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Two-dimensional Ising Model on a 4–6–12 Lattice

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Abstract

We have considered a two-dimensional Ising model on a 4–6–12 lattice. The partition function is evaluated exactly by the method of Pfaffian. The Ising model on a ruby lattice is a special case of our model.

1. Introduction

The partition function of the two-dimensional Ising model on a square lattice was first derived by Onsager (1944), beginning the modern era of critical phenomena. However, the original derivation of the Onsager function is very complicated. The solution of the planar dimer problem by Kasteleyn (1961) and by Temperley and Fisher (1961) was the next major advance in exact statistical mechanics. Kasteleyn (1963) showed that the Ising model partition function on a square lattice can be expressed as a dimer problem on a decorated lattice and was able to rederive Onsager's result. His method is much simpler than the original derivation and can be generalized to any planar lattice.

A dimer is an object which occupies two adjacent lattice sites. The dimer problem is to determine the number of ways of covering a given lattice with dimers such that all sites are occupied and no two dimers overlap. It has been shown by Kasteleyn (1961) and Temperley and Fisher (1961) that the number of dimer coverings is analogous to the Pfaffian calculation of an antisymmetric square matrix A. The usefulness of the Pfaffian method is due to the fact that this number is the square-root of the determinant of A, the determinant being comparatively simple to evaluate. The purpose of the present paper is to use Pfaffian techniques to obtain the partition function of an Ising model on a 4–6–12 lattice.

2. Ising Model

The Pfaffian method (McCoy and Wu 1973) is used to exactly calculate the partition function of an Ising model on the 4–6–12 lattice shown in Fig. 1. The Ising model on a ruby lattice (Lin and Ma 1983) is a special case of our model.

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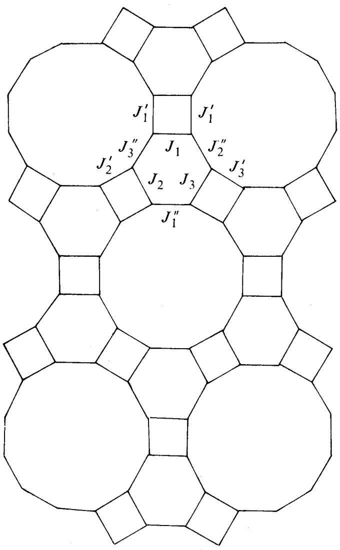

Fig. 1. A 4–6–12 lattice where the interaction between spins is anisotropic.

The Ising model consists of a lattice of N spin variables σ_i which may take only the values $+1$ and -1. The energy of a lattice spin state $(\sigma_1, ..., \sigma_N)$ is

$$E = -\sum_{\mathrm{NN}} J_{ij}\sigma_i\sigma_j, \tag{1}$$

where the sum is taken over all pairs i and j that are nearest neighbours (NN) in the lattice, and a periodic boundary condition is assumed. The partition function is

$$Z = \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \exp\left(\sum_{\mathrm{NN}} K_{ij}\sigma_i\sigma_j\right), \tag{2}$$

where $K = J/kT$, k is Boltzmann's constant and T is the absolute temperature. We assume that the interaction between spins is anisotropic and there are nine different coupling constants (J_i, J_i', J_i'') as shown in Fig. 1.

The partition function can be written as (Van der Waerden 1941; Newell and Montroll 1953)

$$Z = 2^N\left(\prod_{i=1}^{3} \cosh K_i \cosh K_i' \cosh K_i''\right)^{N/6}$$

$$\times \sum n(r,s,t,u,v,w,m,n,p)(x_1^r x_2^s x_3^t y_1^u y_2^v y_3^w z_1^m z_2^n z_3^p)^{-1}, \tag{3}$$

where

$$x_i = \coth K_i, \qquad y_i = \coth K_i', \qquad z_i = \coth K_i''$$

and $n(r,s,t,u,v,w,m,n,p)$ is the number of closed graphs with $(r+s+ \ldots +p)$ bonds, r is in the horizontal direction and u is in the vertical direction, etc.

The partition function can be evaluated by the standard method of Pfaffian and dimer city (Kasteleyn 1963) as follows. A unit cell is shown in Fig. 2 which corresponds to a 36th-order matrix with elements

$$a(i,j) = -a^*(j,i). \tag{4}$$

A periodic boundary condition is assumed and the sign of each element is identified by an arrow such that its direction from i to j implies $\operatorname{sgn} a(i,j) = +1$. A polygon with an odd number of clockwise sides is called clockwise odd and arrows are arranged

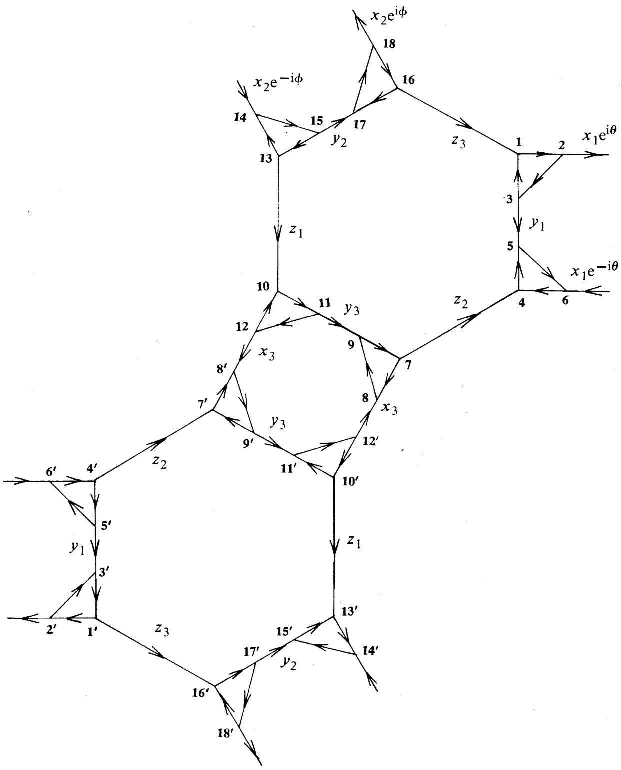

Fig. 2. A unit cell of the 4–6–12 lattice which corresponds to a 36th-order matrix.

so that every closed polygon is clockwise odd. The matrix elements associated with positive signs are shown explicitly in Fig. 2, except those with values of unity. For example, we have

$$a(1,2) = 1, \qquad a(3,5) = y_1, \qquad a(2,6') = x_1 e^{i\theta}.$$

We write

$$N^{-1}\log Z = \log 2 + \tfrac{1}{6}\log\left(\prod_i \cosh K_i \cosh K_i' \cosh K_i''\right)$$

$$+ \frac{M}{2N(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log\{(x_1 x_2 x_3 y_1 y_2 y_3 z_1 z_2 z_3)^{-4}\det A\}\, d\theta\, d\phi$$

$$= \log 2 + (96\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} \log\left\{\det A\left(\prod_i \sinh K_i \sinh K_i' \right.\right.$$

$$\left.\left. \times \sinh K_i''\right)^4\right\} d\theta\, d\phi, \tag{5}$$

where $M = N/12$ is the number of unit cells in this lattice and $\det A$ is the determinant of the matrix $a(i,j)$.

After a straightforward and long calculation, we get

$$N^{-1}\log Z = \log 2+(96\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} \log F(\theta,\phi)\, \mathrm{d}\theta\, \mathrm{d}\phi, \tag{6}$$

where

$$\begin{aligned} 2048F(\theta,\phi) = {} & a(1,2,3)\cos 2\phi + a(2,1,3)\cos 2\theta + a(3,1,2)\cos 2(\theta-\phi) \\ & -2b(1,2,3)\cos(\phi-2\theta) - 2b(2,1,3)\cos(\theta-2\phi) - 2b(3,1,2)\cos(\theta+\phi) \\ & -2c(1,2,3)\cos\phi - 2c(2,1,3)\cos\theta - 2c(3,1,2)\cos(\theta-\phi) + d. \end{aligned} \tag{7}$$

The parameters in this equation are defined as follows:

$$\begin{aligned} a(i,j,k) &= (S''_1 S''_2 S''_3 S'_i S_j S_k)^2, \\ b(i,j,k) &= S_1 S_2 S_3 S_i S'_j S'_k (S''_j S''_k)^2 \\ &\quad \times \{C''^2_i + 2C''_i(C_j C'_j + C_k C'_k) + 2C_j C'_j C_k C'_k + 1\}, \\ c(i,j,k) &= 2fg(i,j,k) - 2g(j,i,k)g(k,i,j) - b(i,j,k), \\ d &= 2\{f^2 + g^2(1,2,3) + g^2(2,1,3) + g^2(3,1,2)\} \\ &\quad - a(1,2,3) - a(2,1,3) - a(3,1,2), \end{aligned}$$

where

$$\begin{aligned} f = {} & (C_1 C'_1 + C_2 C'_2 + C_3 C'_3 + C_1 C_2 C_3 C'_1 C'_2 C'_3)(C''_1 C''_2 C''_3 + C''_1 + C''_2 + C''_3) \\ & + C_1 C_2 C_3 S'_1 S'_2 S'_3 S''_1 S''_2 S''_3 \\ & + (C_1 C_2 C'_1 C'_2 + C_1 C_3 C'_1 C'_3 + C_2 C_3 C'_2 C'_3 + 1) \\ & \times (C''_1 C''_2 + C''_1 C''_3 + C''_2 C''_3 + 1), \end{aligned}$$

$$\begin{aligned} g(i,j,k) = {} & C_i S_j S_k \{S'_i C'_j C'_k S''_1 S''_2 S''_3 + C'_i S'_j S'_k (C''_1 C''_2 C''_3 + C''_i - C''_j - C''_k)\} \\ & + S_j S_k S'_j S'_k \{(C''_j + C''_k) C''_i - C''_j C''_k - 1\}, \end{aligned}$$

$$S_i = \sinh 2K_i, \qquad S'_i = \sinh 2K'_i, \qquad S''_i = \sinh 2K''_i,$$
$$C_i = \cosh 2K_i, \qquad C'_i = \cosh 2K'_i, \qquad C''_i = \cosh 2K''_i.$$

It can be shown that

$$F(\theta,\phi) \geqslant F(0,0) = (P/32)^2,$$

where

$$P = f - g(1,2,3) - g(2,1,3) - g(3,1,2),$$

and the equality holds if and only if $\theta = \phi = 0$. Therefore the critical temperature T_c is determined by $P = 0$.

3. Summary

We have calculated the partition function of an Ising model on a 4–6–12 lattice. This lattice is of interest because it has a large unit cell and because it contains many of the more common-place lattices as special cases. For example, our result agrees with the known results (Syozi 1972; Lin and Ma 1983) for the triangular lattice ($J_i = J_i'' = \infty$), hexagonal lattice ($J_i' = J_i'' = \infty$), ruby lattice ($J_i'' = \infty$), Kagomé lattice ($J_i = J_i' = \infty$), and the 3–12 lattice ($J_i = \infty$).

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