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Bianchi Type I Cosmological Models with Perfect Fluid and Magnetic Field

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Abstract

We investigate Bianchi type I cosmological models containing a perfect fluid and with an incident magnetic field directed along the x axis. The solutions obtained represent an expansion scalar θ bearing a constant ratio to the anisotropy in the direction of a space-like unit vector λ^i.

1. Introduction

Bianchi type I cosmological models which are the anisotropic generalization of FRW models with flat space slices have been widely studied. The evidence for an intergalactic homogeneous magnetic field, speculated to be of primordial nature, has prompted a number of workers to successfully incorporate a homogeneous magnetic field in Bianchi type I spaces. Thorne (1967) investigated LRS Bianchi type I cosmological models containing a magnetic field directed along one axis with a barotropic fluid. Magnetic Bianchi type I cosmological models satisfying a barotropic equation of state have been considered by Jacobs (1968, 1969). A qualitative analysis of Bianchi type I models with a magnetic field has also been given by Collins (1972). Hughston and Jacobs (1970) have shown that the existence of a homogeneous primordial magnetic field in the Universe is limited to Bianchi types I, II, III, VI_0 or VII_0. Electromagnetic Bianchi type I models with stiff matter have recently been studied by Lorenz (1981).

In the present paper we investigate the Bianchi type I cosmological models containing matter in the form of a perfect fluid with an incident magnetic field directed along the x axis. We obtain the solutions of the Einstein–Maxwell equations assuming that the expansion scalar θ bears a constant ratio to the anisotropy in the direction of a unit space-like vector λ^i, the anisotropy being given by $\sigma_{ij}\lambda^i\lambda^j$. This assumption is similar to the condition that σ/θ is constant, as proposed by Collins *et al.* (1980). The solutions obtained show that the matter content reduces to a stiff equation of state asymptotically, assuming that the cosmological constant Λ is zero.

2. Field Equations

The line element describing Bianchi type I space–time is taken in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2 dy^2 + C^2 dz^2, \tag{1}$$

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where the metric potentials A, B and C are functions of time t. The matter distribution consists of an electrically neutral perfect fluid with infinite electrical conductivity and a source free magnetic field given by the energy–momentum tensor

$$T_i^j = (\varepsilon + p)v_i v^j + p g_i^j + E_i^j, \tag{2}$$

where

$$E_i^j = \bar{\mu}\{h_l h^l (v_i v^j + \tfrac{1}{2} g_i^j) - h_i h^j\}, \tag{3}$$

$$h_i = (2\bar{\mu})^{-1} v^j (-g)^{\frac{1}{2}} \varepsilon_{ijkm} F^{km}. \tag{4}$$

In equations (2)–(4) p is the thermodynamic pressure, ε the matter density, E_{ij} the electromagnetic energy tensor, F_{ij} the electromagnetic field tensor, h_i the magnetic flux vector, $\bar{\mu}$ the magnetic permeability, ε_{ijkm} the Levi–Civita tensor density and v^i the flow vector of the fluid. The magnetic field is taken along the x direction so that the only nonzero component of F_{ij} is F_{23}. Maxwell's equation

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{5}$$

gives rise to

$$F_{23} = K \quad \text{(constant)}. \tag{6}$$

The field equations

$$-8\pi G T_i^j = R_i^j - \tfrac{1}{2} R g_i^j + \Lambda g_i^j \tag{7}$$

lead to

$$8\pi G p = \frac{1}{A^2}\left\{-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{4\pi G K^2 A^2}{\bar{\mu} B^2 C^2}\right\} - \Lambda, \tag{8}$$

$$= \frac{1}{A^2}\left\{-\left(\frac{A_4}{A}\right)_4 - \frac{C_{44}}{C} - \frac{4\pi G K^2 A^2}{\bar{\mu} B^2 C^2}\right\} - \Lambda, \tag{9}$$

$$= \frac{1}{A^2}\left\{-\left(\frac{A_4}{A}\right)_4 - \frac{B_{44}}{B} - \frac{4\pi G K^2 A^2}{\bar{\mu} B^2 C^2}\right\} - \Lambda, \tag{10}$$

$$8\pi G \varepsilon = \frac{1}{A^2}\left\{\frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{B_4 C_4}{BC} - \frac{4\pi G K^2 A^2}{\bar{\mu} B^2 C^2}\right\} + \Lambda. \tag{11}$$

The suffix 4 on A, B and C stands for ordinary differentiation with respect to t. Eliminating p from (8)–(10), we get

$$\left(\frac{A_4}{A}\right)_4 - \frac{B_{44}}{B} - \frac{B_4 C_4}{BC} + \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{8\pi G K^2 A^2}{\bar{\mu} B^2 C^2} = 0, \tag{12}$$

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} = 0. \tag{13}$$

The conservation equation for the energy–momentum tensor T^j_i,

$$T^j_{i;j} = 0,$$

leads to

$$\mathrm{d}\varepsilon/\mathrm{d}\tau + (\varepsilon + p)\theta = 0, \tag{14}$$

where τ is the cosmic time given by $\tau = \int A\,\mathrm{d}t$. From equation (13) we have

$$\sigma^2_2 = \sigma^3_3 + M/R^3, \tag{15}$$

which is equivalent to

$$\sigma^1_1 = -2\sigma^3_3 - M/R^3, \tag{16}$$

where σ^j_i is the shear tensor of the fluid flow, M is a constant and $R^3 = ABC$.
Equations (12), (15) and (16) give

$$\frac{\mathrm{d}\sigma^3_3}{\mathrm{d}\tau} + \sigma^3_3\,\theta = \frac{8\pi GK^2}{3\bar{\mu}B^2C^2}, \tag{17}$$

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}\tau} + 2\sigma^2\theta = \frac{16\pi GK^2}{\sqrt{3}\,\bar{\mu}B^2C^2}\left(\sigma^2 - \frac{M^2}{4R^6}\right)^{\frac{1}{2}}, \tag{18}$$

where the expansion scalar is given by $\theta = 3R\,\mathrm{d}R/\mathrm{d}\tau$. From equation (17) we deduce that the vanishing of any component of the shear tensor leads to the vanishing of the magnetic field, whereas equation (18) tells us that σR^3 must not be constant for a non-vanishing magnetic field. From (18) we also conclude that in the expansion stage of the model ($\theta > 0$), σ^2 is a decreasing function of τ when $K = 0$. The effect of the magnetic field is to arrest the decrease.
From equation (11) we get

$$\frac{3\sigma^2}{\theta^2} = 1 - \frac{24\pi G\varepsilon}{\theta^2} - \frac{12\pi GK^2}{\bar{\mu}B^2C^2\theta^2} + \frac{3\Lambda}{\theta^2}, \tag{19}$$

which implies that for $\Lambda \leqslant 0$

$$0 < 3\sigma^2/\theta^2 < 1, \qquad 0 < 24\pi G\varepsilon/\theta^2 < 1,$$

assuming $\varepsilon > 0$. The presence of the magnetic field lowers the upper limit of anisotropy compared with those limits in the perfect fluid case. Equations (14), (18) and (19) give rise to

$$\mathrm{d}\theta/\mathrm{d}\tau = -12\pi G(\varepsilon + p) - 3\sigma^2 - 16\pi G\rho, \tag{20}$$

which is in fact also a consequence of the Raychaudhuri equation

$$\mathrm{d}\theta/\mathrm{d}\tau + \tfrac{1}{3}\theta^2 + 2\sigma^2 + 4\pi G(\varepsilon + 3p + 2\rho) + \Lambda = 0, \tag{21}$$

ρ being the magnetic energy density given by

$$\rho = K^2/2\bar{\mu}B^2C^2.$$

The dominant energy condition (Hawking and Ellis 1973) gives rise to

$$\varepsilon+p+2\rho \geqslant 0, \qquad \varepsilon+p \geqslant 0,$$

$$\varepsilon-p+2\rho \geqslant 0, \qquad \varepsilon-p \geqslant 0.$$

Since $\rho > 0$, it is sufficient to have

$$\varepsilon+p \geqslant 0, \qquad \varepsilon-p \geqslant 0.$$

Equation (14) shows that the matter density of the Universe is decreasing with time during its expansion stage provided the energy condition $\varepsilon+p > 0$ holds. When p satisfies the barotropic equation of state $p = (\gamma-1)\varepsilon$, $1 \leqslant \gamma \leqslant 2$, equation (14) gives

$$\varepsilon = \varepsilon_0 R_0^{3\gamma}/R^{3\gamma},$$

where ε_0 and R_0 are the present day values of ε and R, and this shows that the matter density ε varies as $R^{-3\gamma}$. From equation (20) we find that θ is a decreasing function of time in the case where the above energy condition holds. The rate of decrease is greater when the magnetic field is present. The model, therefore, starts from the big bang singularity and continues to expand until $\tau = \infty$ for $\Lambda \leqslant 0$, whereas for $\Lambda > 0$ the expansion stops at $\tau = \tau_0$ and the model may go into the contraction phase to attain the second singularity. However, from the Raychaudhuri equation (21) we see that, if $\varepsilon+3p > 0$, θ decreases with time when $\Lambda \geqslant 0$ and there is a big bang, whereas for $\Lambda < 0$ the solutions in general will not have a big bang.

The present day value of the deceleration parameter q_0 for $p = 0$ is given by

$$q_0 = \frac{4\pi G\varepsilon_0}{H_0^2} + 9\left(\frac{\sigma_0}{\theta_0}\right)^2 + \frac{16\pi G\rho_0}{3H_0^2} - 1,$$

where H_0, σ_0 and ρ_0 are present day values of the Hubble parameter, the shear and the magnetic energy density respectively. Taking $H_0 = 75\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ (Rees 1980), $\rho_0 = 5\times10^{-35}\ \mathrm{g\,cm^{-3}}$ (Misner *et al.* 1973) and the present day observed anisotropy σ_0/θ_0 of the microwave background radiation to be 10^{-3}, we find that the second and third terms in q_0 are of order 10^{-6} and can be ignored. For a q_0 value of nearly one, ε_0 should be about $1{\cdot}5\times10^{-29}\ \mathrm{g\,cm^{-3}}$. When we take $\varepsilon_0 = 5\times10^{-30}\ \mathrm{g\,cm^{-3}}$ (Adler *et al.* 1965), we find that $q_0 = -0{\cdot}3$ and $\Lambda = -9{\cdot}4\times10^{-35}\ \mathrm{s^{-2}}$. The negative value of Λ shows that the model will continue to expand indefinitely.

In order to have exact solutions of the differential equations (12) and (13), we require one more condition. For this purpose we take the volume expansion θ to have a constant ratio to the anisotropy in the direction of the unit space-like vector λ^i, i.e. $\theta/\sigma_{ij}\lambda^i\lambda^j$ is constant. Here we take $\lambda_i = (k_1 A, k_2 B, k_3 C, 0)$, where k_1, k_2 and k_3 are constants satisfying $k_1^2+k_2^2+k_3^2 = 1$. It is interesting to note that with this condition the metric (1) in the case of perfect fluid distribution and $\Lambda = 0$ always represents a stiff fluid distribution, as we shall see in Section 3. In general the above condition gives rise to

$$A = B^m C^n, \tag{22}$$

with m and n constants depending on k_1, k_2 and k_3. Without loss of generality we can take $m > n$. Equation (12) with (22) leads to

$$(m+n-1)\left(\frac{B_{44}}{B}+\frac{B_4 C_4}{BC}\right)+\frac{8\pi G K^2}{\bar{\mu}} B^{2(m-1)} C^{2(n-1)} = 0. \tag{23}$$

Obviously we need $m+n \neq 1$ for the magnetic field to be nonzero. Integrating (13) and (23) we get the line element (1) in the following forms:

(i) For $m+n \neq 0$ we have

$$\begin{aligned} ds^2 = {} & \frac{1}{f^2(T)}\left(dX^2 - \frac{dT^2}{a^2 f(T)^{4/(m+n)} e^{2(m-n)T/(m+n)}}\right) \\ & + \frac{1}{f(T)^{2/(m+n)}}\left(e^{2nT/(m+n)} dY^2 + e^{-2mT/(m+n)} dZ^2\right), \end{aligned} \tag{24}$$

where $f(T) = e^{-bT} + L e^{bT}$ and a, b and L are constants;

$$\begin{aligned} ds^2 = {} & \operatorname{cosec}^2 T\left(dX^2 - \frac{dT^2}{k^2(\sin T)^{4/(m+n)} e^{-2\alpha(m-n)T}}\right), \\ & + (\operatorname{cosec} T)^{2/(m+n)}\left(e^{-2\alpha n T} dY^2 + e^{2m\alpha T} dZ^2\right), \end{aligned} \tag{25}$$

where k and α are constants;

$$\begin{aligned} ds^2 = {} & (1-MT)^{-2}\left(dX^2 - \frac{dT^2}{\beta^2(1-MT)^{4/(m+n)} e^{-2(m-n)T/(m+n)}}\right) \\ & + (1-MT)^{-2/(m+n)}\left(e^{-2nT/(m+n)} dY^2 + e^{2mT/(m+n)} dZ^2\right), \end{aligned} \tag{26}$$

where M and β are constants.

(ii) For $m+n = 0$ we have

$$\begin{aligned} ds^2 = {} & -e^{2S^2T^2} T^{2N} dT^2 + T^2 dX^2 \\ & + e^{S^2T^2}\left(T^{N+1/m} dY^2 + T^{N-1/m} dZ^2\right), \end{aligned} \tag{27}$$

where S and N are constants.

3. Discussion

In the model (24) we can take $b > 0$ without loss of generality. The dominant energy conditions $\varepsilon + p \geqslant 0$ and $\varepsilon - p \geqslant 0$ put restrictions on the constants and the time period in which the model exists. However, if we choose

$$L < 0, \quad m+n > 0, \quad b^2 > \tfrac{1}{4}\{m+n+(m-n)^2\}, \quad \Lambda \geqslant 0,$$

the energy conditions are always satisfied [see equations (A1)–(A3)]. The model has the following behaviour:

(a) When $L > 0$, the positiveness of the magnetic energy ρ implies that either $m+n > 1$ or $m+n < 0$ (see equation A3). The model has a singularity at $T = -\infty$

provided that $b(m+n+2) > m-n$ for $m+n > 0$ and $b(m+n+2)+m-n < 0$ for $m+n < 0$. The model starts expanding from its singular state and continues to expand until

$$T = \frac{1}{2b}\log\left(\frac{b(m+n+2)-m+n}{L\{b(m+n+2)+m-n\}}\right).$$

Thereafter the model starts contracting to attain the second singularity at $T = \infty$. The initial singularity is of the point type, barrel type or cigar type depending on whether $b > m$, $b = m$ or $-n < b < m$ respectively when $m+n > 0$. In the first two cases the model ultimately evolves into a point singularity, while in the last case it evolves into a point, barrel or cigar singularity depending on whether $b \gtreqless n$. However, for $m+n < 0$, the model has a point, barrel or cigar singularity when $b < m$, $b = m$ or $m < b < -n$ respectively. In the first two cases the model ultimately evolves into an infinite pancake singularity of the second kind, i.e. when one of A, B or C is zero and two are infinite. In the third case the model evolves into a singularity of cigar type, an infinite pancake of the first kind (i.e. when one of A, B or C is zero, one nonzero finite and one infinite) or an infinite pancake of the second kind depending on whether $0 < b < -m$, $b = -m$ or $b > |m|$ (see equation A4).

(b) When $L < 0$, we find $0 < m+n < 1$ due to $\rho > 0$ so that $m > 0$. The model has a singularity at $T = -\infty$ provided $b(m+n+2) > m-n$. It starts expanding from its singular state and continues until $T = (1/2b)\log(-1/L)$. There is an apparent singularity in the model at this value of T; however, it is not a real singularity. The hypersurface $T = (1/2b)\log(-1/L)$ is null. The singularity in the model is of the point type, barrel type or cigar type depending on whether $b > m$, $b = m$ or $-n < b < m$. It is of interest to note that the model starts contracting from the state of infinite dispersion at $T = (1/2b)\log(-1/L)$ and goes on contracting until collapse at $T = \infty$.

In this case the model for which $L > 0$ approaches a stiff fluid distribution at the initial singularity and at late times also when $\Lambda = 0$. Also, in the model for which $L < 0$ and $-\infty < T < (1/2b)\log(-1/L)$, the matter distribution reduces to a stiff fluid at the initial singularity when $\Lambda = 0$. At the initial singularities these models have Kasner asymptotics for

$$b = \frac{(m^2-n^2+m-n)\pm\{(m^2-n^2+m-n)^2+4mn(2m+2n+1)\}^{\frac{1}{2}}}{2(2m+2n+1)}. \tag{28}$$

Therefore matter can be neglected near the initial singularity although there it approaches the stiff fluid condition. In the model for which $L < 0$ and $T > (1/2b)\log(-1/L)$, the matter content reduces to that of the stiff fluid condition at $T = \infty$ in the absence of the cosmological constant Λ. When $L = 0$, i.e. when the magnetic field is absent and $\Lambda = 0$, the model reduces to that of a stiff perfect fluid distribution and this model starting from its singular state at $T = -\infty$ continues to expand until $T = \infty$. It should be noted that the singularities occurring at infinite values of the coordinate time T, discussed above, occur at finite proper times.

In the model (25) the energy condition $\varepsilon - p \geqslant 0$ is always satisfied when $\Lambda \geqslant 0$, while $\varepsilon + p \geqslant 0$ imposes restrictions on the time period during which the model exists [see equations (A7)–(A9)]. However, this is not of much cosmological interest because of the periodic functions involved.

In the model (26) we see that the positiveness of the magnetic energy ρ requires that $0 < m+n < 1$ (see equation A4). The energy condition $\varepsilon+p \geqslant 0$ imposes restrictions on the time period during which the model exists while $\varepsilon-p \geqslant 0$ is always satisfied provided $\Lambda \geqslant 0$ [see equations (A12)–(A14)]. The model has a singularity at $T = -\infty$, which is a point singularity if $n \leqslant 0$ and cigar singularity if $n > 0$. The model starts expanding from its singular state and continues until $T = 1/M$. There is an apparent singularity at $T = 1/M$; however, it is not a real singularity. The hypersurface $T = 1/M$ is null and is in fact a horizon. It is interesting to note that the model starts contracting from the state of infinite dispersion at $T = 1/M$ and, after attaining its maximum contraction rate, stops contracting and begins to expand. The expansion rate after attaining its maximum gradually decreases until at $T = \infty$ the expansion ceases (see equation A15). The models for $-\infty < T < 1/M$ and for $T > 1/M$ are distinct but one may expect the former to be analytically extended across the horizon $T = 1/M$.

The matter distribution approaches a stiff equation of state near the initial singularity for the model for which $-\infty < T < 1/M$ and at late times for the model for which $T > 1/M$ with $\Lambda = 0$. The free gravitational field is of Petrov type D in the asymptotic limit which otherwise is of Petrov type I. In this case also when the magnetic field is absent (i.e. $M = 0$) and $\Lambda = 0$ the model reduces to a stiff perfect fluid distribution. In this case the model has a singularity at $T = -\infty$ and it starts expanding from its singular state at $T = -\infty$ and continues until $T = \infty$. In this case the singularity occurring at $T = -\infty$ corresponds to finite proper time.

In the model (27) the energy conditions $\varepsilon+p \geqslant 0$ and $\varepsilon-p \geqslant 0$ are always satisfied with $N > 1/4m^2$ and $\Lambda \geqslant 0$. The model has a singularity at $T = 0$ provided that $N > 1$. It is a point or barrel singularity respectively if $N > |1/m|$ or $N = |1/m|$, and a cigar singularity if $-|1/m| < N < |1/m|$. The model starts expanding from its singular state and continues until $T = \infty$. The matter always dominates the magnetic field when $N > 1/4m^2$ and $\Lambda \geqslant 0$ [see equations (A18)–(A21)]. The matter distribution approaches a stiff fluid at the initial singularity and at $T = \infty$ with $\Lambda = 0$. There exists a particle but not an event horizon along the x axis. When the magnetic field is absent, i.e. when $S = 0$, the metric describes a perfect fluid distribution. There is a stiff equation of state when $\Lambda = 0$.

In the models (24), (26) and (27) we find that E/ε tends to a constant at the initial singularity and at late times where $2E^2 = E_{ab}E^{ab}$, E_{ab} being the electric part of the free gravitational field [see equations (A2), (A6), (A13), (A17), (A19) and (A23)]. This shows that the matter and free gravitational field are equally dominant, except when (28) holds in the model (24) when matter is negligible compared with the free gravitational field. Since $\rho/\varepsilon \to 0$ as $\tau \to 0$ the magnetic field is negligible in comparison with the matter field near the initial singularity, except when (28) holds and then they are comparable. Moreover, in the above models, σ/θ tends to a constant asymptotically and hence the models do not approach isotropy. However, we can make σ/θ less than any small arbitrary positive number by suitably choosing constants appearing in the models and for a finite period of time the models will be approximately FRW. We also find that in these models σ/θ is always a constant in the absence of a magnetic field, which shows that anisotropy in such models never dies out.

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Appendix

Here we give expressions for the matter distribution (p, ε), the magnetic energy ρ and kinematic quantities for different models. These are as follows:

For the model (24) we have

$$
\begin{aligned}
8\pi G p = {} & a^2 f^{2+4/(m+n)} \mathrm{e}^{2(m-n)T/(m+n)} \left(\frac{b^2}{(m+n)^2} \frac{(\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT})^2}{f^2} \right. \\
& - \frac{b(m-n)(m+n+1)}{(m+n)^2} \frac{\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT}}{f} + \frac{2Lb^2(m+n-1)}{(m+n)f^2} \\
& \left. + \frac{2b^2}{m+n} - \frac{mn}{(m+n)^2} \right) - \Lambda,
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
8\pi G\varepsilon = {} & a^2 f^{2+4/(m+n)} \mathrm{e}^{2(m-n)T/(m+n)} \left(\frac{b^2(2m+2n+1)}{(m+n)^2} \frac{(\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT})^2}{f^2} \right. \\
& \left. - \frac{b(m-n)(m+n+1)}{(m+n)^2} \frac{\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT}}{f} - \frac{2Lb^2(m+n-1)}{(m+n)f^2} - \frac{mn}{(m+n)^2} \right) + \Lambda,
\end{aligned} \tag{A2}
$$

$$
8\pi G\rho = \frac{2L(m+n-1)a^2 b^2}{m+n} f^{4/(m+n)} \mathrm{e}^{2(m-n)T/(m+n)}, \tag{A3}
$$

$$
\theta = |af^{1+2/(m+n)}| \mathrm{e}^{(m-n)T/(m+n)} \left(\frac{b(m+n+2)}{m+n} \frac{\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT}}{f} - \frac{m-n}{m+n} \right), \tag{A4}
$$

$$
\sigma = \sqrt{\tfrac{1}{3}}\, |af^{1+2/(m+n)}| \mathrm{e}^{(m-n)T/(m+n)} \left(\frac{b^2(m+n-1)^2}{(m+n)^2} \frac{(\mathrm{e}^{-bT} - L\,\mathrm{e}^{bT})^2}{f^2} + \right.
$$

$$+\frac{b(m-n)(m+n-1)}{m+n}\frac{e^{-bT}-Le^{bT}}{f}+\frac{m^2+n^2+mn}{(m+n)^2}\Bigg)^{1/2}, \tag{A5}$$

$$E=\sqrt{\tfrac{1}{3}}\,a^2 f^{2+4/(m+n)}\,e^{2(m-n)T/(m+n)}\Bigg\{\Bigg(\frac{b^2(m+n-1)}{m+n}+\frac{2mn}{(m+n)^2}$$

$$\times\frac{(m+n-2)(m+n+1)b^2}{(m+n)^2}\frac{(e^{-bT}-Le^{bT})^2}{f^2}$$

$$+\frac{m^2-n^2+2m-2n}{(m+n)^2}\frac{e^{-bT}-Le^{bT}}{f}\Bigg)^2+3\Bigg\}^{\frac{1}{2}}. \tag{A6}$$

For the model (25) we have

$$8\pi Gp=k^2\sin T^{2+4/(m+n)}\,e^{-2\alpha(m-n)T}\Bigg(\frac{2-(m+n)(m+n-1)}{2(m+n)^2}\cot^2 T$$

$$-\frac{\alpha(m-n)(m+n+1)}{m+n}\cot T-\frac{m+n+3}{2(m+n)}-\alpha^2 mn\Bigg)-\Lambda, \tag{A7}$$

$$8\pi G\varepsilon=k^2\sin T^{2+4/(m+n)}\,e^{-2\alpha(m-n)T}\Bigg(\frac{2+(m+n)(m+n+3)}{2(m+n)^2}\cot^2 T$$

$$-\frac{\alpha(m-n)(m+n+1)}{m+n}\cot T+\frac{m+n-1}{2(m+n)}-\alpha^2 mn\Bigg)+\Lambda, \tag{A8}$$

$$8\pi G\rho=-\frac{k^2(m+n-1)}{2(m+n)}(\sin T)^{4/(m+n)}\,e^{-2\alpha(m-n)T}, \tag{A9}$$

$$\theta=|k\sin T^{1+2/(m+n)}|\,e^{-\alpha(m-n)T}\Bigg(\alpha(m-n)-\frac{m+n+2}{m+n}\cot T\Bigg), \tag{A10}$$

$$\sigma=\sqrt{\tfrac{1}{3}}\,|k(\sin T)^{1+2/(m+n)}|\,e^{-\alpha(m-n)T}\Bigg(\frac{(m+n+1)^2}{(m+n)^2}\cot^2 T$$

$$+\frac{\alpha(m-n)(m+n-1)}{m+n}\cot T+\alpha^2(m^2+n^2+mn)\Bigg)^{\frac{1}{2}}. \tag{A11}$$

For the model (26) we have

$$8\pi Gp=\beta^2(1-MT)^{2+4/(m+n)}e^{-2(m-n)T/(m+n)}\Bigg(\frac{\{2-(m+n)(m+n-1)\}M^2}{2(m+n)^2(1-MT)^2}$$

$$+\frac{M(m-n)(m+n+1)}{(m+n)^2(1-MT)}-\frac{mn}{(m+n)^2}\Bigg)-\Lambda, \tag{A12}$$

$$8\pi G\varepsilon=\beta^2(1-MT)^{2+4/(m+n)}e^{-2(m-n)T/(m+n)}\Bigg(\frac{\{2+(m+n)(m+n+3)\}M^2}{2(m+n)^2(1-MT)^2}$$

$$+\frac{M(m-n)(m+n+1)}{(m+n)^2(1-MT)}-\frac{mn}{(m+n)^2}\Bigg)+\Lambda, \tag{A13}$$

$$8\pi G\rho = -\frac{(m+n-1)M^2\beta^2}{2(m+n)}(1-MT)^{4/(m+n)}\,\mathrm{e}^{-2(m-n)T/(m+n)}, \tag{A14}$$

$$\theta = |\beta(1-MT)^{1+2/(m+n)}|\,\mathrm{e}^{-(m-n)T/(m+n)}\left(\frac{(m+n+2)M}{(m+n)(1-MT)} + \frac{m-n}{m+n}\right), \tag{A15}$$

$$\sigma = \sqrt{\tfrac{1}{3}}\,|\beta(1-MT)^{1+2/(m+n)}|\,\mathrm{e}^{-(m-n)T/(m+n)}\left(\frac{(m+n+1)^2M^2}{(m+n)^2(1-MT)^2}\right.$$
$$\left. - \frac{(m-n)(m+n-1)M}{(m+n)^2(1-MT)} + \frac{m^2+n^2+mn}{(m+n)^2}\right)^{\frac{1}{2}}, \tag{A16}$$

$$E = \sqrt{\tfrac{1}{3}\beta^2}(1-MT)^{2+4/(m+n)}\,\mathrm{e}^{-2(m-n)T/(m+n)}\left\{\left(\frac{(m+n+2)(1-m-n)M^2}{(m+n)^2(1-MT)^2}\right.\right.$$
$$\left.\left. + \frac{(m^2-n^2-2m+2n)M}{(m+n)^2(1-MT)} + \frac{2mn}{(m+n)^2}\right)^2 + 3\right\}^{\frac{1}{2}}. \tag{A17}$$

For the model (27) we have

$$8\pi Gp = \mathrm{e}^{-2S^2T^2}\{S^4T^{2-2N} + S^2(N-1)T^{-2N}$$
$$+\tfrac{1}{4}(N^2+4N-1/m^2)T^{-2(N+1)}\} - \Lambda, \tag{A18}$$

$$8\pi G\varepsilon = \mathrm{e}^{-2S^2T^2}\{S^4T^{2-2N} + S^2(N+1)T^{-2N}$$
$$+\tfrac{1}{4}(N^2+4N-1/m^2)T^{-2(N+1)}\} + \Lambda, \tag{A19}$$

$$8\pi G\rho = S^2\mathrm{e}^{-2S^2T^2}\,T^{-2N}, \tag{A20}$$

$$\theta = \mathrm{e}^{-S^2T^2}\{2S^2T^{1-N} + (N+1)T^{-(N+1)}\}, \tag{A21}$$

$$\sigma = \sqrt{\tfrac{1}{3}}\,\mathrm{e}^{-S^2T^2}\{S^4T^{2-2N} + S^2(N-2)T^{-2N}$$
$$+\tfrac{1}{4}(N^2+4N+4-1/m^2)T^{-2(M+1)}\}^{\frac{1}{2}}, \tag{A22}$$

$$E = \sqrt{\tfrac{1}{3}}\,\mathrm{e}^{-2S^2T^2}\,T^{-2N}[\{(1/T^2)(N^2-2N-1/m^2)$$
$$+4S^4T^2 - 4S^2(2-N)\}^2 + 3]^{\frac{1}{2}}. \tag{A23}$$

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