chou–Yang geometrical model is simpler to apply than the Glauber multiple-scattering model.

Acknowledgment

We thank Dr J. W. G. Wignall for a careful reading of the manuscript.

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Manuscript received 3 June 1983, accepted 10 February 1984