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Higher Frequency Spectrum of a Diffuse Linear Pinch with Multiple Ion Species

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Abstract

A cold plasma model which takes into account finite ion cyclotron frequency effects and multiple ion species has been developed for wave propagation in arbitrary magnetic field geometries. This model has been used to derive an elegant system of normal mode equations for a low-β diffuse linear pinch. From a soluble model, general features of the spectrum are discussed up to and including the ion cyclotron range of frequencies. It is indicated that in the vicinity of the ion–ion hybrid cutoff frequency, there could exist global eigenmodes which might be useful for supplementary heating of diffuse linear pinches.

1. Introduction

Studies of wave propagation in diffuse pinches are valuable in the development and evaluation of radio-frequency (r.f.) heating schemes and novel diagnostic techniques for magnetically confined fusion plasmas. Promising r.f. heating results have been obtained experimentally from the exploitation of the ion–ion hybrid resonance in tokamak plasmas (Hosea *et al.* 1982; Equipe TFR 1982). In the ion cyclotron range of frequencies, as well as at lower frequencies in the Alfvén wave heating schemes (Appert *et al.* 1982*a*; de Chambrier *et al.* 1982; Ross *et al.* 1982), the geometry of the magnetic field configuration is important in determining eigenmode structures, a correct description of which is necessary for accurate antenna–plasma coupling calculations in various heating schemes. The ideal magnetohydrodynamic (MHD) model, which has been extensively applied in plasma stability studies, lends itself conveniently to the consideration of geometric effects. Unfortunately, this model is limited to plasmas with only one ion species and is restricted to cases of very low frequencies, compared with the ion cyclotron frequencies. It is the purpose of the present paper to develop new plasma models which are more suitable for wave propagation and r.f. heating studies at higher frequencies.

In the next section, a model for low-β plasmas, which includes finite ion cyclotron frequency effects and multiple ion species, is constructed from multi-fluid equations. Linearized equations are obtained for arbitrary geometry of the magnetic field configuration. The model developed is essentially a cold plasma model where plasma pressure is considered unimportant in determining the structure of higher frequency eigenmodes in a low-β plasma.

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In Section 3, the linearized equations are reduced, in normal mode analysis of a diffuse linear pinch, to an elegant system of two coupled first order ordinary differential equations. This system is an extension of the corresponding pressureless system for ideal MHD (Appert *et al.* 1974).

A spectral analysis is carried out in Section 4 for a soluble model of a diffuse linear pinch with two ion species. The new features which emerge are discussed in relation to the existing literature. The final section is a discussion of the significance of the new plasma model and its possible further developments.

2. Plasma Model and Basic Equations

In stable magnetically confined plasmas, the plasma gas pressure is usually much smaller than the total magnetic pressure. For such low-β plasmas, it is the magnetic pressure rather than the plasma pressure which is responsible for the propagation of shear Alfvén waves and fast magnetosonic waves. Hence, as has been justified in many previous studies on r.f. heating (Adam and Jacquinot 1977; Messiaen *et al.* 1978; Appert and Vaclavik 1982), a pressureless cold plasma model will provide a good approximate description of these modes of propagation. In the construction of a plasma model which extends the low-β ideal MHD model, many of the assumptions made in justifying the ideal MHD model will be made here. The important exceptions are (1) the low frequency assumption is relaxed by the inclusion of finite ion cyclotron frequency effects, and (2) the plasma is allowed to be composed of multiple ion species. The latter feature requires the development of an equation to replace the conventional Ohm law, which is appropriate only in a one fluid description.

In a pressureless plasma, the equation of motion for particles of species α with mass m_α and charge q_α is

$$m_\alpha \, \mathrm{d}\boldsymbol{v}_\alpha/\mathrm{d}t = q_\alpha(\boldsymbol{E}+\boldsymbol{v}_\alpha \times \boldsymbol{B}), \tag{1}$$

where $\boldsymbol{v}_\alpha$ is the macroscopic fluid velocity. This set of equations is coupled to Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \qquad \nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{j}_{\mathrm{ex}} + \partial \boldsymbol{D}/\partial t), \tag{2a, b}$$

where $\boldsymbol{j}_{\mathrm{ex}}$ is the extraneous current density and $\boldsymbol{D}$ is the electric displacement related in the usual way to the induced current density $\boldsymbol{j}_{\mathrm{in}}$, which is defined by

$$\boldsymbol{j}_{\mathrm{in}} \equiv \sum_\alpha q_\alpha n_\alpha \boldsymbol{v}_\alpha, \tag{3}$$

where n_α is the number density of particles of species α. Total mass density ρ_0 and averaged macroscopic fluid velocity $\boldsymbol{v}$ may be introduced by the definitions

$$\rho_0 \equiv \sum_\alpha m_\alpha n_\alpha, \qquad \rho_0 \boldsymbol{v} \equiv \sum_\alpha m_\alpha n_\alpha \boldsymbol{v}_\alpha. \tag{4a, b}$$

To obtain a set of one fluid equations it is helpful, for clarity, to recapitulate what is normally done for the case of a plasma with one ion species. Multiplication of equation (1) by n_α, followed by summation over the particle species, leads to the usual momentum transport equation. Linearization of this equation about a plasma equilibrium with current density $\boldsymbol{j}_0$ and magnetic field $\boldsymbol{B}_0$ leads to an averaged linearized equation of motion

$$\rho_0\, \partial \boldsymbol{v}/\partial t = \boldsymbol{j} \times \boldsymbol{B}_0 + \boldsymbol{j}_0 \times \boldsymbol{B}, \tag{5}$$

where, here and below, vectors without the zero subscript denote perturbed quantities. We note that this equation is identical to the linearized equation of motion for the pressureless ideal MHD model. It is obtained when one makes the usual assumption that terms such as $\boldsymbol{v}.\nabla\boldsymbol{v}_0$ are small in comparison with $\partial \boldsymbol{v}/\partial t$; that is, the equilibrium drift velocity $\boldsymbol{v}_0$ associated with $\boldsymbol{j}_0$ is small. The set of one-fluid equations of motion and Maxwell's equations is usually closed by a form of Ohm's law, which may be obtained from equation (1) by multiplying it by $q_\alpha n_\alpha/m_\alpha$ and summing over particle species. For the case of the ideal MHD model, some terms in this equation are then dropped, following physical arguments, to give the linearized equation

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}_0 = 0. \tag{6}$$

This procedure of closing the set of one-fluid equations fails for the case of a plasma with more than one ion species, because the quasi-neutrality condition, $\Sigma_\alpha\, q_\alpha n_\alpha = 0$, is not sufficient to allow an elimination of variables giving the usual forms of a generalized Ohm law. In the following discussion, a linearized equation is derived which will fulfil the same function in closing the set of equations as a linearized Ohm law.

A local orthonormal coordinate system $(r, \perp, \|)$, based on a magnetic field line, may be introduced and the perturbed electric field may be written as

$$\boldsymbol{E} = E_r \boldsymbol{e}_r + E_\perp(\boldsymbol{b} \times \boldsymbol{e}_r) + E_\| \boldsymbol{b}, \tag{7}$$

where $\boldsymbol{b} = \boldsymbol{B}_0/B_0$, $\boldsymbol{e}_r$ is a unit vector normal to the magnetic surface r constant, and $\boldsymbol{b} \times \boldsymbol{e}_r$, orthonormal to $\boldsymbol{b}$, is tangential to the surface. Consistent with the assumptions made in obtaining (5), the plasma equilibrium is regarded as stationary and the linearized equations of (1), Fourier analysed in time according to $\exp(-\mathrm{i}\omega t)$, are solved individually in this system of coordinates in the guiding centre approximation (see Appendix 1). Appropriate summations of these solutions give

$$-\mathrm{i}\omega\rho_0 \boldsymbol{v} = \mathbf{T}.\boldsymbol{E}, \tag{8}$$

where

$$\mathbf{T} = \varepsilon_0\, \omega B_0 \{D(\mathbf{I} - \boldsymbol{bb}) + \mathrm{i} S \boldsymbol{b} \times \mathbf{I}\}. \tag{9}$$

Here $\mathbf{I}$ is the unit matrix and, without confusion with the electric displacement D in (2),

$$S \equiv -\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega^2 - \Omega_\alpha^2}, \qquad D \equiv \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2\, \Omega_\alpha/\omega}{\omega^2 - \Omega_\alpha^2}. \tag{10a, b}$$

In the expression (10a), a unit constant has been dropped from S, since it is significant only in a very low density plasma. The plasma frequencies $\omega_{\mathrm{p}\alpha}$ and cyclotron frequencies Ω_α are defined in the usual ways:

$$\omega_{\mathrm{p}\alpha}^2 = q_\alpha^2\, n_\alpha/\varepsilon_0\, m_\alpha, \qquad \Omega_\alpha = q_\alpha B_0/m_\alpha. \tag{11a, b}$$

We note that equations (8) and (9) together indicate that $\boldsymbol{v}.\boldsymbol{B}_0 = 0$, which is consistent with (5) if $\boldsymbol{j}_0 \times \boldsymbol{B}_0 = 0$, which is the pressureless approximation adopted in this paper. Equation (8) will be seen to be a suitable generalization of the linearized Ohm law. Indeed, noting that as $\omega \to 0$, then $S \to \rho_0/\varepsilon_0 B_0^2$ and $D \to 0$, one finds $\boldsymbol{v} = \boldsymbol{E} \times \boldsymbol{B}_0/B_0^2$ which is equivalent to (6) when $\boldsymbol{v}.\boldsymbol{B}_0 = 0$. Just as the linearized Ohm law (6) is not simply another version of the linearized equation of motion (5), even though they are derived from the same initial set of equations (1), we also note that equation (8), being obtained from a different summation (averaging) procedure, is an equation independent from (5) and contains additional information despite its rather deceptive appearance. Equation (8) allows the set of linearized equations to be closed.

To make the connection with the ideal MHD formulation more transparent, a linearized fluid displacement vector ξ is introduced by $\boldsymbol{v} = \partial\xi/\partial t$. When the appropriate equations are collected together, a basic system of linearized equations in ξ, $\boldsymbol{E}$ and $\boldsymbol{B}$ is obtained:

$$-\mu_0 \rho_0 \omega^2 \xi = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}, \tag{12}$$

$$-\rho_0 \omega^2 \xi = \mathbf{T}.\boldsymbol{E}, \tag{13}$$

$$\mathrm{i}\omega \boldsymbol{B} = \nabla \times \boldsymbol{E}. \tag{14}$$

The tensor $\mathbf{T}$ is related to the 'cold' plasma dielectric tensor $\mathbf{K}$ (Stix 1962) by the equation

$$\mathbf{T} = \mathrm{i}\,\varepsilon_0\, \omega \boldsymbol{B}_0 \times \mathbf{K}. \tag{15}$$

In fact, if the solutions of the linearized equations of (1) were added together in a different but more conventional way (see Appendix 1) and then substituted into (2), the resulting equation would be

$$\nabla \times \boldsymbol{B} = (-\mathrm{i}\,\omega/c^2)\mathbf{K}.\boldsymbol{E}. \tag{16}$$

From the foregoing remark, the system of equations (14) and (16) is evidently equivalent to the system (12)–(14) adopted above. The first system may be regarded as a generalization of the cold plasma theory to include the structure of a helical magnetic field configuration, which leads to a geometrically modified form of the dielectric tensor. The second system may be regarded as a generalization of pressureless ideal MHD theory to include finite frequency effects and multiple ion species. In some antenna–plasma coupling calculations, it may be more straightforward and natural to use the former system of equations. However, for the purposes of the present paper, the latter system is used in order to show a clear and unambiguous connection with ideal MHD theory and to extend the elegant formulation of that theory (Appert *et al.* 1974).

3. Normal Mode Equations for a Diffuse Linear Pinch

Within the limitations imposed by the simplifying assumptions made in the previous section, the plasma model developed is applicable to arbitrary magnetic field geometry. In this section, the model is used to derive normal mode equations for a diffuse linear pinch. The cylindrical approximation for toroidal pinches has been shown (Appert *et al.* 1982*b*) to be adequate for the description of low frequency Alfvén wave propagation in low-β tokamaks. Also, on account of the shapes of the magnetic isobar surfaces, it is likely to be a good approximation for high-β toroidal plasmas, such as reverse field pinches, even at higher frequencies of wave propagation.

In the most general and elegant ideal MHD formulation of a diffuse linear pinch (Appert *et al.* 1974), the normal mode equations have been reduced to a pair of coupled first order ordinary differential equations. The problem of high frequency wave propagation in helical magnetic field configurations, however, has been considered only approximately by using special cases of the system of equations equivalent to (14) and (16) (Adam and Jacquinot 1977; Messiaen *et al.* 1978; Sy and Cotsaftis 1979) or by using the pressureless ideal MHD equations with Ohm's law augmented perturbatively by a Hall term (Appert and Vaclavik 1982). In the present section, this problem is treated in a more general way and the elegant formulation for the ideal MHD model (Appert *et al.* 1974) is shown to be extendable to the case of high frequency wave propagation in a diffuse linear pinch with multiple ion species.

On account of the helical structure of the equilibrium magnetic field $\boldsymbol{B}_0$, given in cylindrical coordinates (r, θ, z) by

$$\boldsymbol{B}_0 = B_{0\theta}(r)\boldsymbol{e}_\theta + B_{0z}(r)\boldsymbol{e}_z\,, \tag{17}$$

the components of a vector in (r, θ, z) coordinates are related to the components in local field line coordinates $(r, \perp, \|)$ by a transformation matrix $\mathbf{M}$ given by

$$\mathbf{M} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & B_{0z}/B_0 & B_{0\theta}/B_0 \\ 0 & -B_{0\theta}/B_0 & B_{0z}/B_0 \end{bmatrix}. \tag{18}$$

It is convenient to introduce the new dependent variables

$$Q \equiv -E_\perp/\mathrm{i}\,\omega B_0\,, \qquad R \equiv E_r/\mathrm{i}\,\omega B_0\,. \tag{19a, b}$$

These variables are related to the displacement vector $\boldsymbol{\xi}$ through equation (13), from which it may be deduced that

$$\xi_r = (\varepsilon_0 B_0^2/\rho_0)(SQ - \mathrm{i}\,DR)\,, \qquad \xi_\perp = (\varepsilon_0 B_0^2/\rho_0)(\mathrm{i}\,DQ + SR)\,. \tag{20a, b}$$

At low frequencies, when $|\omega/\Omega_\alpha| \to 0$, it may be shown from equations (10) that $S = \rho_0/\varepsilon_0 B_0^2$ and $D = 0$ and hence $\xi_r = Q$ and $\xi_\perp = R$. That is, Q may be regarded as a finite frequency generalization of the radial displacement ξ_r, which is an important variable in ideal MHD theory. The introduction of another variable, which is the magnetic pressure perturbation along $\boldsymbol{B}_0$,

$$P \equiv \boldsymbol{B}.\boldsymbol{B}_0/\mu_0\,, \tag{21}$$

leads to a form of equation (12) given by

$$\mathbf{T}.\boldsymbol{E} = -\nabla P + \mu_0^{-1}(\boldsymbol{B}.\nabla)\boldsymbol{B}_0 + \mu_0^{-1}(\boldsymbol{B}_0.\nabla)\boldsymbol{B}\,. \tag{22}$$

The aim of the subsequent elimination is to show that, in close analogy to ideal MHD theory, equations (14) and (22) may be reduced to an elegant system of coupled first order ordinary differential equations in P and Q. For this purpose, a Fourier analysis in the ignorable coordinates θ and z is made by taking a perturbed field component to vary as $\exp(\mathrm{i}m\theta + \mathrm{i}kz)$. It is then convenient to introduce the functions

$$F \equiv mB_{0\theta}/r + kB_{0z}, \qquad G \equiv kB_{0\theta} - mB_{0z}/r. \tag{23a, b}$$

From the condition of a pressureless equilibrium,

$$\mathrm{d}B_0^2/\mathrm{d}r = -2B_{0\theta}^2/r, \tag{24}$$

it may be shown from the definition (21) that

$$\mu_0 P = \mathrm{i}GB_0 R - \frac{B_0^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rQ) + \frac{B_{0\theta}^2 Q}{r}. \tag{25}$$

In an algebra which closely parallels that for the ideal MHD equation (Appert *et al.* 1974), it may be shown (Appendix 2) from (14) and (22) that

$$AB_0 R = -\mathrm{i}GP - \frac{\mathrm{i}}{\mu_0}\left(\frac{2FB_{0\theta}B_{0z}}{r} + \frac{B_0^3\,\omega^2 D}{c^2}\right)Q, \tag{26}$$

and the desired system of coupled equations is

$$\frac{A}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rQ) = C_1 Q - C_2 P, \tag{27}$$

$$A\,\mathrm{d}P/\mathrm{d}r = C_3 Q - C_1 P, \tag{28}$$

where

$$A \equiv \mu_0^{-1}\{(B_0^2\omega^2 S/c^2) - F^2\}, \tag{29a}$$

$$C_1 \equiv \frac{2B_{0\theta}^2\omega^2 S}{\mu_0 rc^2} + \frac{G\omega^2 B_0 D}{\mu_0 c^2} - \frac{2mFB_{0\theta}}{\mu_0 r^2}, \tag{29b}$$

$$C_2 \equiv (\omega^2 S/c^2) - \{k^2 + (m^2/r^2)\}, \tag{29c}$$

$$C_3 \equiv A\left\{A + \frac{1}{\mu_0}\left(\frac{2B_{0\theta}^2}{rB_0}\right)^2 + \frac{r}{\mu_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_{0\theta}}{r}\right)^2\right\} - \left(\frac{B_0^2\omega^2 D}{\mu_0 c^2} + \frac{2B_{0\theta}B_{0z}F}{\mu_0 rB_0}\right)^2, \tag{29d}$$

with F and G given by equations (23), and S and D by (10). At low frequencies we have $B_0^2\omega^2 S/\mu_0 c^2 = \rho_0\omega^2$, $Q = \xi_r$ etc.; these equations reduce to the corresponding equations of pressureless ideal MHD theory (Appert *et al.* 1974). The equations derived by previous authors, including finite ion cyclotron frequency effects (Adam and Jacquinot 1977; Messiaen *et al.* 1978; Sy and Cotsaftis 1979; Appert and Vaclavik 1982), may be obtained as special cases of the present system.

Equations (27) and (28) may be combined to give a single equation having the formal structure of a Sturm–Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rA}{C_3}\frac{\mathrm{d}P}{\mathrm{d}r}\right) + \left\{\frac{rN}{C_3} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rC_1}{C_3}\right)\right\}P = 0, \tag{30}$$

where $N \equiv (C_2 C_3 - C_1^2)/A$. Such an equation is useful for calculation of the detailed structures of the eigenmodes. However, the coefficients in the equation are rather complicated, with the possibilities of zeros and infinities corresponding to the existence of cutoffs and resonances. Attempts to obtain general spectral properties directly from this equation can lead to incorrect results, as has been shown, for example, in the discussion of the continuous spectra of the ideal MHD case by Appert *et al.* (1974), who have also indicated that the system of coupled equations such as (27) and (28) is much more reliable for such purposes.

4. Spectrum of a Soluble Model

A spectral problem, which is of practical significance for r.f. heating of a diffuse linear pinch, may be posed as follows. Given a magnetic field configuration and a set of mode numbers (m, k), what sorts of waves can propagate as the driving frequency ω is varied over a range validly covered by the plasma model described by equations (27) and (28)? Such a question is meaningful, strictly speaking, only for a specific magnetic field configuration. General statements could perhaps be made for certain classes of equilibria, which are generically similar in some restricted sense. In this section, the problem is discussed for a simple analytically soluble model, which will give insight into the solutions of more realistic and complicated cases.

Let us consider a 'feeble' diffuse linear pinch with uniform density, a uniform axial magnetic field and a small uniform axial current density such that

$$\varepsilon \equiv B_{0\theta}(a)/B_{0z} \ll 1\,, \tag{31}$$

where a is the plasma radius. To order ε^2, the equilibrium has negligible plasma pressure, the magnetic field strength B_0 is uniform and hence S, D, F, A and C_3 in equations (29) are all constants independent of the radius. In this case, to order ε^2, the model is analytically soluble and further calculations show that (30) can be reduced to a standard Bessel differential equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}P}{\mathrm{d}r}\right) + \left(k_{\mathrm{r}}^2 - \frac{m^2}{r^2}\right)P = 0\,, \tag{32}$$

where the radial wavenumber is defined by

$$k_{\mathrm{r}}^2 = \frac{\omega^2 S}{c^2} - k^2 - \frac{\omega^2 B_0 D}{\mu_0 A c^2}\left(\frac{\omega^2 B_0 D}{c^2} + \frac{2kB_{0\theta}}{r}\right), \tag{33}$$

which is independent of the radius. The physical solutions to (32), which are finite on the axis $r = 0$, are $\mathrm{J}_m(|k_{\mathrm{r}}|r)$ when $k_{\mathrm{r}}^2 > 0$ corresponding to oscillatory eigenmodes and $\mathrm{I}_m(|k_{\mathrm{r}}|r)$ when $k_{\mathrm{r}}^2 < 0$ corresponding to evanescent eigenmodes. The precise discrete set of eigenvalues k_{r}^2 depends on the actual boundary conditions. For example, for the case of a perfectly conducting cylinder at the plasma surface we have $k_{\mathrm{r}}^2 a^2 = \zeta_n^2$ $(n = 1, 2, 3, \ldots)$, where ζ_n are the zeros of the J_m Bessel function. Otherwise, k_{r}^2 is determined from more complicated boundary equations. Nevertheless, it is possible to obtain a general picture of the positive ω^2 spectrum and to predict the types of eigenmodes that may be excited at a given frequency by simply plotting k_{r}^2 given by (33) as a function of ω.

Consider a plasma with two ion species where, for definiteness, $\alpha \equiv \Omega_1/\Omega_2 \leqslant 1$. In the computation of (33), it is convenient to normalize the frequencies with respect to the cyclotron frequency of the first ion species Ω_1 by introducing $\nu \equiv \omega/\Omega_1$. Similarly, the wavenumbers may be normalized with respect to $1/a$, the reciprocal of the plasma radius, by defining $K \equiv ka$ and $K_r \equiv k_r a$. It follows from equations (10) and the quasi-neutrality condition that

$$\frac{a^2\omega^2 S}{c^2} = -K_c^2\, \nu^2 \frac{(\eta+\alpha-\alpha\eta)(\nu^2-\nu_{HB}^2)}{(\nu^2-1)(\nu^2-\alpha^2)}, \tag{34}$$

$$\frac{a^2\omega^2 D}{c^2} = -K_c^2\, \nu^2 \frac{\nu(\nu^2-1+\eta-\alpha^2\eta)}{(\nu^2-1)(\nu^2-\alpha^2)}. \tag{35}$$

In these expressions, η is the charge concentration of the first ion species,

$$\eta \equiv q_1 n_1/(q_1 n_1 + q_2 n_2),$$

and ν_{HB} is the dimensionless Buchsbaum ion–ion hybrid frequency defined by

$$\nu_{HB}^2 \equiv \alpha\nu_c/(\eta+\alpha-\alpha\eta); \qquad \nu_c \equiv 1-\eta+\alpha\eta, \tag{36}$$

where ν_c is its associated cutoff frequency. A characteristic normalized wavenumber K_c has been introduced, in convenient units, by

$$K_c^2 = 1{\cdot}93 n_e a^2 Z_1/A_1, \tag{37}$$

where Z_1 and A_1 are respectively the charge and atomic number of the first ion species, n_e is the electron number density in units of $10^{21}\ \mathrm{m}^{-3}$ and a (cm) is the plasma radius.

Since for a given K, K_r generally increases with ν, apart from regions of cutoffs and resonances, it is convenient for clarity of illustration to plot $K_r^2/\nu^2 K_c^2$ against ν. Application of expressions (34) and (35) to equation (33) leads to the radial dispersion relation,

$$K_r^2/\nu^2 K_c^2 = \mathcal{N}/\mathcal{D}, \tag{38}$$

where

$$\mathcal{N} \equiv -\nu^2 K_c^2(\nu^2-\nu_c^2) + 2K(K+\varepsilon m)(\eta+\alpha-\alpha\eta)(\nu^2-\nu_{HB}^2)$$
$$+2\varepsilon K\nu(\nu^2-1+\eta-\alpha^2\eta) + K^3(K+2\varepsilon m)(\nu^2-1)(\nu^2-\alpha^2)/\nu^2 K_c^2, \tag{39a}$$

$$\mathcal{D} \equiv -\nu^2 K_c^2(\eta+\alpha-\alpha\eta)(\nu^2-\nu_{HB}^2) - K(K+2\varepsilon m)(\nu^2-1)(\nu^2-\alpha^2). \tag{39b}$$

In the absence of a current ($\varepsilon = 0$), the radial dispersion relation is degenerate in the sense that it is independent of the azimuthal mode number m; this case is illustrated in Fig. 1. The presence of a current ($\varepsilon \neq 0$) removes this degeneracy as shown in Fig. 2. It is evident from these plots that the frequency spectrum generally has a resonance (singular point)–cutoff (turning point) doublet around the ion cyclotron frequencies and a cutoff–resonance–cutoff triplet at much lower frequencies. The components of the higher frequency doublet ν_{BR} and ν_{BC} (Fig. 1) are respectively the ion–ion hybrid resonance frequency and its associated cutoff. Exploitation of

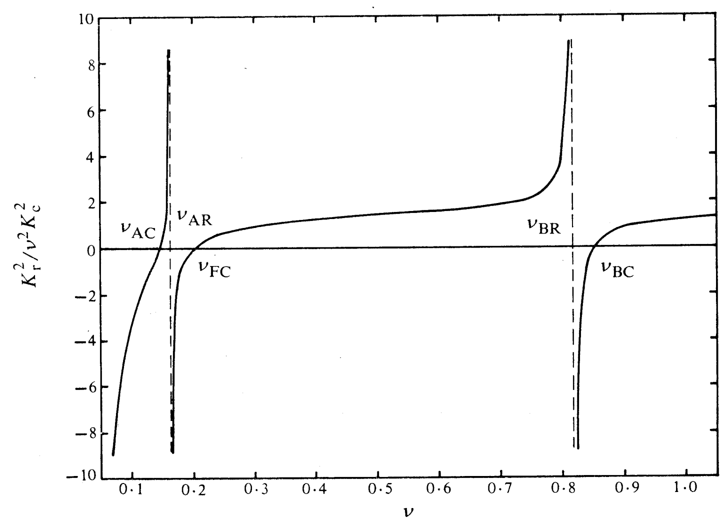

Fig. 1. Dispersion curves for a two ion species, currentless plasma ($\varepsilon = 0$). The parameters are the electron density $n_e = 10^{20}\,\mathrm{m}^{-3}$, plasma radius $a = 10$ cm, $\alpha = 0 \cdot 5$, $\eta = 0 \cdot 3$ and $k = 0 \cdot 1\,\mathrm{cm}^{-1}$.

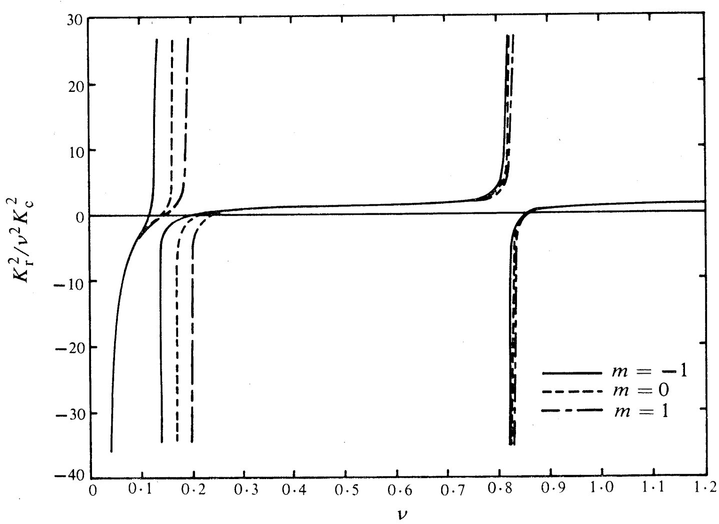

Fig. 2. Modification of the dispersion curves due to the presence of an axial current with $\varepsilon = 0 \cdot 2$; other parameters are as in Fig. 1.

such a structure in the frequency spectrum has led to successful r.f. heating of magnetically confined plasmas in the ion cyclotron range of frequencies (Hosea *et al.* 1982; Equipe TFR 1982). In the case of the lower frequency triplet, ν_{AR} is the shear Alfvén resonance frequency, ν_{AC} is its associated cutoff and ν_{FC} is the usual fast magnetosonic wave cutoff. This spectral structure has recently been intensively investigated for application in the Alfvén wave heating schemes (de Chambrier *et al.* 1982; Ross *et al.* 1982).

The eigenmodes in the frequency range $\nu_{AC} < \nu < \nu_{AR}$, which could exist even in a plasma with only one ion species, have been variously called global Alfvén waves (Appert *et al.* 1982*a*) or discrete Alfvén waves (de Chambrier *et al.* 1982). In the absence of current ($\varepsilon = 0$), they have been discussed by Stix (1957), who called them ion cyclotron waves and called their associated resonance ν_{AR}, the perpendicular ion cyclotron resonance. However, it follows from equations (39) that it is only in the limit $K \gg 1$ that the eigenfrequencies approach the ion cyclotron frequency (Stix 1957); otherwise, the finite ion cyclotron effect is only one of a number of possible effects which determine the dispersive properties of this branch of eigenmodes (Appert and Vaclavik 1982). For cases where K is finite and of order unity, the eigenfrequencies are normally much less than the ion cyclotron frequency, and it is therefore preferable to call these modes global or discrete Alfvén waves.

For typical densities of magnetically confined plasmas, the eigenmodes in the branch $\nu_{FC} \ll \nu < \nu_{BR}$ have very large radial wavenumbers K_r, with many nodes in the eigenfunctions. Coupling of r.f. energy to these high order eigenmodes from an external antenna is usually much less efficient than coupling to low order eigenmodes. Excitation of low order eigenmodes in the vicinity of ν_{BR} and ν_{BC} has recently been shown to be possible (Kieu and Sy 1983) for currentless non-uniform plasmas. The study of eigenmodes in a current-carrying plasma column using the system of equations (28) and (29) has recently been carried out by Cramer and Donnelly (1984). These eigenmodes might be useful for supplementary heating of plasma ions in diffuse linear pinches.

5. Discussion

A plasma model has been developed which includes finite ion cyclotron frequency effects and the possibility of multiple ion species. This model, which may be regarded as an extension of the pressureless ideal MHD model, has been shown to be equivalent to the cold plasma model with an arbitrary magnetic field structure. Although the model is still incomplete in the sense that a number of physical effects has been neglected, it is believed nevertheless that it will be a useful approximate model for the study of certain aspects of r.f. heating schemes for magnetically confined plasmas.

The pressureless approximation, which has been made in nearly all previous studies on r.f. heating in the literature, is justifiable for low-β plasmas, particularly at higher frequencies. The normal mode equations generally possess spatial singularities, which are associated with the existence of continuous spectra in a nonuniform plasma. Among many possibilities, such singularities may be removed by dissipative processes or finite Larmor radius effects. Exactly which effects are dominant will depend on the specific physical situation considered and a general discussion on the regularization problem is not appropriate at this stage. Nevertheless, despite the mathematical problems associated with the presence of the singularities, useful estimates on energy absorption by the plasma in an r.f. heating situation may be obtained (Chen and Hasegawa 1974; Tataronis 1975), since the continuous spectra themselves lead to 'phase-mix' dissipation even in a conservative system. Such problems will be discussed elsewhere.

In the case of a diffuse linear pinch with multiple ion species, it has been shown (Section 3) that the new plasma model admits a system of normal mode equations which have the same elegant structure as that for ideal MHD theory (Appert *et al.* 1974). Such a formulation will facilitate the numerical solution of the problem for a given plasma equilibrium, since a number of computing codes already exist for the ideal MHD case. The case of a soluble model with uniform density and a small uniform current has been used (Section 4) to illustrate some new features in the spectrum at higher frequencies. The presence of another ion species has led to ion–ion hybrid resonances and associated cutoffs in the ion cyclotron range of frequencies. The similarities and differences between the spectral structure in the ion cyclotron range of frequencies and the spectral structure at the lower frequencies of the Alfvén wave heating schemes have been made evident. The simple illustrative model used has led only to degenerate 'point-like' characteristic frequencies. In more realistic plasmas with non-uniform densities and magnetic field strengths, these characteristic frequencies form continuous frequency bands, corresponding to the shear Alfvén continuum ν_{AR}, the ion–ion hybrid continuum ν_{BR} and their associated cutoff continua ν_{AC}, ν_{FC} and ν_{BC}.

Since the ion–ion hybrid global eigenmodes can have frequencies close to an ion cyclotron frequency by a suitable choice of relative ion charge concentrations, it might be possible to combine good plasma–antenna coupling from eigenmode excitations with a significant wave–particle absorption mechanism to produce optimal ion heating in a diffuse pinch. Such problems appear to be worth while subjects for further investigation.

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Appendix 1. Guiding Centre Motion in a Helical Magnetic Field

The linearized equations of motion of (1) are

$$-\mathrm{i}\,\omega m\boldsymbol{v} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}_0)\,, \tag{A1}$$

where the subscript α has been omitted, without confusion, for the moment. Equation (A1) is solved locally, in the small Larmor radius approximation, along a helical magnetic field line in the orthonormal coordinate system with unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\perp \equiv \boldsymbol{b}\times\boldsymbol{e}_r$ as $\boldsymbol{b} = \boldsymbol{B}_0/B_0$ as introduced in (7). If one writes

$$v^{\pm} = v_r \pm \mathrm{i}\,\varepsilon v_\perp \quad \text{and} \quad E^{\pm} = E_r \pm \mathrm{i}\,\varepsilon E_\perp\,, \tag{A2}$$

where $\varepsilon = q/|\,q\,|$, then equation (A1) has the solution

$$v^{\pm} = \frac{\mathrm{i}\,q}{m(\omega \pm \varepsilon\Omega)}E^{\pm} \quad \text{and} \quad v_{\|} = \frac{\mathrm{i}\,q}{m\omega}E_{\|}\,, \tag{A3}$$

where $\Omega = qB_0\,/\,m$. Written in the orthonormal coordinate system with

$$\boldsymbol{v} = v_r\boldsymbol{e}_r + v_\perp\boldsymbol{e}_\perp + v_{\|}\boldsymbol{b}$$

and so on, and summed over particle species α, the solution for the one fluid velocity $\boldsymbol{v}$ as defined by (4) is given by

$$-\mathrm{i}\,\omega\rho_0\,\boldsymbol{v} = \mathbf{T}.\boldsymbol{E}\,, \tag{A4}$$

where the quasi-neutrality condition has been used and, in the localized field line coordinates,

$$\mathbf{T} \equiv \varepsilon_0\,\omega B_0\begin{bmatrix} D & -\mathrm{i}\,S & 0 \\ \mathrm{i}\,S & D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A5}$$

with S and D as defined in equations (10). This result, written in a coordinate invariant form, is equation (9).

If on the other hand, instead of summing for the one fluid velocity by $\rho_0\,\boldsymbol{v} = \Sigma_\alpha\, m_\alpha n_\alpha \boldsymbol{v}_\alpha$ as done above, we sum for the induced current by $\boldsymbol{j} = \Sigma_\alpha\, q_\alpha n_\alpha \boldsymbol{v}_\alpha$, then we have

$$\boldsymbol{j} = -\mathrm{i}\,\omega\varepsilon_0\,\mathbf{K}.\boldsymbol{E}\,, \tag{A6}$$

where $\mathbf{K}$ is the cold plasma dielectric tensor (Stix 1962), when $\boldsymbol{j}$ and $\boldsymbol{E}$ are expressed in localized field line coordinates. Substitution of this result into (2) gives equation (16) as stated in Section 2. Equations (A6) and (A4) are consistent if $\boldsymbol{E}.\boldsymbol{B}_0 = 0$, which is the case when the frequencies are much less than the plasma frequencies.

Appendix 2. Derivation of the Normal Mode Equations

From Faraday's law $\nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$ and the definitions (19), it follows that

$$B_r = \mathrm{i}\,FQ\,, \tag{A7a}$$

$$B_\theta = \mathrm{i}kB_0\,R - \mathrm{d}(B_{0\theta}\,Q)/\mathrm{d}r\,, \tag{A7b}$$

$$B_z = -\mathrm{i}\,mB_0\,R/r - (1/r)\mathrm{d}(rB_{0z}\,Q)/\mathrm{d}r\,, \tag{A7c}$$

where F is defined in equation (23a). Application of these expressions to (22), together with the observations that

$$(\mathbf{T}\cdot\boldsymbol{E})_\theta = (B_{0z}/B_0)(\mathbf{T}\cdot\boldsymbol{E})_\perp \quad \text{and} \quad (\mathbf{T}\cdot\boldsymbol{E})_z = -(B_{0\theta}/B_0)(\mathbf{T}\cdot\boldsymbol{E})_\perp$$

gives, after a little algebra,

$$\frac{\mathrm{i}\,G}{\mu_0\,r}\frac{\mathrm{d}}{\mathrm{d}r}(rQ) - \frac{1}{\mu_0}\left(\frac{\omega^2 S}{c^2} - k^2 - \frac{m^2}{r^2}\right)B_0\,R = \frac{\mathrm{i}}{\mu_0}\left(\frac{\omega^2 B_0\,D}{c^2} + \frac{2kB_{0\theta}}{r}\right)Q\,, \tag{A8}$$

$$\frac{F^2}{\mu_0\,r}\frac{\mathrm{d}}{\mathrm{d}r}(rQ) - \frac{\mathrm{i}\,G}{\mu_0}\frac{\omega^2 S}{c^2}B_0\,R = -\left(k^2 + \frac{m^2}{r^2}\right)P + \left(\frac{2mFB_{0\theta}}{r^2\mu_0} - \frac{G\omega^2 B_0\,D}{c^2}\right)Q. \tag{A9}$$

Combinations of equation (A9) with (25) give (26) and (27) in Section 3, where it is useful to note the following simple identities:

$$B_{0\theta}\,F - B_{0z}\,G = mB_0^2/r\,, \qquad B_{0z}\,F + B_{0\theta}\,G = kB_0^2\,, \tag{A10a, b}$$

$$F^2 + G^2 = B_0^2(k^2 + m^2/r^2)\,. \tag{A10c}$$

Finally, the radial component of (22), together with equations (26), (27), (A7) and (A10) after some calculation, gives (28).

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