Aust. J. Phys., 1984, **37**, 615–37

Dispersion in a Relativistic Quantum Electron Gas. I General Distribution Functions

Leith M. Hayes and D. B. Melrose

School of Physics, University of Sydney, Sydney, N.S.W. 2006.

Abstract

The covariant response tensor for a relativistic electron gas is calculated in two ways. One involves introducing a four-dimensional generalization of the electron–positron occupation number, and the other is a covariant generalization of a method due to Harris. The longitudinal and transverse parts are evaluated for an isotropic electron gas in terms of three plasma dispersion functions, and the contributions from Landau damping and pair creation to the dispersion curve are identified separately. The long-wavelength limit and the non-quantum limit, with first quantum corrections, are found. The plasma dispersion functions are evaluated explicitly for a completely degenerate relativistic electron gas, and a detailed form due to Jancovici is reproduced.

1. Introduction

The literature on the (dielectric) responses of a relativistic quantum electron gas is relatively sparse. Tsytovich (1961) calculated the response functions using an averaged propagator method for an arbitrary electron gas described in terms of the occupation numbers $n^{+}(\boldsymbol{p})$ and $n^{-}(\boldsymbol{p})$ for electrons and positrons; he also derived explicit results for the isotropic case and briefly discussed the limit of an ultra-relativistic Boltzmann gas. Jancovici (1962), using a method based on a quasi-boson Hamiltonian, derived explicit expressions for a relativistic, completely degenerate electron gas. Hakim and Heyvaerts (1978), using a method involving the one-particle Wigner function, developed a covariant theory (in the Lorentz gauge) and considered the quantum corrections to the response of a non-quantum electron gas. Delsante and Frankel (1980) and Kowalenko (1982) used a relativistic version of Harris' (1969) non-relativistic quantum approach to derive expressions for the longitudinal response of a degenerate electron gas and of a relativistic pair plasma at zero temperature.

Our main interest in the present paper and in an accompanying paper (Melrose and Hayes 1984, see p. 639) is in defining plasma dispersion functions for a relativistic quantum electron gas, and in discussing their approximations in the non-quantum and non-relativistic limits. In this paper we discuss more general results; we specialize to thermal distributions in the accompanying paper.

In Section 2 we calculate the response functions in covariant form by introducing a four-dimensional occupation number $N(P)$ which is similar to the four-dimensional distribution $F(P)$ used in treating the response of a non-quantum electron gas in a covariant and gauge invariant manner (Melrose 1982). The longitudinal and

0004-9506/84/060615$02.00

transverse response functions for an isotropic electron gas are then written down; in Appendix 1 these functions are rederived using a covariant version of Harris' (1969) method.

In Section 3 the integrals over angle are carried out and it is found that the responses may be described in terms of three transcendental functions, denoted by $S^{(0)}(k)$, $S^{(1)}(k)$ and $S^{(2)}(k)$, with $S^{(1)}(k)$ and $S^{(2)}(k)$ appearing only in one particular linear combination in both the longitudinal and transverse dielectric functions.

In Section 4 the contributions to dissipation from Landau damping (LD) and pair creation (PC) are considered separately. Two important approximations, the long-wavelength and the non-quantum limits, are treated in Section 5. In Section 6 we apply our results to a completely degenerate electron gas and rederive Jancovici's (1962) expressions, whose validity in detail has been questioned by Kowalenko (1982). Finally we present some detailed results for the dispersion curves in a high temperature pair plasma.

Our notation is that used by Melrose (1982) with units chosen such that $\hbar = c = 1$.

2. Covariant Form of the Response Tensor

In this section we define a four-dimensional occupation number $N(P)$ for an electron–positron gas and write a covariant form of the linear response tensor (Tsytovich 1961) in terms of it. We then show that this form reproduces the appropriate non-quantum result written in terms of $F(P)$ (Melrose 1982).

Let electrons be described by $\zeta = 1$ and positrons by $\zeta = -1$. Their usual occupation numbers are written $n^{\zeta}(\boldsymbol{p})$ with $\boldsymbol{p}$ being the physical momentum (and not minus the physical momentum for positrons). The 4-momentum $P = (E, \boldsymbol{P})$ is related to the physical 4-momentum $p = (\varepsilon, \boldsymbol{p})$ by $P = \zeta p$, i.e.

$$E = \zeta\varepsilon, \qquad \boldsymbol{P} = \zeta\boldsymbol{p}. \tag{1}$$

We define $N(P)$ by

$$N(P) = \sum_{\zeta} \frac{2\pi m}{\varepsilon}\,\delta(E-\zeta\varepsilon)\,n^{\zeta}(\zeta\boldsymbol{p}), \tag{2}$$

which may be written in the alternative form

$$N(P) = \sum_{\zeta} 4\pi m\,\delta(P^2-m^2)\,\vartheta(\zeta\varepsilon)\,n^{\zeta}(\zeta\boldsymbol{p}), \tag{2'}$$

where $\vartheta(\zeta\varepsilon)$ is the Heaviside step function. The definition (2) corresponds to

$$\int_0^{\infty} \frac{\mathrm{d}E}{2\pi} N(P) = \frac{m}{\varepsilon} n^{+}(\boldsymbol{p}), \qquad \int_{-\infty}^{0} \frac{\mathrm{d}E}{2\pi} N(P) = \frac{m}{\varepsilon} n^{-}(-\boldsymbol{p}), \tag{3a, b}$$

where the factors m/ε are analogous to the corresponding factor γ^{-1} appearing in the relation between $F(P)$ and $f(\boldsymbol{p})$ (cf. Melrose 1982; equation 23).

Following Tsytovich (1961) we derive the linear response tensor $\alpha^{\mu\nu}(k)$, with the wave 4-vector $k = (\omega, \boldsymbol{k})$, by starting from the vacuum polarization tensor, namely

$$\alpha^{\mu\nu}(k) = -\mathrm{i}\,e^2 Sp \int \frac{\mathrm{d}^4P}{(2\pi)^4}\,\gamma^{\mu}G(P)\gamma^{\nu}G(P-k), \tag{4}$$

where Sp denotes the trace over γ matrices. The propagator in vacuo is re-interpreted as a propagator statistically averaged over the electron gas. As shown in Appendix 1, the resulting form is

$$G(P) = (\gamma^\mu P_\mu + m)\left(\frac{1}{P^2 - m^2 + \mathrm{i}0} + \mathrm{i}\frac{N(P)}{2m}\right). \tag{5}$$

The term involving $N(P)$ arises from the electron gas; it does not affect the non-resonant part of the propagator, which arises from the principal value part of the vacuum term. The resonant part of the vacuum propagator is replaced by itself times a factor

$$1 - 2\sum_\zeta \vartheta(\zeta\varepsilon)\, n^\zeta(\zeta \boldsymbol{p}).$$

The unit term describes the vacuum response, which we omit hereafter. The non-resonant part of $G(P)$ is then given by the principal part of the term involving $1/(P^2 - m^2)$ in (5) and the resonant part is given by the term involving $N(P)$.

As pointed out by Tsytovich (1961) the Feynman prescription for evaluating the resonant part of (5) is acausal, and the only physical part of (4) is the hermitian part which arises from the resonant part of one propagator and the non-resonant part of the other. Writing

$$P' = P - k, \qquad P'' = P + k, \tag{6}$$

there are two contributions which give

$$\alpha^{\mu\nu}(k) = \frac{2e^2}{m}\int \frac{\mathrm{d}^4P}{(2\pi)^4}\, F^{\mu\nu}(P,P')\left(\frac{N(P)}{P'^2 - m^2} + \frac{N(P')}{P^2 - m^2}\right) \tag{7a}$$

$$= \frac{2e^2}{m}\int \frac{\mathrm{d}^4P}{(2\pi)^4}\, N(P)\left(\frac{F^{\mu\nu}(P,P')}{P'^2 - m^2} + \frac{F^{\mu\nu}(P,P'')}{P''^2 - m^2}\right), \tag{7b}$$

with

$$F^{\mu\nu}(P,P') = \tfrac{1}{4}Sp\{\gamma^\mu(\gamma^\tau P_\tau + m)\,\gamma^\nu(\gamma^\tau P'_\tau + m)\}$$
$$= P^\mu P'^\nu + P^\nu P'^\mu + g^{\mu\nu}(m^2 - PP'). \tag{8}$$

The alternative form (7b) follows from (7a) by shifting the origin of integration for the final term and using obvious symmetry properties of (8).

A further form may be derived from (7b) by replacing the variable of integration P by $-P$. Under $P \to -P$ we have $P' \to -P''$ and $P'' \to -P'$; the symmetry properties apparent from (8) then imply that the quantity in large parentheses in (7b) is invariant under $P \to -P$. It follows that $2N(P)$ in (7b) may be replaced by $N(P) + N(-P)$.

On performing the integral over E, equations (3) imply that the resulting expression depends on the occupation numbers only in the combination

$$\bar{n}(\boldsymbol{p}) = n^+(\boldsymbol{p}) + n^-(-\boldsymbol{p}). \tag{9}$$

An obvious physical interpretation of this result is that an electron (charge $-e$) with momentum $\boldsymbol{p}$ and a positron (charge e) with physical momentum $-\boldsymbol{p}$ correspond to the same current.

The reduction of (7b) to the non-quantum limit involves two steps. First one makes the identification

$$2\int\frac{\mathrm{d}^4P}{(2\pi)^4}N(P)=\int\mathrm{d}^4P\,F(P) \tag{10}$$

between $N(P)$ and $F(P)$. The factor of 2 on the left arises from the two spin states of the electron, and the factor $(2\pi)^{-4}$ is omitted on the right by choice of convention. A minor complication concerns the positrons. One may regard the non-quantum limit as requiring that there be a negligible number of positrons (Hakim and Heyvaerts 1978). Alternatively one may regard electrons and positrons as separate types of classical particles, and separate $F(P)$ into two independent components for the two corresponding distributions.

The second step is to expand in $\hbar$ and retain only the lowest order non-vanishing terms. In (7b) $\hbar$ appears only in $P'=P-\hbar k$ and $P''=P+\hbar k$. Using the δ function in (2′) one has

$$P'^2-m^2=-2kP+k^2,\qquad P''^2-m^2=2kP+k^2,$$

where the dependence on $\hbar$ is implicit. Then on expanding in $\hbar$ one finds

$$\frac{F^{\mu\nu}(P,P')}{P'^2-m^2}+\frac{F^{\mu\nu}(P,P'')}{P''^2-m^2}\approx-a^{\mu\nu}(k,P/m), \tag{11}$$

with (Melrose 1982)

$$a^{\mu\nu}(k,P/m)=g^{\mu\nu}+\frac{P^\mu k^\nu+P^\nu k^\mu}{Pk}+\frac{k^2P^\mu P^\nu}{(Pk)^2}. \tag{12}$$

Then (7a) reduces to

$$\alpha^{\mu\nu}(k)=-\frac{e^2}{m}\int\mathrm{d}^4P\,F(P)\,a^{\mu\nu}(k,P/m), \tag{13}$$

which reproduces the relevant non-quantum result (cf. equation 22 of Melrose 1982).

3. Response Functions for Isotropic Distributions

The longitudinal (L) and transverse (T) response functions may be derived from (7a) using a method given by Melrose (1982). The result is

$$\alpha^{\mathrm{L,T}}(k)=2e^2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\bar{n}(\varepsilon)\bigg(\frac{\varepsilon-\varepsilon'}{\omega^2-(\varepsilon-\varepsilon')^2}a_{+}^{\mathrm{L,T}}(\boldsymbol{p},\boldsymbol{k})$$

$$+\frac{\varepsilon+\varepsilon'}{\omega^2-(\varepsilon+\varepsilon')^2}a_{-}^{\mathrm{L,T}}(\boldsymbol{p},\boldsymbol{k})\bigg), \tag{14}$$

with

$$a_{\pm}^{\mathrm{L}}(\boldsymbol{p},\boldsymbol{k})=1\mp\frac{1}{\varepsilon\varepsilon'}\left(\varepsilon^2+\boldsymbol{p}.\boldsymbol{k}-2\frac{(\boldsymbol{p}.\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\right), \tag{15a}$$

$$a_{\pm}^{\mathrm{T}}(\boldsymbol{p},\boldsymbol{k})=1\mp\frac{1}{\varepsilon\varepsilon'}\left(m^2-\boldsymbol{p}.\boldsymbol{k}+\frac{(\boldsymbol{p}.\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\right), \tag{15b}$$

where ε denotes $\varepsilon(\boldsymbol{p}) = (m^2 + |\boldsymbol{p}|^2)^{\frac{1}{2}}$ and ε' denotes $\varepsilon(\boldsymbol{p}')$ with $\boldsymbol{p}' = \boldsymbol{p}-\boldsymbol{k}$. We also change notation, writing $\bar{n}(\boldsymbol{p})$ as $\bar{n}(\varepsilon)$ for an isotropic distribution. The result (14) is essentially that given by Tsytovich (1961).

The integral over $\boldsymbol{p}$ involves angular integrals, which may be chosen as polar angles of $\boldsymbol{p}$ relative to $\boldsymbol{k}$. The integral over azimuthal angle is trivial, and the remaining integrals over $|\boldsymbol{p}|$ and the polar angle may be rewritten as integrals over ε and ε' with $\varepsilon_- \leqslant \varepsilon' \leqslant \varepsilon_+$ and

$$\varepsilon_\pm = (\varepsilon^2 \pm 2|\boldsymbol{p}||\boldsymbol{k}| + |\boldsymbol{k}|^2)^{\frac{1}{2}}. \tag{16}$$

This gives

$$\alpha^{\mathrm{L,T}}(k) = \frac{e^2}{4\pi^2|\boldsymbol{k}|}\int \mathrm{d}\varepsilon\,\bar{n}(\varepsilon)\int_{\varepsilon_-}^{\varepsilon_+}\mathrm{d}\varepsilon'\left\{\left(\frac{1}{\omega-\varepsilon+\varepsilon'}-\frac{1}{\omega+\varepsilon-\varepsilon'}\right)\varepsilon\varepsilon' a_+^{\mathrm{L,T}}(\boldsymbol{p},\boldsymbol{k})\right.$$
$$\left.+\left(\frac{1}{\omega-\varepsilon-\varepsilon'}-\frac{1}{\omega+\varepsilon+\varepsilon'}\right)\varepsilon\varepsilon' a_-^{\mathrm{L,T}}(\boldsymbol{p},\boldsymbol{k})\right\}. \tag{17}$$

Using the relation

$$\int \mathrm{d}\varepsilon'\frac{\varepsilon'^n}{\omega-\varepsilon+\varepsilon'} = \frac{\varepsilon'^n}{n}-\frac{(\omega-\varepsilon)\varepsilon'^{n-1}}{n-1}+\ldots$$
$$+(-1)^n(\omega-\varepsilon)^n\int\frac{\mathrm{d}\varepsilon'}{\omega-\varepsilon+\varepsilon'},$$

we may rewrite (17) in the form

$$\alpha^{\mathrm{L,T}}(k) = \frac{e^2}{4\pi^2|\boldsymbol{k}|}\int \mathrm{d}\varepsilon\,\bar{n}(\varepsilon)\left\{c^{\mathrm{L,T}}(\varepsilon,k)\right.$$
$$+b_+^{\mathrm{L,T}}(\varepsilon,k)\int_{\varepsilon_-}^{\varepsilon_+}\mathrm{d}\varepsilon'\left(\frac{1}{\varepsilon'-\varepsilon+\omega}+\frac{1}{\varepsilon'+\varepsilon-\omega}\right)$$
$$\left.+b_-^{\mathrm{L,T}}(\varepsilon,k)\int_{\varepsilon_-}^{\varepsilon_+}\mathrm{d}\varepsilon'\left(\frac{1}{\varepsilon'+\varepsilon+\omega}+\frac{1}{\varepsilon'-\varepsilon-\omega}\right)\right\}, \tag{18}$$

with

$$c^{\mathrm{L}}(\varepsilon,k) = 4|\boldsymbol{p}|\omega^2/|\boldsymbol{k}|, \qquad c^{\mathrm{T}}(\varepsilon,k) = -2|\boldsymbol{p}|(\omega^2+|\boldsymbol{k}|^2)/|\boldsymbol{k}|, \tag{19a, b}$$

$$b_\pm^{\mathrm{L}}(\varepsilon,k) = \frac{\omega^2}{2|\boldsymbol{k}|^2}(\omega^2-|\boldsymbol{k}|^2\mp 4\omega\varepsilon+4\varepsilon^2), \tag{19c}$$

$$b_\pm^{\mathrm{T}}(\varepsilon,k) = -\frac{(\omega^2-|\boldsymbol{k}|^2)}{4|\boldsymbol{k}|^2}\left(\omega^2+|\boldsymbol{k}|^2+\frac{4m^2|\boldsymbol{k}|^2}{\omega^2-|\boldsymbol{k}|^2}\mp 4\omega\varepsilon+4\varepsilon^2\right). \tag{19d}$$

The integral over ε' may now be performed giving

$$\alpha^{\mathrm{L}}(k) = \frac{e^2\bar{n}_0\,\omega^2}{m|\boldsymbol{k}|^2}+\frac{e^2 m\omega^2}{2\pi^2|\boldsymbol{k}|^3}\{\tfrac{1}{4}(\omega^2-|\boldsymbol{k}|^2)S^{(0)}(k)$$
$$-m\omega S^{(1)}(k)+m^2S^{(2)}(k)\}, \tag{20a}$$

$$\alpha^{\mathrm{T}}(k) = -\frac{e^2\bar{n}_0(\omega^2+|\boldsymbol{k}|^2)}{2m|\boldsymbol{k}|^2} - \frac{e^2m(\omega^2-|\boldsymbol{k}|^2)}{4\pi^2|\boldsymbol{k}|^2}\{(-\varepsilon_0^2+\tfrac{1}{4}\omega^2+\tfrac{1}{2}|\boldsymbol{k}|^2)S^{(0)}(k)$$

$$-m\omega S^{(1)}(k)+m^2S^{(2)}(k)\}, \tag{20b}$$

where

$$\bar{n}_0 = 2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{m}{\varepsilon}\bar{n}(\varepsilon) = \frac{m}{\pi^2}\int \mathrm{d}\varepsilon\,|\boldsymbol{p}|\,\bar{n}(\varepsilon) \tag{21}$$

is the proper number density, and with

$$\varepsilon_0 = \left(\frac{m^2}{1-\omega^2/|\boldsymbol{k}|^2}+\tfrac{1}{4}|\boldsymbol{k}|^2\right)^{\frac{1}{2}}. \tag{22}$$

The plasma dispersion functions introduced in (20) are

$$S^{(0)}(k) = \int\frac{\mathrm{d}\varepsilon}{m}\,\bar{n}(\varepsilon)\ln\Lambda_1\,, \tag{23a}$$

$$S^{(1)}(k) = \int\frac{\mathrm{d}\varepsilon}{m^2}\,\bar{n}(\varepsilon)\ln\Lambda_2\,, \tag{23b}$$

$$S^{(2)}(k) = \int\frac{\mathrm{d}\varepsilon}{m^3}\,\bar{n}(\varepsilon)\ln\Lambda_1\,, \tag{23c}$$

with

$$\Lambda_1 = \frac{(\varepsilon_+-\varepsilon+\omega)(\varepsilon_+-\varepsilon-\omega)(\varepsilon_++\varepsilon-\omega)(\varepsilon_++\varepsilon+\omega)}{(\varepsilon_--\varepsilon+\omega)(\varepsilon_--\varepsilon-\omega)(\varepsilon_-+\varepsilon-\omega)(\varepsilon_-+\varepsilon+\omega)}, \tag{24a}$$

$$\Lambda_2 = \frac{(\varepsilon_+-\varepsilon+\omega)(\varepsilon_++\varepsilon-\omega)(\varepsilon_--\varepsilon-\omega)(\varepsilon_-+\varepsilon+\omega)}{(\varepsilon_--\varepsilon+\omega)(\varepsilon_-+\varepsilon-\omega)(\varepsilon_+-\varepsilon-\omega)(\varepsilon_++\varepsilon+\omega)}. \tag{24b}$$

Alternative forms of the functions (24), along with another function Λ_3, are written down in Appendix 2.

In the next section we use the functions (23) to discuss Landau damping and pair creation.

4. Landau Damping and Pair Creation

The kinematics for Landau damping (LD) and pair creation (PC) were discussed by Tsytovich (1961): LD corresponds to the resonances at $\varepsilon-\varepsilon' = \pm\omega$ and pair creation to the resonances at $\varepsilon+\varepsilon' = \pm\omega$. LD is allowed only for $\omega < |\boldsymbol{k}|$ and PC only for $\omega^2 > 4m^2+|\boldsymbol{k}|^2$; there is no damping for $|\boldsymbol{k}|^2 < \omega^2 < 4m^2+|\boldsymbol{k}|^2$.

The resonant parts of the dielectric functions are not included correctly in (4), due to the propagator $G(P)$ being of the Feynman type and hence acausal (positrons propagate backwards in time). However, the causal condition may be imposed on (7a) and (7b) and subsequent forms of $\alpha^{\mu\nu}(k)$ by replacing ω by $\omega+\mathrm{i}0$, according to the usual Landau prescription.

By inspection of (18) we may identify the functions

$$S_{\mathrm{LD}\pm}^{(n)}(k) = \frac{1}{m^{n+1}}\int \mathrm{d}\varepsilon\, \varepsilon^n \bar{n}(\varepsilon)\int_{\varepsilon_-}^{\varepsilon_+}\mathrm{d}\varepsilon'\,\frac{1}{\varepsilon'-\varepsilon\pm(\omega+\mathrm{i}0)}, \tag{25a}$$

$$S_{\mathrm{PC}\pm}^{(n)}(k) = \frac{1}{m^{n+1}}\int \mathrm{d}\varepsilon\, \varepsilon^n \bar{n}(\varepsilon)\int_{\varepsilon_-}^{\varepsilon_+}\mathrm{d}\varepsilon'\,\frac{1}{\varepsilon'+\varepsilon\mp(\omega+\mathrm{i}0)}, \tag{25b}$$

which separate the dispersion into parts due to LD and PC. The functions (23) are related to the functions (25) by

$$S^{(n)}(k) = S_{\mathrm{LD}+}^{(n)}(k)+S_{\mathrm{PC}+}^{(n)}(k)+(-1)^n\{S_{\mathrm{LD}-}^{(n)}(k)+S_{\mathrm{PC}-}^{(n)}(k)\}\,. \tag{26}$$

The relation (26) allows us to identify the imaginary parts of $S^{(n)}(k)$ in terms of the contributions from LD and PC. After using the Plemelj formula

$$\frac{1}{x+\mathrm{i}0} = \mathrm{P}(x^{-1})-\mathrm{i}\pi\delta(x),$$

where P denotes the Cauchy principal value, and integrating over ε', the relevant imaginary parts are given by

$$\begin{aligned}\operatorname{Im} S_{\mathrm{LD}\pm}^{(n)}(k) &= \mp\frac{\pi}{m^{n+1}}\int_{\varepsilon_-\pm\omega}^{\varepsilon_+\pm\omega}\mathrm{d}\varepsilon\,\varepsilon^n\bar{n}(\varepsilon)\\ &= \mp\frac{\pi}{m^{n+1}}\int_{\varepsilon_0}^{\infty}\mathrm{d}\varepsilon\,(\varepsilon\pm\tfrac{1}{2}\omega)^n\,\bar{n}(\varepsilon\pm\tfrac{1}{2}\omega),\end{aligned} \tag{27a}$$

$$\begin{aligned}\operatorname{Im} S_{\mathrm{PC}\pm}^{(n)}(k) &= \pm\frac{\pi}{m^{n+1}}\int_{\pm\omega-\varepsilon_+}^{\pm\omega-\varepsilon_-}\mathrm{d}\varepsilon\,\varepsilon^n\,\bar{n}(\varepsilon)\\ &= \pm\frac{\pi}{m^{n+1}}\vartheta(\pm\omega)\int_{\frac{1}{2}|\omega|-\varepsilon_0}^{\frac{1}{2}|\omega|+\varepsilon_0}\mathrm{d}\varepsilon\,\varepsilon^n\,\bar{n}(\varepsilon)\\ &= \pm\frac{\pi}{m^{n+1}}\vartheta(\pm\omega)\int_0^{\varepsilon_0}\mathrm{d}\varepsilon\,\{(\tfrac{1}{2}|\omega|+\varepsilon)^n\,\bar{n}(\tfrac{1}{2}|\omega|+\varepsilon)\\ &\qquad +(\tfrac{1}{2}|\omega|-\varepsilon)^n\,\bar{n}(\tfrac{1}{2}|\omega|-\varepsilon)\},\end{aligned} \tag{27b}$$

where $\vartheta(\omega)$ is the step function.

5. Long-wavelength and Non-quantum Limits

The approximate cases of most interest are the long-wavelength limit and the non-quantum limit along with the first quantum corrections to it.

Long-wavelength Limit

The long-wavelength limit is obtained by expanding $\ln\Lambda_1$ and $\ln\Lambda_2$ in powers of $|\boldsymbol{k}|$. On retaining terms up to $O(|\boldsymbol{k}|^5)$, inserting the expansions into equations (23) and thence into (20), one finds to $O(|\boldsymbol{k}|^2)$

$$\operatorname{Re}\alpha^{\mathrm{L,T}}(k) = A^{\mathrm{L,T}}(\omega)+|\boldsymbol{k}|^2B^{\mathrm{L,T}}(\omega), \tag{28}$$

with

$$A^{\mathrm{L}}(\omega) = A^{\mathrm{T}}(\omega) = -\frac{2e^2}{3\pi^2}\int d\varepsilon\,|\boldsymbol{p}|\,\bar{n}(\varepsilon)\frac{3\varepsilon^2-|\boldsymbol{p}|^2}{4\varepsilon^2-\omega^2}, \tag{29a}$$

$$\begin{aligned} B^{\mathrm{L}}(\omega) = -\frac{2e^2}{\pi^2\omega^2}\int d\varepsilon\,|\boldsymbol{p}|\,\bar{n}(\varepsilon)\bigg(&\frac{|\boldsymbol{p}|^2}{4\varepsilon^2-\omega^2}(1-3|\boldsymbol{p}|^2/5\varepsilon^2) \\ &+\frac{\omega^2}{(4\varepsilon^2-\omega^2)^2}(\tfrac{5}{3}|\boldsymbol{p}|^2-2\varepsilon^2-3|\boldsymbol{p}|^4/5\varepsilon^2) \\ &+\frac{16|\boldsymbol{p}|^2\omega^2}{(4\varepsilon^2-\omega^2)^3}(\tfrac{1}{3}\varepsilon^2-\tfrac{1}{5}|\boldsymbol{p}|^2)\bigg), \end{aligned} \tag{29b}$$

$$\begin{aligned} B^{\mathrm{T}}(\omega) = -\frac{2e^2}{\pi^2\omega^2}\int d\varepsilon\,|\boldsymbol{p}|\,\bar{n}(\varepsilon)\bigg(&\frac{|\boldsymbol{p}|^2}{4\varepsilon^2-\omega^2}(\tfrac{1}{3}-|\boldsymbol{p}|^2/5\varepsilon^2) \\ &+\frac{\omega^2}{(4\varepsilon^2-\omega^2)^2}(\tfrac{1}{3}|\boldsymbol{p}|^2-2\varepsilon^2-|\boldsymbol{p}|^4/5\varepsilon^2) \\ &+\frac{16|\boldsymbol{p}|^2\omega^2}{3(4\varepsilon^2-\omega^2)^3}(\varepsilon^2-\tfrac{1}{5}|\boldsymbol{p}|^2)\bigg). \end{aligned} \tag{29c}$$

We have used these expressions in treating the example discussed in Section 7 below.

The region of validity of the small $|\boldsymbol{k}|$ expansion is restricted by the location of the singular points in the integrands of (17) or (18). This region depends on the form of the distribution function.

Let us start by considering the contribution to the integrals of particles with given $\boldsymbol{p}$ and ε. Then in (17), the small $|\boldsymbol{k}|$ expansion is valid for

$$|\boldsymbol{k}| < \big|(|\boldsymbol{p}|^2u^2 \pm 2\omega\varepsilon+\omega^2)^{\frac{1}{2}} \pm |\boldsymbol{p}|\,u\big|, \tag{30}$$

which must be satisfied for all four choices of sign. The most restrictive condition is for $\pm u = -1$, where u is the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{p}$. Then (30) leads to two conditions:

$$|\boldsymbol{k}| < \big|(|\boldsymbol{p}|^2+2\omega\varepsilon+\omega^2)^{\frac{1}{2}}-|\boldsymbol{p}|\big|, \tag{31a}$$

$$|\boldsymbol{k}| < \big|(|\boldsymbol{p}|^2-2\omega\varepsilon+\omega^2)^{\frac{1}{2}}-|\boldsymbol{p}|\big|. \tag{31b}$$

Without loss of generality we may now assume $\omega > 0$. The square root in (31a) is then always real, and that in (31b) is imaginary for $\varepsilon - m < \omega < \varepsilon + m$, and real otherwise. For $\omega < 2m$, the right-hand sides of (31) are always nonzero. For $\omega > 2m$, the RHS of (31b) becomes zero at $\omega = 2\varepsilon$, so there can be no long-wavelength expansion for particles with an energy of $\frac{1}{2}\omega$. We may re-write (31b) as

$$\begin{aligned} |\boldsymbol{k}| &< |\boldsymbol{p}|-(|\boldsymbol{p}|^2-2\omega\varepsilon+\omega^2)^{\frac{1}{2}}, && 0 < \omega < \varepsilon - m; \\ &< (2\omega\varepsilon-\omega^2)^{\frac{1}{2}}, && \varepsilon - m < \omega < 2m; \\ &< |\boldsymbol{p}|-(|\boldsymbol{p}|^2-2\omega\varepsilon+\omega^2)^{\frac{1}{2}}, && \varepsilon + m < \omega < 2\varepsilon; \\ &< (|\boldsymbol{p}|^2-2\omega\varepsilon+\omega^2)^{\frac{1}{2}}-|\boldsymbol{p}|, && 2\varepsilon < \omega. \end{aligned} \tag{32}$$

If we have a distribution of particles over the whole momentum range then the regions in which small $|\boldsymbol{k}|$ expansions are valid will be given by the minimum values of the right-hand sides of (31a) and (32). These conditions then become

$$|\boldsymbol{k}| < \omega; \tag{33a}$$

$$\begin{aligned} |\boldsymbol{k}| &< \omega, & \omega &< m; \\ &< (2\omega m-\omega^2)^{\frac{1}{2}}, & m &< \omega < 2m; \end{aligned} \tag{33b}$$

and we see that no long-wavelength expansion is possible for $\omega > 2m$. These limits are plotted in Fig. 1.

More realistic distributions of particles cover a range of momenta from zero to some effective maximum p_{m}. For a degenerate distribution p_{m} would be the Fermi momentum, while for a Boltzmann distribution p_{m} may be identified as several times the thermal momentum. Since for $\omega > 2m$ the right-hand side of (31b) becomes zero at $\varepsilon = \frac{1}{2}\omega$, there can be no long-wavelength expansion if the distribution includes particles with this energy, i.e. if the maximum energy ε_{m} is greater than $\frac{1}{2}\omega$. So there is no expansion for $2m < \omega < 2\varepsilon_{\mathrm{m}}$.

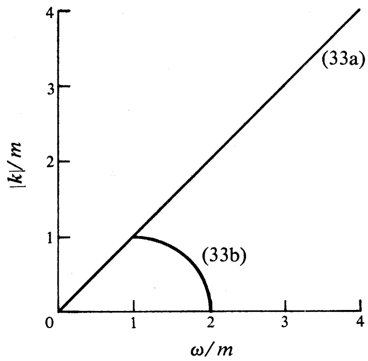
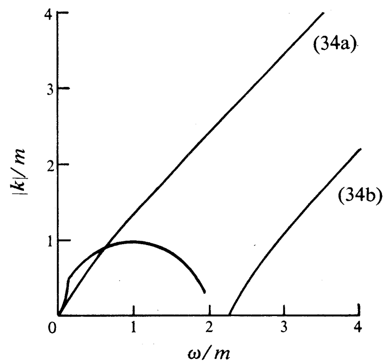


Fig. 1. Conditions (33a) and (33b) for the boundaries of the region of validity of the long-wavelength expansion are plotted for the LD and PC contributions to the dielectric functions when all values of particle momentum ($|p|$ from 0 to ∞) are allowed.

Fig. 2. Conditions (34a) and (34b) for the boundaries of the region of validity of the long-wavelength expansion are plotted for the LD and PC contributions when only values of momentum from 0 to p_{m} are allowed.



Subject to this condition, the regions of validity for a small $|\boldsymbol{k}|$ expansion are again given by finding the minimum values of (31a) and (32) over the momentum range from zero to p_{m}. These conditions become

$$|\boldsymbol{k}| < (p_{\mathrm{m}}^2+2\omega\varepsilon_{\mathrm{m}}+\omega^2)^{\frac{1}{2}}-p_{\mathrm{m}}; \tag{34a}$$

$$
\begin{aligned}
|\boldsymbol{k}| &< \min\{p_{\mathrm{m}}-(p_{\mathrm{m}}^2-2\omega\varepsilon_{\mathrm{m}}+\omega^2)^{\frac{1}{2}},\ (2\omega m-\omega^2)^{\frac{1}{2}}\}, && 0<\omega<\varepsilon_{\mathrm{m}}-m;\\
&< (2\omega m-\omega^2)^{\frac{1}{2}}, && \varepsilon_{\mathrm{m}}-m<\omega<2m;\\
&< (p_{\mathrm{m}}^2-2\omega\varepsilon_{\mathrm{m}}+\omega^2)^{\frac{1}{2}}-p_{\mathrm{m}}, && 2\varepsilon_{\mathrm{m}}<\omega. \qquad (34\mathrm{b})
\end{aligned}
$$

These conditions are plotted in Fig. 2 for $p_{\mathrm{m}} = 0\cdot 5m$.

In the non-quantum limit, the long-wavelength expansion corresponds to an expansion in the ratio of particle speed to phase speed, which in this case is $|\boldsymbol{k}|p_{\mathrm{m}}/\omega\varepsilon_{\mathrm{m}}$. Taking the non-quantum limit $\omega \ll \varepsilon$, we find that (34a) reduces to

$$|\boldsymbol{k}| < \omega\varepsilon_{\mathrm{m}}/p_{\mathrm{m}}.$$

In the non-quantum case, this condition applies to the contribution to the dispersion from LD. Hence we interpret (34a) as the condition for the small $|\boldsymbol{k}|$ expansion which applies to the LD contribution, and (34b) as the condition which applies to the PC contribution.

Non-quantum Limit

In developing approximations for the non-quantum limit we start from $\ln\Lambda_1$ and $\ln\Lambda_2$ in the forms

$$\ln\Lambda_1 = \ln\left(\frac{\{\omega-|\boldsymbol{k}|v+(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}\{\omega+|\boldsymbol{k}|v-(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}}{\{\omega-|\boldsymbol{k}|v-(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}\{\omega+|\boldsymbol{k}|v+(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}}\right), \qquad (35\mathrm{a})$$

$$\ln\Lambda_2 = \ln\left(\frac{\{\omega+|\boldsymbol{k}|v-(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}\{\omega+|\boldsymbol{k}|v+(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}}{\{\omega-|\boldsymbol{k}|v-(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}\{\omega-|\boldsymbol{k}|v+(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon\}}\right). \qquad (35\mathrm{b})$$

In the non-quantum limit the terms $(\omega^2-|\boldsymbol{k}|^2)/2\varepsilon$ do not contribute. Hence this limit requires

$$|\omega-|\boldsymbol{k}|v| \gg |\omega^2-|\boldsymbol{k}|^2|/2\varepsilon. \qquad (36)$$

In this limit we find

$$
\begin{aligned}
\ln\Lambda_1 &= \frac{\omega^2-|\boldsymbol{k}|^2}{2\varepsilon}\left(\frac{1}{\omega-|\boldsymbol{k}|v}-\frac{1}{\omega+|\boldsymbol{k}|v}\right)\\
&\quad+\frac{(\omega^2-|\boldsymbol{k}|^2)^3}{12\varepsilon^3}\left(\frac{1}{(\omega-|\boldsymbol{k}|v)^3}-\frac{1}{(\omega+|\boldsymbol{k}|v)^3}\right)+\dots, \qquad (37\mathrm{a})\\
\ln\Lambda_2 &= 2\ln\left(\frac{\omega+|\boldsymbol{k}|v}{\omega-|\boldsymbol{k}|v}\right)\\
&\quad+\frac{(\omega^2-|\boldsymbol{k}|^2)^2}{4\varepsilon^2}\left(\frac{1}{(\omega-|\boldsymbol{k}|v)^2}-\frac{1}{(\omega+|\boldsymbol{k}|v)^2}\right)+\dots. \qquad (37\mathrm{b})
\end{aligned}
$$

The requirement (36) may be interpreted as follows. When a particle emits (upper sign) or absorbs (lower sign) a quantum $(\omega,\boldsymbol{k})$ the resonance condition $\varepsilon(\boldsymbol{p}\mp\boldsymbol{k}) = \varepsilon(\boldsymbol{p})\mp\omega$ requires, to lowest order in $\hbar$,

$$\omega = \boldsymbol{k}\,.\,\boldsymbol{v}\pm(2\varepsilon)^{-1}(|\boldsymbol{k}\,.\,\boldsymbol{v}|^2-|\boldsymbol{k}|^2)+\dots. \qquad (38)$$

The correction term to the classical resonance condition $\omega = \boldsymbol{k}.\boldsymbol{v}$ corresponds to the quantum recoil which splits the LD resonance into two parts. Hence, the singularities at $\omega - |\boldsymbol{k}|v = \pm(\omega^2 - |\boldsymbol{k}|^2)/2\varepsilon$ in (35) may be attributed to resonance in emission and absorption when the effects of the quantum recoil are retained.

Other explicit quantum terms in the dielectric functions (20) occur in the coefficients of $S^{(0)}(k)$: in the non-quantum limit the coefficient in (20a) vanishes and that in (20b) reduces to $-m^2/(1-\omega^2/|\boldsymbol{k}|^2)$.

In the non-quantum limit it is conventional to rewrite $\bar{n}(\boldsymbol{p})$ in terms of the classical distribution function $f(\boldsymbol{p})$. We need to introduce distributions $f^+(\boldsymbol{p})$ for electrons and $f^-(\boldsymbol{p})$ for positrons. Then in classical notation $\bar{n}(\boldsymbol{p})$ should be replaced according to

$$f^+(\boldsymbol{p}) + f^-(-\boldsymbol{p}) = 2\bar{n}(\boldsymbol{p})/(2\pi)^3\,, \tag{39}$$

where the factor of 2 arises from the two spin states.

6. Completely Degenerate Limit

The completely degenerate limit corresponds to

$$\bar{n}(\varepsilon) = 1\,, \qquad \varepsilon < \varepsilon_{\mathrm{F}}; \tag{40a}$$

$$= 0\,, \qquad \varepsilon > \varepsilon_{\mathrm{F}}; \tag{40b}$$

where ε_{F} is the Fermi energy. The functions $S^{(n)}(k)$ may be evaluated explicitly in this case. Some of the details of the evaluation are outlined in Appendix 2.

Denoting any quantity evaluated at $\varepsilon = \varepsilon_{\mathrm{F}}$ by a subscript F, we find that

$$S^{(0)}(k) = \frac{\varepsilon_{\mathrm{F}}}{m}\ln\Lambda_{1\mathrm{F}} - \frac{\omega}{2m}\ln\Lambda_{2\mathrm{F}} + \frac{\varepsilon_0}{m}\ln\Lambda_{3\mathrm{F}} + \frac{2|\boldsymbol{k}|}{m}\ln\left(\frac{\varepsilon_{\mathrm{F}}+p_{\mathrm{F}}}{m}\right), \tag{41a}$$

$$S^{(1)}(k) = \tfrac{1}{2}\left(\frac{\varepsilon_{\mathrm{F}}}{m}\right)^2\ln\Lambda_{1\mathrm{F}} - \tfrac{1}{2}\left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_{2\mathrm{F}} + \frac{\omega}{2m}\frac{\varepsilon_0}{m}\ln\Lambda_{3\mathrm{F}}$$
$$+\frac{2|\boldsymbol{k}|}{\omega}\left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\omega}{|\boldsymbol{k}|}\right)^2\frac{\varepsilon_0}{m}\right\}\ln\left(\frac{\varepsilon_{\mathrm{F}}+p_{\mathrm{F}}}{m}\right), \tag{41b}$$

$$S^{(2)}(k) = \frac{|\boldsymbol{k}|p_{\mathrm{F}}\varepsilon_{\mathrm{F}}}{3m^3} + \tfrac{1}{3}\left(\frac{\varepsilon_{\mathrm{F}}}{m}\right)^3\ln\Lambda_{1\mathrm{F}} - \tfrac{1}{3}\left(\frac{\omega}{2m}\right)\left\{\left(\frac{\omega}{2m}\right)^2 + 3\left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_{2\mathrm{F}}$$
$$+\tfrac{1}{3}\left(\frac{\varepsilon_0}{m}\right)\left\{3\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_{3\mathrm{F}}$$
$$+\frac{|\boldsymbol{k}|}{3m}\left[1 + 2\left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\frac{2\omega^2+|\boldsymbol{k}|^2}{|\boldsymbol{k}|^2}\right\}\right]\ln\left(\frac{\varepsilon_{\mathrm{F}}+p_{\mathrm{F}}}{m}\right). \tag{41c}$$

When substituted into (20) these reproduce results obtained by Jancovici (1962), allowing for differences in notation and a factor of e^2 which he has omitted in his equations (A1) and (A4). Note in particular that

$$\bar{n}_0 = \frac{m^3}{2\pi^2}\left\{\frac{\varepsilon_{\mathrm{F}}\,p_{\mathrm{F}}}{m^2} - \ln\left(\frac{\varepsilon_{\mathrm{F}}+p_{\mathrm{F}}}{m}\right)\right\}, \tag{42}$$

and $\ln\{(\varepsilon_F+p_F)/m\} = \sinh^{-1}(p_F/m)$. Equations (41) also reproduce a result due to Kowalenko (1982) for the longitudinal dielectric function, subject to appropriate choices of signs of some of his square roots.

7. High Temperature Pair Plasma

The functions $S^{(n)}(k)$ have also been evaluated in the limit of a high temperature pair plasma. This corresponds to a Fermi–Dirac distribution with zero chemical potential, with $n^+(\boldsymbol{p}) = n^-(\boldsymbol{p})$ and

$$\bar{n}(\varepsilon) = 2n^+(\boldsymbol{p}) = 2(e^\rho+1)^{-1}, \tag{43}$$

where $\rho = m/T$ is the inverse temperature. However, we have made several approximations in treating this case: we assume the long-wavelength limit and expand in $\omega/2m$ for $\omega < 2m$ and in $2m/\omega$ for $\omega > 2m$. The results, which are quite cumbersome, are written down in Appendix 3.

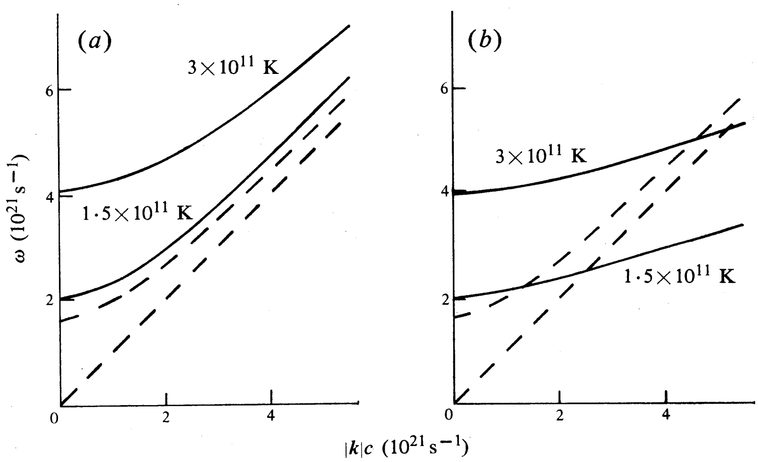


Fig. 3. Dispersion curves are shown for (*a*) longitudinal and (*b*) transverse waves in a pair plasma [cf. the distribution (43)] at $1\cdot5\times10^{11}$ and 3×10^{11} K. The lower and upper dashed curves correspond to $\omega = |\boldsymbol{k}|c$ and $\omega^2 = 4m^2c^4/\hbar + |\boldsymbol{k}|^2$ respectively, which separate the LD region to the right from the PC region to the left.

The dispersion relations for longitudinal and transverse waves can be found using an iterative procedure and, in the cases considered numerically, the iterative procedure was found to converge rapidly. The region of validity for our various approximations restricts our results to temperatures around 10^{11} K. The dispersion relations for (*a*) longitudinal and (*b*) transverse waves are plotted in Fig. 3 for $T = 1\cdot5\times10^{11}$ K and $3\cdot0\times10^{11}$ K. Also shown are the threshold curves $\omega = |\boldsymbol{k}|$ and $\omega^2 = 4m^2+|\boldsymbol{k}|^2$; LD occurs to the right of these curves and PC to the left of them. The dispersion curves for longitudinal waves cross from the LD region, through the dissipation-free region to the PC region, whereas the dispersion curves for transverse waves are entirely within the PC region.

8. Discussion

Our main emphasis in this paper is on the plasma dispersion functions which appear in a relativistic quantum treatment. In the accompanying paper (Melrose and Hayes 1984) we apply these results to thermal distributions of particles. Here we discuss the general features of the dispersion functions: resonances and the effect of particles on pair creation.

The plasma dispersion functions contain singularities, and when any particular singularity occurs in the physical regime $0 < |\boldsymbol{p}| < \infty$ it corresponds to a resonance. In the non-quantum case the only singularities are at $v = \pm\omega/|\boldsymbol{k}|$; for $|\omega| < |\boldsymbol{k}|$ this resonance is due to LD. In the relativistic quantum case there are eight singularities. Four of these correspond to LD and the other four to PC. The four LD singularities arise from pairwise splitting of the two non-quantum LD singularities; as suggested in Section 5, this splitting may be attributed to the different quantum recoils in emission and absorption. As in the non-quantum case, the singularities are related pairwise through the transformations $\omega \to -\omega$, and only half of them can correspond to resonances for a given sign of ω.

In Appendix 2 the singularities appear as zeros of the arguments of logarithms and are labelled $i = 1$–4 (cf. equations A26). These may be rewritten in terms of $|\boldsymbol{v}|$, $|\boldsymbol{p}|$ or ε using (A23). In terms of the resonant energies there are LD resonances at $\varepsilon = \varepsilon_0 \pm \frac{1}{2}\omega$ and PC resonances at $\varepsilon = \frac{1}{2}\omega \pm \varepsilon_0$. In the range $|\boldsymbol{k}|^2 < \omega^2 < 4m^2 + |\boldsymbol{k}|^2$ the parameter ε_0 is imaginary and there is no resonance. The resonant energies correspond to two roots in each of the regions of LD and PC and if we denote these roots by $\varepsilon_{\mathrm{R}1,2}$ then they are given by

$$\varepsilon_{\mathrm{R}1,2} = \varepsilon_1, \varepsilon_4, \qquad \text{LD}; \tag{44a}$$

$$= \varepsilon_1, \varepsilon_2, \qquad \text{PC}, \ \omega > 0; \tag{44b}$$

$$= \varepsilon_4, \varepsilon_3, \qquad \text{PC}, \ \omega < 0. \tag{44c}$$

The corresponding resonant speeds are

$$v_{1,2} = -v_{3,4} = \frac{|\boldsymbol{k}|/2m \pm \omega\varepsilon_0/|\boldsymbol{k}|m}{\omega/2m \pm \varepsilon_0/m}. \tag{45}$$

In the non-quantum limit these reduce to $v_{1,2} = -v_{3,4} = \omega/|\boldsymbol{k}|$.

The threshold for PC is at $\varepsilon_0 = 0$, when v_1 and v_2 are equal to the inverse of the phase speed. Hence, the resonant speed in the PC case is near the speed $|\boldsymbol{k}|/\omega$ which one would define by taking the ratio of the momentum to the energy of the photon as though it were a material particle. Again, except at threshold, the resonance is split in two.

Another interesting point concerns the effect of the particles on dissipation due to pair creation. Let us discuss this for the case of a completely degenerate electron gas. Dissipation due to LD in this case requires that the electron be in an occupied state, i.e. $\varepsilon < \varepsilon_{\mathrm{F}}$, before absorbing the quantum, and be in a previously empty state, i.e. $\varepsilon + \omega > \varepsilon_{\mathrm{F}}$, after absorbing the quantum. However, in PC there is no initial particle and there is a pair in the final state. There is no restriction on the positron (all positron states are initially unoccupied) but the electron must be in a previously unoccupied state, i.e. $\varepsilon > \varepsilon_F$. This implies that the presence of electrons suppresses dissipation due to pair creation below the value which one would have in vacuo. The sign of 'dissipation' due to PC in a plasma is opposite to that for LD and appears to cause growth. However, the actual effect is simply to reduce the dissipation which would occur in vacuo.

Acknowledgments

The initial work by L.M.H. on this topic was under the supervision of Dr N. E. Frankel. The introduction of the variable t (see Appendix 2) arose from a suggestion by Dr R. C. McPhedran.

References

Berestetskii, V. B., Lifshitz, E. M., and Pitaevskii, L. P. (1971). 'Relativistic Quantum Theory', Part I (Pergamon: Oxford).
Delsante, A. E., and Frankel, N. E. (1980). *Ann. Phys. (New York)* **125**, 135.
Hakim, R., and Heyvaerts, J. (1978). *Phys. Rev.* A **18**, 1250.
Harris, E. G. (1969). *Adv. Plasma Phys.* **3**, 157.
Jancovici, B. (1962). *Nuovo Cimento* **25**, 428.
Kowalenko, V. (1982). Ph.D. thesis, Univ. of Melbourne.
Melrose, D. B. (1982). *Aust. J. Phys.* **35**, 41.
Melrose, D. B., and Hayes, L. M. (1984). *Aust. J. Phys.* **37**, 639.
Melrose, D. B., and Parle, A. J. (1983). *Aust. J. Phys.* **36**, 799.
Tsytovich, V. N. (1961). *Sov. Phys. JETP* **13**, 1249.

Appendix 1

The electron propagator in coordinate space may be written as the vacuum expectation value (see e.g. Berestetskii *et al.* 1971; p. 251)

$$G(x-x') = -\mathrm{i}\,\mathrm{Tr}[\hat{\rho}_0\, T\{\hat{\psi}(x)\hat{\bar{\psi}}(x')\}], \tag{A1}$$

where

$$\hat{\rho}_0 = |0\rangle\langle 0| \tag{A2}$$

is the vacuum density operator, and T denotes the chronological product. Using a similar notation to equations (9) of Melrose and Parle (1983), the second quantized wavefunctions are written in the form

$$\hat{\psi}(x) = \sum_{q,\zeta} \hat{a}_q^\zeta\, \psi_q^\zeta(\boldsymbol{x})\exp(-\mathrm{i}\,\zeta\varepsilon_q t), \tag{A3a}$$

$$\hat{\bar{\psi}}(x) = \sum_{q,\zeta} \hat{\bar{a}}_q^\zeta\, \bar{\psi}_q^\zeta(\boldsymbol{x})\exp(\mathrm{i}\,\zeta\varepsilon_q t), \tag{A3b}$$

where q labels the quantum numbers collectively, with $\varepsilon_q = \varepsilon = (m^2+|\boldsymbol{p}|^2)^{\frac{1}{2}}$ here, and where $\hat{a}_q^+$, $\hat{a}_q^-$, $\hat{\bar{a}}_q^+$ and $\hat{\bar{a}}_q^-$ are the electron annihilation, positron creation, electron creation and positron annihilation operators respectively. The statistical average is achieved by replacing $\hat{\rho}_0$ by the density matrix $\hat{\rho}$ for the electron gas. Then (A1) becomes

$$G(x-x') = -\mathrm{i}\sum_{\zeta,q}\sum_{\zeta',q'} \mathrm{Tr}[\hat{\rho}\{\hat{a}_q^\zeta \hat{\bar{a}}_{q'}^{\zeta'}\,\vartheta(t-t') - \hat{\bar{a}}_{q'}^{\zeta'}\hat{a}_q^\zeta\,\vartheta(t'-t)\}]$$
$$\times\, \psi_q^\zeta(\boldsymbol{x})\,\bar{\psi}_{q'}^{\zeta'}(\boldsymbol{x}')\exp(-\mathrm{i}\,\zeta\varepsilon_q t)\exp(\mathrm{i}\,\zeta'\varepsilon_{q'} t'), \tag{A4}$$

where the extra minus sign arises from the anticommutation of the operators, namely

$$[\hat{a}_{q'}^{\zeta'}, \hat{a}_q^{\zeta}]_+ = \delta^{\zeta'\zeta}\,\delta_{q'q}. \tag{A5}$$

The trace (Tr) over states is performed most conveniently by introducing a Fock-space representation $|n_q^+, n_q^-\rangle$ for electrons and positrons; $\hat{\rho}$ may then be identified as the sum over all states of the outer product $|n_q^+, n_q^-\rangle\langle n_q^+, n_q^-|$. Then using the familiar relations $\langle n|aa^\dagger|n\rangle = n$ and $\langle n|a^\dagger a|n\rangle = 1-n$ for anticommuting creation $a^\dagger$ and annihilation a operators, one has

$$\mathrm{Tr}[\hat{\rho}\hat{a}_q^{\zeta}\hat{\bar{a}}_{q'}^{\zeta'}] = \delta^{\zeta\zeta'}\,\delta_{qq'}\{\tfrac{1}{2}(1+\zeta)-\zeta n_q^{\zeta}\}, \tag{A6a}$$

$$\mathrm{Tr}[\hat{\rho}\hat{\bar{a}}_{q'}^{\zeta'}\hat{a}_q^{\zeta}] = \delta^{\zeta\zeta'}\,\delta_{qq'}\{\tfrac{1}{2}(1-\zeta)+\zeta n_q^{\zeta}\}. \tag{A6b}$$

The next steps are to replace the step function by its integral representation,

$$\vartheta(t) = \mathrm{i}\int\frac{\mathrm{d}\Omega}{2\pi}\frac{\exp(-\mathrm{i}\,\Omega t)}{\Omega+\mathrm{i}0}, \tag{A7}$$

to identify the $\psi_q^{\zeta}(x)$ with the plane-wave functions,

$$\psi_q^{\zeta}(x) = \chi_\sigma^{\zeta}(\boldsymbol{p})\exp(\mathrm{i}\,\zeta\boldsymbol{p}.\boldsymbol{x}), \tag{A8}$$

and to perform the sum over spin σ using

$$\sum_\sigma \chi_\sigma^{\zeta}(\boldsymbol{p})\,\bar{\chi}_\sigma^{\zeta}(\boldsymbol{p}) = \frac{\zeta\gamma^\mu p_\mu+m}{\zeta\varepsilon}. \tag{A9}$$

The sum over q then reduces to an integral over $\mathrm{d}^3p/(2\pi)^3$ giving

$$\begin{aligned} G(x-x') = \int\frac{\mathrm{d}\Omega}{2\pi}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sum_\zeta\frac{\zeta\gamma^\mu p_\mu+m}{\zeta\varepsilon}\frac{1}{\Omega+\mathrm{i}0}&\exp\{-\mathrm{i}\,\zeta\varepsilon(t-t')\}\exp\{\mathrm{i}\,\zeta\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{x}')\}\\ \times[\{\tfrac{1}{2}(1+\zeta)-\zeta n^{\zeta}(\zeta\boldsymbol{p})\}&\exp\{-\mathrm{i}\,\Omega(t-t')\}\\ -\{\tfrac{1}{2}(1-\zeta)+\zeta n^{\zeta}(\zeta\boldsymbol{p})\}&\exp\{\mathrm{i}\,\Omega(t-t')\}]. \end{aligned} \tag{A10}$$

Finally a Fourier transform of (A10) is carried out,

$$G(P) = \int \mathrm{d}^4(x-x')\exp\{\mathrm{i}\,P(x-x')\}\,G(x-x'), \tag{A11}$$

and straightforward manipulation allows one to reduce $G(P)$ to

$$G(P) = (\gamma^\mu P_\mu+m)\left(\frac{1}{P^2-m^2+\mathrm{i}0}+\mathrm{i}\,\pi\delta(E-\zeta\varepsilon)\,n^{\zeta}(\zeta\boldsymbol{p})/\zeta\right), \tag{A12}$$

with $P = \zeta p$. The form (5) then follows from the definition (2) of $N(P)$.

Alternative Derivation of $\alpha^{\mu\nu}(k)$

Harris' (1969) non-relativistic quantum method may be generalized as follows to calculate $\alpha^{\mu\nu}(k)$. The Dirac Hamiltonian in second quantized form is separated into an unperturbed part

$$\hat{H}_0(t) = \sum_{\zeta q}\zeta\varepsilon_q\,\hat{\bar{a}}_q^{\zeta}(t)\,\hat{a}_q^{\zeta}(t), \tag{A13}$$

and a second part of first order in the test field $A_\mu(x)$

$$\hat{H}_1(t) = e \sum_{\varepsilon' q'} \sum_{\varepsilon q} \hat{\bar{a}}_{q'}^{\zeta'}(t)\, \hat{a}_q^{\zeta}(t) \int \mathrm{d}^3x\, \bar{\psi}_{q'}^{\zeta'}(\boldsymbol{x})\, \gamma^\mu \psi_q^{\zeta}(\boldsymbol{x})\, A_\mu(x) \times \exp\{\mathrm{i}(\zeta'\varepsilon_{q'} - \zeta\varepsilon_q)t\}. \tag{A14}$$

In Harris' method the temporal variation is included in the operators, or equivalently in the density matrix

$$\Psi_{q'q}^{\zeta'\zeta}(t) = \mathrm{Tr}[\hat{\rho}\hat{\bar{a}}_{q'}^{\zeta'}(t)\, \hat{a}_q^{\zeta}(t)]. \tag{A15}$$

The evolution of the products of operators in (A15) is found using the equation of motion

$$\frac{\mathrm{d}\hat{P}(t)}{\mathrm{d}t} = \mathrm{i}[\hat{H}(t), \hat{P}(t)], \tag{A16}$$

for any operator $\hat{P}(t)$. Using (A13), (A14), (A16) and the anticommutation relations (A5), one finds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \hat{\bar{a}}_{q'}^{\zeta'}(t)\, \hat{a}_q^{\zeta}(t) = {} & \mathrm{i}(\zeta'\varepsilon_{q'} - \zeta\varepsilon_q)\hat{\bar{a}}_{q'}^{\zeta'}(t)\, \hat{a}_q^{\zeta}(t) \\ & + \mathrm{i}\, e \sum_{\zeta'', q''} \int \mathrm{d}^3x'\, A_\nu(x') \\ & \times [\hat{\bar{a}}_{q''}^{\zeta''}(t)\, \hat{a}_q^{\zeta}(t)\, \bar{\psi}_{q''}^{\zeta''}(\boldsymbol{x}')\, \gamma^\nu \psi_{q'}^{\zeta'}(\boldsymbol{x}') \exp\{\mathrm{i}(\zeta\varepsilon_q - \zeta''\varepsilon_{q''})t\} \\ & - \hat{\bar{a}}_{q'}^{\zeta'}(t)\, \hat{a}_{q''}^{\zeta''}(t)\, \bar{\psi}_q^{\zeta}(\boldsymbol{x}')\, \gamma^\nu \psi_{q''}^{\zeta''}(\boldsymbol{x}') \exp\{\mathrm{i}(\zeta''\varepsilon_{q''} - \zeta'\varepsilon_{q'})t\}]. \end{aligned} \tag{A17}$$

Now we make a perturbation expansion in powers of A_ν. The zero order density matrix is given by (A6b):

$$[\Psi_{q'q}^{\zeta'\zeta}(t)]^{(0)} = \Psi_q^{\zeta} \delta^{\zeta'\zeta} \delta_{q'q}, \tag{A18a}$$

$$\Psi_q^{\zeta} = \tfrac{1}{2}(1-\zeta) + \zeta n_q^{\zeta}, \tag{A18b}$$

and the first order term then follows from (A17):

$$[\Psi_{q'q}^{\zeta'\zeta}(t)]^{(1)} = -e \int \frac{\mathrm{d}\Omega}{2\pi} \exp(-\mathrm{i}\Omega t) \frac{\Psi_q^{\zeta} - \Psi_{q'}^{\zeta'}}{\Omega - \zeta\varepsilon_q + \zeta'\varepsilon_{q'}} \int \mathrm{d}^3x'\, A_\nu(x', \Omega) \times \bar{\psi}_q^{\zeta}(\boldsymbol{x}')\, \gamma^\nu \psi_{q'}^{\zeta'}(\boldsymbol{x}') \exp\{\mathrm{i}(\zeta'\varepsilon_{q'} - \zeta\varepsilon_q)t\}. \tag{A19}$$

The linear response is described in terms of the linear current

$$\begin{aligned} [J^\mu(x)]^{(1)} & = -e\, \mathrm{Tr}[\hat{\rho}\hat{\bar{\psi}}(x)\, \gamma^\mu \hat{\psi}(x)]^{(1)} \\ & = -e \sum_{\zeta', q'} \sum_{\zeta, q} \bar{\psi}_{q'}^{\zeta'}(\boldsymbol{x}')\, \gamma^\mu \psi_q^{\zeta}(\boldsymbol{x}) \exp\{-\mathrm{i}(\zeta'\varepsilon_{q'} - \zeta\varepsilon_q)t\}[\Psi_{q'q}^{\zeta'\zeta}(t)]^{(1)}, \end{aligned} \tag{A20}$$

where we have inserted (A3) and retained the linearized part of (A15). After inserting (A19) in (A20), identifying the wavefunctions as in (A8), and writing

$$[J^\mu(x)]^{(1)} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \exp(\mathrm{i}\, kx)\, \alpha^{\mu\nu}(k)\, A_\nu(k), \tag{A21}$$

one identifies

$$\alpha^{\mu\nu}(k) = e^2 \sum_{\zeta',\zeta} Sp \int \frac{d^3p}{(2\pi)^3} \frac{\frac{1}{2}(\zeta'-\zeta)+\zeta n^{\zeta}(\zeta p)-\zeta' n^{\zeta'}(\zeta' p')}{\omega-\zeta\varepsilon+\zeta'\varepsilon'}$$

$$\times \left(\frac{1}{\zeta\zeta'\varepsilon\varepsilon'}\gamma^{\mu}(\zeta\gamma^{\tau}p_{\tau}+m)\gamma^{\nu}(\zeta'\gamma^{\tau}p'_{\tau}+m)\right). \tag{A22}$$

In (A22) it is assumed that the occupation numbers are independent of spin so that the sum over spin states may be performed using (A9), and $\zeta' p' = \zeta p - k$ is implicit. Apart from notation, (A22) is equivalent to (7a).

Appendix 2

Here we write down some alternative forms of the functions Λ_1 and Λ_2 defined by (24), and of a third function Λ_3. We then carry out the integrals involved in deriving (41).

Alternative Forms for Λ_1, Λ_2, Λ_3

We introduce four different variables which are monotonic functions of the energy ε of a particle: these are ε itself and $|\boldsymbol{p}|$, $v = |\boldsymbol{p}|/\varepsilon$ and t, given by

$$\frac{\varepsilon}{m} = \frac{1+t^2}{1-t^2}, \qquad \frac{|\boldsymbol{p}|}{m} = \frac{2t}{1-t^2}, \qquad v = \frac{2t}{1+t^2}. \tag{A23}$$

The functions (24) and Λ_3 may be expressed in the following forms:

$$\Lambda_1 = \frac{(t+t_1)(t+t_2)(t+t_3)(t+t_4)}{(t-t_1)(t-t_2)(t-t_3)(t-t_4)} = \frac{(p+p_1)(p+p_2)}{(p-p_1)(p-p_2)}$$

$$= \frac{4\varepsilon^2\omega^2-(\omega^2-|\boldsymbol{k}|^2-2|\boldsymbol{p}||\boldsymbol{k}|)^2}{4\varepsilon^2\omega^2-(\omega^2-|\boldsymbol{k}|^2+2|\boldsymbol{p}||\boldsymbol{k}|)^2}, \tag{A24a}$$

$$\Lambda_2 = \frac{(t+t_1)(t+t_2)(t-t_3)(t-t_4)}{(t-t_1)(t-t_2)(t+t_3)(t+t_4)} = \frac{(v+v_1)(v+v_2)}{(v-v_1)(v-v_2)}$$

$$= \frac{4(\varepsilon\omega+|\boldsymbol{p}||\boldsymbol{k}|)^2-(\omega^2-|\boldsymbol{k}|^2)^2}{4(\varepsilon\omega-|\boldsymbol{p}||\boldsymbol{k}|)^2-(\omega^2-|\boldsymbol{k}|^2)^2}, \tag{A24b}$$

$$\Lambda_3 = \frac{(t+t_1)(t-t_2)(t-t_3)(t+t_4)}{(t-t_1)(t+t_2)(t+t_3)(t-t_4)} = \frac{(v+v_1)(v-v_2)}{(v-v_1)(v+v_2)}$$

$$= \frac{[(\omega^2-|\boldsymbol{k}|^2)\varepsilon+\boldsymbol{p}\{(|\boldsymbol{k}|^2-\omega^2)(|\boldsymbol{k}|^2-\omega^2+4m^2)\}^{\frac{1}{2}}]^2-4m^4\omega^2}{[(\omega^2-|\boldsymbol{k}|^2)\varepsilon-\boldsymbol{p}\{(|\boldsymbol{k}|^2-\omega^2)(|\boldsymbol{k}|^2-\omega^2+4m^2)\}^{\frac{1}{2}}]^2-4m^4\omega^2}. \tag{A24c}$$

In terms of the parameters

$$a = (\omega^2-|\boldsymbol{k}|^2)/2m\omega, \qquad b = |\boldsymbol{k}|/\omega, \tag{A25a, b}$$

we have

$$t_1 = \frac{b+(a^2+b^2-1)^{\frac{1}{2}}}{1+a} = -1/t_3\,, \tag{A26a}$$

$$t_2 = \frac{b-(a^2+b^2-1)^{\frac{1}{2}}}{1+a} = -1/t_4\,, \tag{A26b}$$

with ε_i, p_i and v_i for $i = 1,2$ defined by substituting $t = t_i$ in (A23). One finds

$$\varepsilon_{1,2} = \tfrac{1}{2}\omega \pm \varepsilon_0\,, \qquad \varepsilon_{3,4} = -(\tfrac{1}{2}\omega \pm \varepsilon_0)\,, \tag{A27}$$

with

$$\frac{\omega}{2m} = \frac{a}{1-b^2}\,, \qquad \frac{\varepsilon_0}{m} = \frac{b}{1-b^2}(a^2+b^2-1)^{\frac{1}{2}}\,. \tag{A28a, b}$$

Evaluation of Certain Integrals and Sums

The quantities $S^{(n)}(k)$ defined by equations (23) are evaluated in the completely degenerate limit by partially integrating once and writing the remaining integral in terms of the variable t introduced in (A23):

$$S^{(0)}(k) = \frac{\varepsilon_F}{m}\ln\Lambda_{1F} - \sum_{i=1}^{4}\int_0^{t_F} dt\,\frac{1+t^2}{1-t^2}\left(\frac{1}{t+t_i} - \frac{1}{t-t_i}\right), \tag{A29a}$$

$$S^{(1)}(k) = \tfrac{1}{2}\left(\frac{\varepsilon_F}{m}\right)^2\ln\Lambda_{2F} - \tfrac{1}{2}\sum_{i=1}^{4}\eta_i\int_0^{t_F} dt\,\left(\frac{1+t^2}{1-t^2}\right)^2\left(\frac{1}{t+t_i} - \frac{1}{t-t_i}\right), \tag{A29b}$$

$$S^{(2)}(k) = \tfrac{1}{3}\left(\frac{\varepsilon_F}{m}\right)^3\ln\Lambda_{1F} - \tfrac{1}{3}\sum_{i=1}^{4}\int_0^{t_F} dt\,\left(\frac{1+t^2}{1-t^2}\right)^3\left(\frac{1}{t+t_i} - \frac{1}{t-t_i}\right), \tag{A29c}$$

with $\eta_i = 1$ for $i = 1,2$ and $\eta_i = -1$ for $i = 3,4$, and with the t_i given by equations (A26). The t integrals are elementary. Writing

$$I_i^{(n)}(t) = \int dt\,\left(\frac{1+t^2}{1-t^2}\right)^{1+n}\left(\frac{1}{t+t_i} - \frac{1}{t-t_i}\right), \tag{A30}$$

we find that

$$I_i^{(0)}(t) = \frac{1+t_i^2}{1-t_i^2}\ln\left|\frac{t+t_i}{t-t_i}\right| - \frac{2t_i}{1-t_i^2}\ln\left|\frac{1+t}{1-t}\right|, \tag{A31a}$$

$$I_i^{(1)}(t) = \left(\frac{1+t_i^2}{1-t_i^2}\right)^2\ln\left|\frac{t+t_i}{t-t_i}\right| - \frac{2t_i}{1-t_i^2}\,\frac{2t}{1-t^2} + \left(\frac{2t_i}{1-t_i^2} + \frac{4t_i}{(1-t_i^2)^2}\right)\ln\left|\frac{1+t}{1-t}\right|, \tag{A31b}$$

$$\begin{aligned} I_i^{(2)}(t) = {} & \left(\frac{1+t_i^2}{1-t_i^2}\right)^3\ln\left|\frac{t+t_i}{t-t_i}\right| - \frac{1}{2}\,\frac{2t_i}{1-t_i^2}\,\frac{4t}{(1-t^2)^2} + \left(\frac{3t_i}{1-t_i^2} - \frac{4t_i}{(1-t_i^2)^2}\right)\frac{2t}{1-t^2} \\ & - \left(\frac{2t_i(3+t_i^2)}{(1-t_i^2)^3} - \frac{3}{2}\,\frac{4t_i}{(1-t_i^2)^3} + \frac{3t_i}{1-t_i^2}\right)\ln\left|\frac{1+t}{1-t}\right|. \end{aligned} \tag{A31c}$$

Using equations (A25)–(A28) we have

$$\frac{1+t_{1,2}^2}{1-t_{1,2}^2} = -\frac{1+t_{3,4}^2}{1-t_{3,4}^2} = \frac{\omega}{2m} \pm \frac{\varepsilon_0}{m}, \tag{A32a}$$

$$\frac{2t_{1,2}}{1-t_{1,2}^2} = \frac{2t_{3,4}}{1-t_{3,4}^2} = \frac{|\boldsymbol{k}|}{2m} \pm \frac{\omega\varepsilon_0}{|\boldsymbol{k}|m}. \tag{A32b}$$

These together with (A24) imply

$$\sum_{i=1}^{4} \frac{1+t_i^2}{1-t_i^2} \ln\left|\frac{t+t_i}{t-t_i}\right| = \frac{\omega}{2m}\ln\Lambda_2 - \frac{\varepsilon_0}{m}\ln\Lambda_3, \tag{A33a}$$

$$\sum_{i=1}^{4} \eta_i \left(\frac{1+t_i^2}{1-t_i^2}\right)^2 \ln\left|\frac{t+t_i}{t-t_i}\right| = \left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_2 - 2\frac{\omega}{2m}\frac{\varepsilon_0}{m}\ln\Lambda_3, \tag{A33b}$$

$$\sum_{i=1}^{4} \left(\frac{1+t_i^2}{1-t_i^2}\right)^3 \ln\left|\frac{t+t_i}{t-t_i}\right| = \frac{\omega}{2m}\left\{\left(\frac{\omega}{2m}\right)^2 + 3\left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_2 - \frac{\varepsilon_0}{m}\left\{3\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\right\}\ln\Lambda_3. \tag{A33c}$$

Other relevant sums are

$$\sum_{i=1}^{4} \frac{2t_i}{1-t_i^2} = \sum_{i=1}^{4} \frac{4t_i}{(1-t_i^2)^2} = 2|\boldsymbol{k}|/m, \tag{A34a}$$

$$\sum_{i=1}^{4} \frac{2t_i(3+t_i^2)}{(1-t_i^2)^3} = \frac{|\boldsymbol{k}|}{m}\left[1+2\left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_0}{m}\right)^2\right\}\right], \tag{A34b}$$

$$\sum_{i=1}^{4} \eta_i \frac{2t_i}{1-t_i^2} = 0, \tag{A34c}$$

$$\sum_{i=1}^{4} \eta_i \frac{4t_i}{(1-t_i^2)^2} = \frac{4|\boldsymbol{k}|}{\omega}\left\{\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\omega}{|\boldsymbol{k}|}\right)^2\left(\frac{\varepsilon_0}{m}\right)^2\right\}. \tag{A34d}$$

The remainder of the derivation of equations (41) involves evaluating t at t_{F} and expressing the result in terms of ε_{F} and p_{F} using (A23).

Appendix 3

The functions $A^{\mathrm{L,T}}(\omega)$ and $B^{\mathrm{L,T}}(\omega)$ defined by (29) may be evaluated for the distribution (43) by expanding in either $\omega/2m$ for $\omega < 2m$, or $2m/\omega$ for $\omega > 2m$.

Expansion in $\omega/2m < 1$

Writing

$$A^{\mathrm{L,T}}(\omega) = -\frac{e^2m^2}{2\pi^2\rho^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{2m}\right)^{2n} f^{\mathrm{L,T}}(\omega,\rho,n), \tag{A35}$$

$$B^{\mathrm{L,T}}(\omega) = -\frac{e^2}{2\pi^2}\sum_{n=0}^{\infty}\left(\frac{\omega}{2m}\right)^{2n} g^{\mathrm{L,T}}(\omega,\rho,n)\,, \tag{A36}$$

we find, setting $r = \omega/2m$,

$$\begin{aligned} f^{\mathrm{L}}(\omega,\rho,n) = f^{\mathrm{T}}(\omega,\rho,n) = {} & (1-r^2)^{1/2}\left\{4\tau(2)\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,n!} + 4\pi\rho^{2n+1}\frac{\tau(1-2n)}{(2n-1)!}\right. \\ & + 4\rho^2\frac{\Gamma(n+\frac{3}{2})}{\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{3}{2})}{(p+1)!\,(n-p-\frac{1}{2})} \\ & \left. + \rho^2\frac{\Gamma(n+\frac{3}{2})}{\sqrt{\pi}\,n!\,(n-\frac{1}{2})}\left(\ln(\pi/\rho) - C - \tfrac{1}{2} - \frac{1}{2n-1}\right)\right\} \\ & + (1-r^2)^{3/2}\left\{-\frac{8\tau(2)\,\Gamma(n+\frac{3}{2})}{3\sqrt{\pi}\,n!} + \tfrac{4}{3}\pi\rho^{2n+3}\frac{\tau(-1-2n)}{(2n+1)!}\right. \\ & + \tfrac{4}{3}\rho^2\frac{\Gamma(n+\frac{5}{2})}{\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{1}{2})}{(p+1)!\,(n-p+\frac{1}{2})} \\ & \left. + \tfrac{2}{3}\rho^2\frac{\Gamma(n+\frac{5}{2})}{\sqrt{\pi}\,n!\,(n+\frac{1}{2})}\left(\ln(\pi/\rho) - C + \tfrac{1}{2} - \frac{1}{2n+1}\right)\right\}, \end{aligned} \tag{A37}$$

$$\begin{aligned} g^{\mathrm{L}}(\omega,\rho,n) = {} & \\ & (1-r^2)^{-1/2}\left\{\tfrac{1}{2}\pi\rho^{2n-1}\frac{\tau(1-2n)}{(2n-1)!} + \frac{\Gamma(n-\frac{1}{2})}{8\sqrt{\pi}\,n!}\left(\ln(\pi/\rho) - C - 1 - \frac{1}{2n-1}\right)\right. \\ & \left. + \frac{\Gamma(n+\frac{1}{2})}{2\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{3}{2})}{p!\,(n-p-\frac{1}{2})}\right\} \\ & + (1-r^2)^{1/2}\left\{\pi\rho^{2n+1}\frac{\tau(-1-2n)}{(2n+1)!} + \frac{\Gamma(n+\frac{1}{2})}{2\sqrt{\pi}\,n!}\left(\ln(\pi/\rho) - C - \frac{1}{2n+1}\right)\right. \\ & \left. + \frac{\Gamma(n+\frac{3}{2})}{\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{1}{2})}{p!\,(n-p+\frac{1}{2})}\right\} \\ & + (1-r^2)^{3/2}\left\{2\tau(2)\frac{\Gamma(n+\frac{3}{2})}{r^2\sqrt{\pi}\,n!} - \pi\rho^{2n+3}\frac{\tau(-1-2n)}{r^2(2n+1)!} - \tfrac{1}{6}\pi\rho^{2n+3}(4n^2+10n-3)\frac{\tau(-3-2n)}{(2n+3)!}\right. \\ & - \frac{3\Gamma(n+\frac{3}{2})}{2\sqrt{\pi}\,n!}\left(\ln(\pi/\rho) - C - \tfrac{10}{9} - \frac{1}{2n+3}\right) \\ & - \tfrac{3}{2}\sqrt{\pi}\,\rho\tau(-1)\frac{\Gamma(n+\frac{5}{2})}{(n+1)!} - \frac{\rho^2\Gamma(n+\frac{5}{2})}{r^2 2\sqrt{\pi}\,n!\,(n+\frac{1}{2})}\left(\ln(\pi/\rho) - C + \tfrac{1}{2} - \frac{1}{2n+1}\right) \\ & - \frac{\rho^2}{r^2}\frac{\Gamma(n+\frac{5}{2})}{\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{1}{2})}{(p+1)!\,(n-p+\frac{1}{2})} - \end{aligned}$$

$$-\frac{\Gamma(n+\frac{5}{2})}{6\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)(4p^2-2p-9)\frac{\Gamma(p-\frac{1}{2})}{p!(n-p+\frac{3}{2})}\bigg\}$$

$$+(1-r^2)^{5/2}\bigg\{-\frac{4\tau(2)\Gamma(n+\frac{5}{2})}{5r^2\sqrt{\pi}\,n!}-\tfrac{3}{5}\pi\rho^{2n+5}\frac{\tau(-3-2n)}{r^2(2n+3)!}$$

$$-\tfrac{2}{5}\pi\rho^{2n+5}(n+1)(n+\tfrac{7}{2})\frac{\tau(-5-2n)}{(2n+5)!}+\tfrac{3}{5}\sqrt{\pi}\,\rho\tau(-1)\frac{\Gamma(n+\frac{7}{2})}{n!(n+2)}$$

$$+\tfrac{2}{5}\sqrt{\pi}\,\rho^3\tau(-1)\frac{\Gamma(n+\frac{7}{2})}{r^2(n+1)!}+\frac{2\Gamma(n+\frac{5}{2})}{5\sqrt{\pi}\,n!}\left(\ln(\pi/\rho)-C-\tfrac{3}{2}-\frac{1}{2n+5}\right)$$

$$+\frac{\rho^2 3\Gamma(n+\frac{7}{2})}{r^2 5\sqrt{\pi}\,n!(n+\frac{3}{2})}\left(\ln(\pi/\rho)-C-\tfrac{1}{2}-\frac{1}{2n+3}\right)$$

$$-\frac{\rho^2 3\Gamma(n+\frac{7}{2})}{r^2 5\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p-\frac{1}{2})}{(p+1)!(n-p+\frac{3}{2})}$$

$$-\frac{2\Gamma(n+\frac{7}{2})}{5\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{(p+1)\Gamma(p-\frac{1}{2})}{p!(n-p+\frac{5}{2})}\bigg\}, \tag{A38}$$

$$g^{\mathrm{T}}(\omega,\rho,n)=$$

$$(1-r^2)^{-1/2}\bigg\{\tfrac{1}{2}\pi\rho^{2n-1}\frac{\tau(1-2n)}{(2n-1)!}+\frac{\Gamma(n-\frac{1}{2})}{8\sqrt{\pi}\,n!}\left(\ln(\pi/\rho)-C-1-\frac{1}{2n-1}\right)$$

$$+\frac{\Gamma(n+\frac{1}{2})}{2\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{3}{2})}{p!(n-p-\frac{1}{2})}\bigg\}$$

$$+(1-r^2)^{3/2}\bigg\{\frac{2\tau(2)\Gamma(n+\frac{3}{2})}{3r^2\sqrt{\pi}\,n!}-\tfrac{1}{6}\pi\rho^{2n+3}(4n^2+10n+3)\frac{\tau(-3-2n)}{(2n+3)!}$$

$$-\tfrac{1}{3}\pi\rho^{2n+3}\frac{\tau(-1-2n)}{r^2(2n+1)!}-\frac{\rho^2\Gamma(n+\frac{5}{2})}{r^2 6\sqrt{\pi}\,n!(n+\frac{1}{2})}\left(\ln(\pi/\rho)-C+\tfrac{1}{2}-\frac{1}{2n+1}\right)$$

$$-\tfrac{1}{2}\sqrt{\pi}\,\rho\tau(-1)\frac{\Gamma(n+\frac{5}{2})}{(n+1)!}-\frac{\Gamma(n+\frac{3}{2})}{2\sqrt{\pi}\,n!}\left(\ln(\pi/\rho)-C-\tfrac{4}{3}-\frac{1}{2n+3}\right)$$

$$-\frac{\rho^2\Gamma(n+\frac{5}{2})}{r^2 3\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+\frac{1}{2})}{(p+1)!(n-p+\frac{1}{2})}$$

$$-\frac{\Gamma(n+\frac{5}{2})}{6\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)(4p^2-2p-3)\frac{\Gamma(p-\frac{1}{2})}{p!(n-p+\frac{3}{2})}\bigg\}$$

$$+(1-r^2)^{5/2}\bigg\{-\frac{4\tau(2)\Gamma(n+\frac{5}{2})}{15r^2\sqrt{\pi}\,n!}-\tfrac{2}{15}\pi\rho^{2n+5}(n+1)(n+\tfrac{7}{2})\frac{\tau(-5-2n)}{(2n+5)!}$$

$$-\tfrac{1}{5}\pi\rho^{2n+5}\frac{\tau(-3-2n)}{r^2(2n+3)!}+\frac{\rho^2\Gamma(n+\frac{7}{2})}{r^2 5\sqrt{\pi}\,n!(n+\frac{3}{2})}\left(\ln(\pi/\rho)-C-\tfrac{1}{2}-\frac{1}{2n+3}\right)+$$

$$+\tfrac{2}{15}\sqrt{\pi}\,\rho^3\tau(-1)\frac{\Gamma(n+\frac{7}{2})}{r^2(n+1)!}+\frac{2\Gamma(n+\frac{5}{2})}{15\sqrt{\pi}\,n!}\left(\ln(\pi/\rho)-C-\tfrac{3}{2}-\frac{1}{2n+5}\right)$$

$$+\tfrac{1}{5}\sqrt{\pi}\,\rho\tau(-1)\frac{\Gamma(n+\frac{7}{2})}{n!(n+2)}-\frac{2\Gamma(n+\frac{7}{2})}{15\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{(p+1)\Gamma(p-\frac{1}{2})}{p!(n-p+\frac{5}{2})}$$

$$-\frac{\rho^2\Gamma(n+\frac{7}{2})}{r^2 5\pi n!}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p-\frac{1}{2})}{(p+1)!(n-p+\frac{3}{2})}\Bigg\}, \tag{A39}$$

with

$$\tau(s)=(1-2^{1-s})\zeta(s), \qquad \tau'(s)=(\mathrm{d}/\mathrm{d}s)\tau(s), \tag{A40}$$

where $\zeta(s)$ is the Riemann zeta function and C is Euler's constant.

Expansion in $2m/\omega<1$

In the limit $\omega \gg 2m$, equations (A35) and (A36) are replaced by

$$A^{\mathrm{L}}(\omega)=A^{\mathrm{T}}(\omega)=-\frac{e^2m^2}{2\pi^2\rho^2}$$

$$\times\Bigg\{\frac{8\tau(2)}{3}-r^2\left(\ln(\pi/2r)-C+2\sum_{p=1}^{\infty}\frac{r^{2p}}{(2p)!}\tau'(-2p)\right)\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{(n-1)\Gamma(n-\frac{3}{2})}{\sqrt{\pi}\,n!}$$

$$+2r^2\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{(p+n-1)\Gamma(p+n-\frac{3}{2})}{\sqrt{\pi}(p+n)!}$$

$$+r^2\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{(n-1)\Gamma(n-\frac{3}{2})}{\sqrt{\pi}\,n!}\left(\ln(\pi/2\rho)-C-\tfrac{1}{2}\psi(n-\tfrac{3}{2})+\tfrac{1}{2}\psi(n+1)-\frac{1}{2n-2}\right)\Bigg\}, \tag{A41}$$

$$B^{\mathrm{L}}(\omega)=-\frac{e^2}{2\pi^2}$$

$$\times\Bigg\{\frac{2\tau(2)}{5r^2}-\left(\ln(\rho/r)+\tfrac{1}{2}\psi(n-\tfrac{3}{2})-\tfrac{1}{2}\psi(n+1)+\frac{2n-1}{2n(n-1)}\right)\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{\Gamma(n-\frac{3}{2})}{4\sqrt{\pi}(n-2)!}$$

$$+\tfrac{1}{30}+\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{(n-1)\Gamma(n-\frac{5}{2})}{2\sqrt{\pi}\,n!}\sum_{p=1}^{\infty}\frac{r^{2p}}{(2p)!}\tau'(-2p)\{p(p-\tfrac{1}{2})-n(n-\tfrac{5}{2})\}$$

$$+\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{(n+p-1)\Gamma(n+p-\frac{5}{2})}{2\sqrt{\pi}(n+p)!}\{(n+p)(n+p-\tfrac{5}{2})-p(p-\tfrac{1}{2})\}\Bigg\}, \tag{A42}$$

$$B^{\mathrm{T}}(\omega)=-\frac{e^2}{2\pi^2}$$

$$\times\Bigg\{\frac{2\tau(2)}{15r^2}-\frac{\rho^2}{12r^2}+\tfrac{1}{15}-\{\ln(\pi/2r)-C\}\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{(n-1)^2\Gamma(n-\frac{3}{2})}{4\sqrt{\pi}\,n!}$$

$$+\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{\Gamma(n-\frac{5}{2})}{2\sqrt{\pi}\,n!}\sum_{p=1}^{\infty}\frac{r^{2p}}{(2p)!}\tau'(-2p)\{(n-2)p(p-\tfrac{1}{2})-(n-1)^2(n-\tfrac{5}{2})\}$$

$$+\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\frac{(n-1)^2\Gamma(n-\frac{3}{2})}{4\sqrt{\pi}\,n!}\left(\ln(\pi/2\rho)-C-\tfrac{1}{2}\psi(n-\tfrac{3}{2})+\tfrac{1}{2}\psi(n+1)-\frac{1}{n-1}\right)$$

$$+\sum_{n=2}^{\infty}\left(\frac{\rho}{r}\right)^{2n}\sum_{p=1}^{\infty}\frac{\rho^{2p}}{(2p)!}\tau'(-2p)\frac{\Gamma(p+n-\frac{5}{2})}{2\sqrt{\pi}(n+p)!}\{(p+n-\tfrac{5}{2})(p+n-1)^2$$

$$\left.-(p+n-2)p(p-\tfrac{1}{2})\}\right\}, \qquad \text{(A43)}$$

where $\psi(x)$ is the digamma function.

Manuscript received 2 July, accepted 6 September 1984