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The Steady Currents Driven in a Conducting Sphere Placed in a Rotating Magnetic Field

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Abstract

The steady currents (and the associated steady magnetic fields) generated in a conducting sphere placed in a rotating magnetic field are calculated in the weakly nonlinear limit. It is found that the steady driven current has a poloidal and a toroidal component. The steady toroidal magnetic field associated with the driven poloidal current has opposite senses above and below the equatorial plane; the net toroidal flux is zero. The relevance of this result to some recent observations in the Rotamak experiment is discussed.

1. Introduction

The force per unit volume acting on the electron fluid in a conducting object is given by

$$\boldsymbol{F} = -ne\boldsymbol{E} + \boldsymbol{J}\times \boldsymbol{B}, \tag{1}$$

where n is the electron number density, e is the electron charge and $\boldsymbol{J}$ is the current density, and we have assumed that the ion contribution to the current is negligible. The ratio between the Hall force $\boldsymbol{J}\times\boldsymbol{B}$ and the electric force $ne\boldsymbol{E}$ is approximately given by

$$\varepsilon \sim JB/neE \sim B/ne\eta\,, \tag{2}$$

where η is the resistivity. By using the classical formula for the resistivity

$$\eta = m_{\rm e}\,\nu_{\rm ei}/ne^2 \tag{3}$$

($m_{\rm e}$ is the electron mass and $\nu_{\rm ei}$ is the electron–ion momentum transfer collision frequency), it can be shown that

$$\varepsilon \sim \omega_{\rm ce}/\nu_{\rm ei}\,, \tag{4}$$

where $\omega_{\rm ce} = eB/m_{\rm e}$ is the electron cyclotron frequency. For metallic conductors $\nu_{\rm ei}$ is large, ε is always much smaller than one and the Hall force is rightfully ignored. However, ε can be larger than one for semiconductors, gas discharges and cosmic and laboratory plasmas. The Hall term becomes dominant in these situations and it has to be retained.

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It is well known that screening currents are induced in conducting objects placed in time-varying magnetic fields. The Hall force resulting from the nonlinear interaction between the time-varying magnetic fields and the screening currents associated with them has in general a steady part as well as a time-varying part. The Hall force drives a 'Hall current' in a system of finite dimensions provided that its curl is nonzero. The steady part of the Hall current can be much larger than the time-varying part since it is limited only by the resistance of its path, whereas the time-varying Hall current is limited by the inductance as well as the resistance of its path. Again we stress that this Hall current is not significant for metallic conductors (for which $\varepsilon \ll 1$) but can be appreciable for semiconductors and plasmas.

The use of rotating magnetic fields to drive steady currents in plasmas (Blevin and Thonemann 1962; Davenport *et al.* 1966; Hugrass *et al.* 1981) is an application of this effect. Consider an infinitely long plasma cylinder, to which is applied a uniform transverse magnetic field that rotates about the axis of the cylinder at an angular frequency ω:

$$\boldsymbol{B} = B_\omega \cos(\omega t-\theta)\,\hat{\boldsymbol{r}} + B_\omega \sin(\omega t-\theta)\,\hat{\boldsymbol{\theta}}\,, \tag{5}$$

where r, θ and z are the standard cylindrical coordinates and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are the corresponding unit vectors. An axial screening current J_z is induced in the plasma, and the effect of this current is to limit the penetration of the rotating field in the plasma. The Hall force acting on the electrons (which results from the nonlinear interaction between the rotating field and the screening current associated with it) has a steady part. The corresponding steady Hall current driven in the plasma is

$$J_\theta = (-1/ne\eta)\langle J_z B_r\rangle\,, \tag{6}$$

where the angle brackets denote time averaging. We note that the ratio of the Hall current to the axial screening current is

$$J_\theta/J_z \sim B_r/ne\eta \sim \omega_{\mathrm{ce}}/\nu_{\mathrm{ei}} = \varepsilon\,. \tag{7}$$

The equations describing this system were solved analytically for $\varepsilon \ll 1$ and $\varepsilon \gg 1$ (Jones and Hugrass 1981) and numerically (Hugrass and Grimm 1981) for arbitrary ε. These studies showed that for $\varepsilon \gg 1$ the azimuthal current is given by

$$J_\theta \approx -ne\omega r \tag{8}$$

and the axial current is of the order $\varepsilon^{-1}J_\theta$. This axial current can be much smaller than the screening current predicted by the linear theory, and the rotating field therefore penetrates into the plasma cylinder much further than the classical skin depth. For $\varepsilon \ll 1$, the axial current is not much different from that obtained using the linear theory. For this weakly nonlinear case, the Hall current is calculated using the zeroth-order fields obtained from the linear analysis; these zeroth-order fields are those associated with the classical skin effect (Jones and Hugrass 1981).

In the Rotamak experiments, the rotating field is utilized to drive the toroidal current in a compact toroidal plasma (Jones 1979; Hugrass *et al.* 1980; Durance *et al.* 1982). For typical Rotamak experiments we have $\varepsilon \gtrsim 1$ and the rotating magnetic field B_ω smaller than (but of the same order as) the equilibrium steady magnetic field. The steady toroidal current is driven by the steady toroidal component of the Hall force resulting from the interaction between the rotating magnetic field and the screening currents it induces in the plasma.

In contrast with the case of a cylindrical plasma, the Hall force drives a poloidal steady current in a compact toroidal plasma in addition to the steady toroidal current. This poloidal component arises because the screening currents are not purely vertical. From symmetry considerations, the poloidal current below the equatorial plane must be a mirror image of that above the equatorial plane. It follows that the associated toroidal magnetic field is antisymmetric with respect to the equatorial plane; this result is obtained naturally from the analysis.

In this paper we will consider the idealized model of a spherical conductor of uniform electron number density and uniform resistivity, to which is applied a uniform transverse magnetic field which rotates about the polar axis. The effect of the steady equilibrium field appropriate to a Rotamak equilibrium will not be considered, and the problem will be treated only for the weakly nonlinear case ($\varepsilon \ll 1$). While it is recognized (Hugrass 1982) that steady magnetic fields have an appreciable effect on the screening currents (and hence the Hall force) and that the quasi-linear case ($\varepsilon \ll 1$) bears little resemblance to the experimental situation, the results obtained using this simple model are not expected to be qualitatively different from what would be obtained using a more exact (and more complicated) model. Furthermore, the disregard of unnecessary details allows a clearer insight into the physical mechanism involved.

The main assumptions and equations describing the model are presented in Section 2 and the zeroth-order fields are obtained in Section 3. The steady Hall currents and the associated steady magnetic fields are calculated in Section 4 and the relevance of these results to the latest observations in the Rotamak is discussed in Section 5.

2. Main Equations and Assumptions

Consider a conducting sphere of radius R, placed in a uniform magnetic field of amplitude B_ω which rotates about the polar axis at an angular frequency ω:

$$\begin{aligned} \boldsymbol{B} = {} & B_\omega \sin\theta\cos(\omega t-\phi)\hat{\boldsymbol{r}} + B_\omega \cos\theta\cos(\omega t-\phi)\hat{\boldsymbol{\theta}} \\ & + B_\omega \sin(\omega t-\phi)\hat{\boldsymbol{\phi}}, \end{aligned} \tag{9}$$

where r, θ and ϕ are the standard spherical coordinates and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are the corresponding unit vectors. We assume that the ions are immobile, singly charged and have a uniform number density n. The electrons are assumed to form a cold fluid of uniform number density n, and the resistivity is assumed to be isotropic and uniform. We also assume that the radius of the sphere is much smaller than the free space wavelength of electromagnetic radiation at the rotating field frequency, so that the displacement current can be ignored. The fields satisfy Maxwell's equations

$$\nabla\times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \tag{10}$$

$$\nabla\times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{11}$$

and the appropriate form of Ohm's law

$$\eta \boldsymbol{J} = \boldsymbol{E} - (1/ne)\boldsymbol{J}\times \boldsymbol{B}. \tag{12}$$

Using equations (10)–(12) we obtain

$$\nabla^2 \boldsymbol{B} - \frac{\mu_0}{\eta}\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{ne\eta}\nabla \times \{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B})\}, \tag{13}$$

for $0 \leqslant r \leqslant R$, and

$$\nabla^2 \boldsymbol{B} = 0, \tag{14}$$

for $r \geqslant R$.

It is extremely difficult to obtain a general analytical solution to the above equations since the right-hand term of equation (13) is nonlinear. We note, however, that the ratio of this nonlinear term to the linear term $\nabla^2 \boldsymbol{B}$ is of the order $\varepsilon = B/ne\eta = \omega_{ce}/\nu_{ei}$. It is therefore possible to obtain an approximate solution valid for the quasi-linear case, $\varepsilon \ll 1$, using a perturbation analysis. For the purpose of this analysis, we express the magnetic field as the sum of a zeroth-order field $\boldsymbol{B}_0$ and a first-order field $\boldsymbol{b}_1$:

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}_1, \tag{15}$$

where b_1 is of order εB_0, and we neglect terms of order ε^2 and higher. The zeroth-order field $\boldsymbol{B}_0$ satisfies the equation

$$\nabla^2 \boldsymbol{B}_0 - \frac{\mu_0}{\eta}\frac{\partial}{\partial t}\boldsymbol{B}_0 = 0 \tag{16}$$

in the region $0 \leqslant r \leqslant R$, and

$$\nabla^2 \boldsymbol{B}_0 = 0 \tag{17}$$

in the region $r \geqslant R$, and matches the externally applied field (equation 9) for $r \gg R$. The first-order field has a steady part $\boldsymbol{b}_{1s}$ and a time-varying part $\boldsymbol{b}_{1t}$. The steady part satisfies the equation

$$\nabla^2 \boldsymbol{b}_{1s} = (1/ne\eta)\langle \nabla \times \{(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0\}\rangle \tag{18}$$

for $0 \leqslant r \leqslant R$, and

$$\nabla^2 \boldsymbol{b}_{1s} = 0 \tag{19}$$

for $r \geqslant R$, and tends to zero for $r \gg R$, where the angle brackets in equation (18) denote time averaging. It is clear from (18) that $b_{1s} \sim \varepsilon B_0$ as required by the perturbation analysis.

3. Zeroth-order Field

The zeroth-order field satisfies equation (16) in the region $0 \leqslant r \leqslant R$ and equation (17) in the region $r \geqslant R$. The solution that matches the externally applied field for $r \gg R$ is given by

$$B_{0r} = \mathrm{Re}\left(C_1 \frac{\mathrm{I}_{3/2}(\gamma r)}{(\gamma r)^{3/2}} B_\omega \sin\theta\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}\right), \tag{20}$$

$$B_{0\theta} = \mathrm{Re}\left\{\tfrac{1}{2}C_1\left(\frac{\mathrm{I}_{1/2}(\gamma r)}{(\gamma r)^{1/2}} - \frac{\mathrm{I}_{3/2}(\gamma r)}{(\gamma r)^{3/2}}\right) B_\omega \cos\theta\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}\right\}, \tag{21}$$

$$B_{0\phi} = \mathrm{Re}\left\{-\tfrac{1}{2}\mathrm{i}\, C_1\left(\frac{\mathrm{I}_{1/2}(\gamma r)}{(\gamma r)^{1/2}} - \frac{\mathrm{I}_{3/2}(\gamma r)}{(\gamma r)^{3/2}}\right) B_\omega\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}\right\} \tag{22}$$

in the region $0 \leqslant r \leqslant R$, and

$$B_{0r} = \mathrm{Re}[\{1 - C_0(R/r)^3\}B_\omega \sin\theta\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}], \tag{23}$$

$$B_{0\theta} = \mathrm{Re}[\{1 + \tfrac{1}{2}C_0(R/r)^3\}B_\omega \cos\theta\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}], \tag{24}$$

$$B_{0\phi} = \mathrm{Re}[-\mathrm{i}\{1 + \tfrac{1}{2}C_0(R/r)^3\}B_\omega \mathrm{e}^{\mathrm{i}(\omega t - \phi)}] \tag{25}$$

in the region $r \geqslant R$, where

$$\gamma = (\mathrm{i}\omega\mu_0/\eta)^{1/2} = \delta^{-1}(1+\mathrm{i}),$$

[$\delta = (2\eta/\omega\mu_0)^{\frac{1}{2}}$ is the classical skin depth]

$$C_0 = 1 + \frac{3}{(\gamma R)^2} - \frac{3}{\gamma R}\frac{\mathrm{I}_{-1/2}(\gamma R)}{\mathrm{I}_{1/2}(\gamma R)},$$

$$C_1 = \frac{3(\gamma R)^{1/2}}{\mathrm{I}_{1/2}(\gamma R)},$$

and $\mathrm{I}_{\pm n/2}(X)$ is the modified Bessel function of the first kind of half-integer order.

The components of the zeroth-order current density are

$$J_{0r} = 0, \tag{26}$$

$$J_{0\theta} = \mathrm{Re}\{\mathrm{i}(C_1/2r)(\gamma r)^{\frac{1}{2}}\mathrm{I}_{3/2}(\gamma r)\, B_\omega \mathrm{e}^{\mathrm{i}(\omega t - \phi)}\}, \tag{27}$$

$$J_{0\phi} = \mathrm{Re}\{(C_1/2r)(\gamma r)^{\frac{1}{2}}\mathrm{I}_{3/2}(\gamma r)\, B_\omega \cos\theta\, \mathrm{e}^{\mathrm{i}(\omega t - \phi)}\} \tag{28}$$

in the region $0 \leqslant r \leqslant R$, and

$$J_{0r} = J_{0\theta} = J_{0\phi} = 0$$

in the region $r \geqslant R$.

4. First-order Steady Current and Magnetic Field

The first-order steady current and magnetic field are independent of ϕ. It is therefore convenient to solve for the toroidal components, $J_{1s\phi}$ and $b_{1s\phi}$. The other components can be obtained from $J_{1s\phi}$ and $b_{1s\phi}$ using Ampere's law.

The first-order steady toroidal current density is given by

$$\begin{aligned} J_{1s\phi} &= (-1/ne\eta)\langle(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0\rangle \cdot \hat{\phi} \\ &= -\tfrac{9}{2}\left(\frac{\omega_{\mathrm{ce}}}{\nu_{\mathrm{ei}}}\right)^2 \left(\frac{(\delta/2r)^2\{\cosh(2r/\delta)\cos(2r/\delta)\} - 2\{\cosh(2r/\delta) - \cos(2r/\delta)\}}{\cosh(2R/\delta) - \cos(2R/\delta)}\right) \\ &\quad \times ne\omega R \sin\theta\,. \end{aligned} \tag{29}$$

The first-order steady toroidal magnetic field satisfies the equation

$$\nabla^2 b_{1s\phi} - b_{1s\phi}/r^2 \sin^2\theta = G_\phi \tag{30}$$

in the region $0 \leqslant r \leqslant R$, and

$$\nabla^2 b_{1s\phi} - b_{1s\phi}/r^2 \sin^2\theta = 0 \tag{31}$$

in the region $r \geqslant R$, where

$$\begin{aligned} G_\phi &= (1/ne\eta)\langle \nabla \times \{(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0\}\rangle . \hat{\boldsymbol{\phi}} \\ &= (C/r^2)[2(2r/\delta)^{-2}\{\cosh(2r/\delta) - \cos(2r/\delta)\} \\ &\qquad -(2r/\delta)^{-1}\{\sinh(2r/\delta) - \sin(2r/\delta)\}]B_\omega \sin 2\theta\,, \end{aligned} \tag{32}$$

$$C = \tfrac{9}{4}\left(\frac{R}{\delta}\right)^2 \frac{\omega_{ce}}{\nu_{ei}} \frac{1}{\cosh(2R/\delta) - \cos(2R/\delta)}\,. \tag{33}$$

The solution that satisfies equation (30) in the region $0 \leqslant r \leqslant R$, and equation (31) in the region $r \geqslant R$, is continuous at $r = R$, tends to zero as $r \to \infty$ and is given by

$$b_{1s\phi} = \tfrac{1}{5}C\{Q(2r/\delta) - (r/R)^2 Q(2R/\delta)\}B_\omega \sin 2\theta \tag{34}$$

in the region $0 \leqslant r \leqslant R$, and

$$b_{1s\phi} = 0 \tag{35}$$

in the region $r \geqslant R$, where C is given by equation (33) and

$$\begin{aligned} Q(X) &= (-3/X^3)(\sinh X - \sin X) + (1/2X^2)(\cosh X - \cos X) \\ &\quad + (1/6X)(\sinh X + \sin X) \\ &\quad + \tfrac{1}{12}(\cosh X + \cos X) + \tfrac{1}{12}X(\sinh X - \sin X) \\ &\quad - \tfrac{1}{6}X^2\left(\frac{X^2}{2.2!} + \frac{X^6}{6.6!} + \frac{X^{10}}{10.10!} + \ldots\right). \end{aligned} \tag{36}$$

It is seen from equations (29) and (34) that the solution depends on two dimensionless numbers, ω_{ce}/ν_{ei} and R/δ. It is convenient to define the normalized toroidal current density

$$J_n = J_{1s\phi}/\{-(\omega_{ce}/\nu_{ei})^2 ne\omega r\} \tag{37}$$

and the normalized toroidal field

$$b_n = b_{1s\phi}/\{(\omega_{ce}/\nu_{ei})B_\omega\}\,. \tag{38}$$

Both of these normalized quantities depend on the ratio R/δ. Fig. 1 shows the normalized current density J_n plotted against r in the equatorial plane $\theta = \frac{1}{2}\pi$ for different values of R/δ. In Fig. 2, the normalized toroidal field b_n is plotted against r in the conical surface $\theta = \frac{1}{4}\pi$ for different values of R/δ. It is seen that b_n equals zero at $r = 0$ and at $r = R$, and attains a maximum value $b_{n,\max}$ at some intermediate value of r. Fig. 3 shows the variation of $b_{n,\max}$ with R/δ. It is seen that $b_{n,\max}$ is small for very small and very large values of R/δ, and has a broad maximum for $R/\delta \sim 4$.

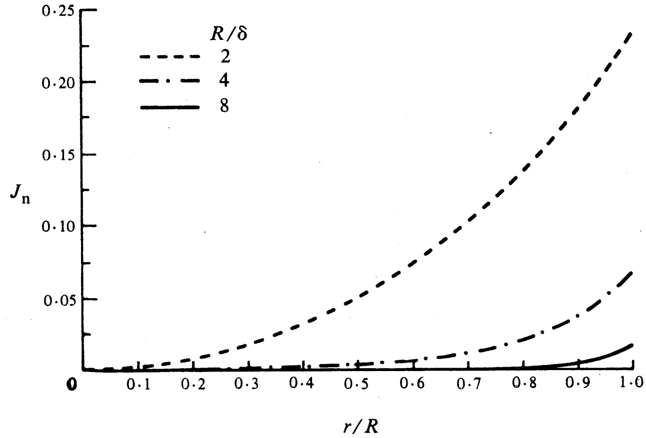

Fig. 1. Normalized toroidal current density plotted against r in the plane $\theta = \frac{1}{2}\pi$ for three values of R/δ.

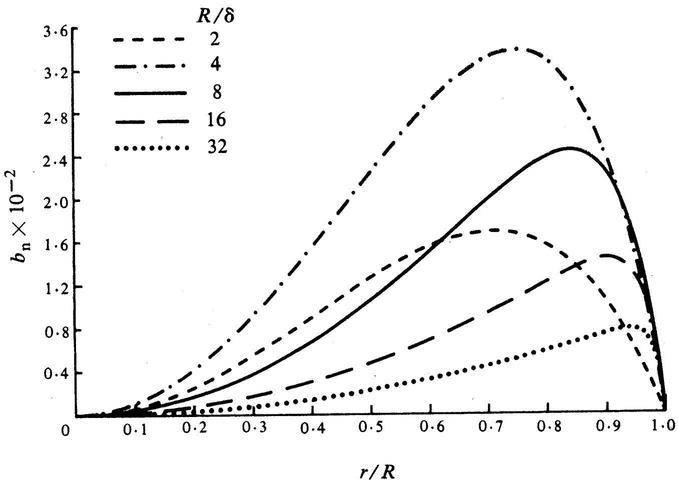

Fig. 2. Normalized steady toroidal field b_n plotted against r at $\theta = \frac{1}{4}\pi$ for different values of R/δ.

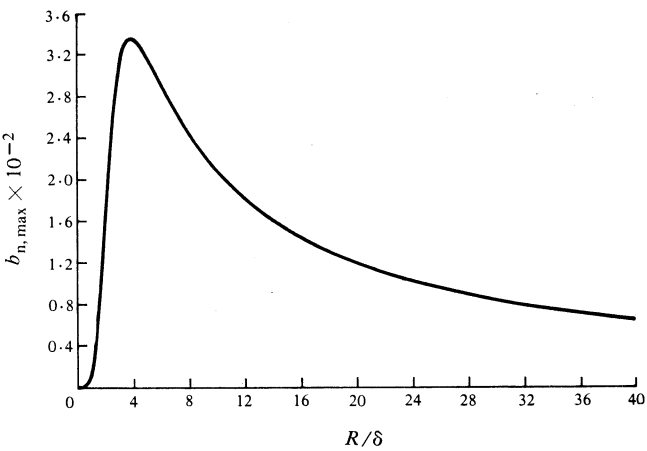

Fig. 3. Maximum value of the normalized steady toroidal field $b_{n,\max}$ plotted against R/δ.

5. Discussion

It has been shown in previous works that a steady azimuthal current is generated in an infinitely long plasma cylinder placed in a rotating magnetic field. This steady current is driven by the Hall force

$$F_\theta = (1/ne)\langle J_z B_r \rangle ,$$

where B_r is the r component of the rotating field and J_z is the screening current induced by the rotating field in the plasma. Similarly, a rotating magnetic field drives a steady toroidal current in spheroidal plasmas. It has been observed in some recent Rotamak III experiments (described in the paper by Durance *et al.* 1982) that a steady toroidal magnetic field is also generated. It has also been observed that the toroidal magnetic field is in a positive sense above the equatorial plane and in a negative sense below the equatorial plane; the net toroidal flux is zero (Durance 1983).

It is suggested here, that this steady toroidal field is produced by a steady poloidal current which is in turn driven by the poloidal component of the Hall force. In contrast with the case of an infinitely long plasma cylinder, the Hall force which arises from the interaction between the rotating field and the screening current it induces in a spheroidal plasma is not purely toroidal, but has a nonzero poloidal component. From symmetry considerations, the poloidal current in the upper hemisphere is a mirror image of that in the lower hemisphere and, consequently, the steady toroidal magnetic field is antisymmetric with respect to the equatorial plane and the net toroidal flux is zero. It is also found that this steady toroidal field has the same sense as observed experimentally ($b_{1s\phi}$ is positive for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and negative for $\frac{1}{2}\pi \leqslant \theta \leqslant \pi$). The results of this work provide a possible explanation for the 'self-generation of a steady toroidal magnetic field in the Rotamak'. It should be admitted, however, that the simple model we have adopted here has a number of drawbacks. The motion of the ions is neglected, only the quasilinear case is treated and the effect of the equilibrium poloidal field is not considered. The experimental results however were obtained for a strongly nonlinear case ($\varepsilon \gg 1$); it follows that a quantitative comparison between theory and experiment is not possible. It is also recognized that a steady toroidal field of the observed topology can be generated by other physical mechanisms, for example, if the ion fluid acquires a rotational motion which violates the Ferraro (1937) isorotation condition. A complete understanding of this interesting phenomenon can only be achieved by more detailed experimental and theoretical work.

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