Aust. J. Phys., 1985, 38, 329–35

Dead Time Correction for a Position Sensitive Detector*

T. C. Brown[A] *and J. K. Mackenzie*[B]

[A] The Statistical Consulting Centre,
University of Melbourne, Parkville, Vic. 3052.
[B] Division of Chemical Physics, CSIRO,
P.O. Box 160, Clayton, Vic. 3168.

Abstract

The dead time of some position sensitive detectors is determined by the arrangement in time of the totality of incoming events, irrespective of the position channel to which each event may be assigned. If the arrangement of events in position is statistically independent of their arrangement in time, the ratios of the means for the different position channels are unaffected by dead time and all channels need correction by the same factor, determined solely by the combined rate for all channels. Expressions for the variance of the individual counts in each channel are given for Type I counters. A symmetric uncertainty in the allocation of each event to its proper channel is considered and has no effect in a first approximation.

1. Introduction

It is familiar to users of particle counters that the observed number of counts (registrations) requires a correction for 'dead time' and that this correction increases proportionally with the count rate. Thus, without appropriate correction, low count rates, quite apart from any background rate, are enhanced relative to peak rates. In recent years the efficiency of data collection has been increased by the use of spatial position sensitive detectors and this has been of great importance where sources have been intrinsically weak as, for example, in some neutron diffraction experiments. In some cases these detectors have been composed of arrays of completely independent counters and, of course, low count rates are enhanced relative to peak rates in the usual way.

There are, however, linear position sensitive detectors available which take the form of a single gas-filled proportional counter, where the position of an incoming particle is determined by means of fast electronics from the time difference between pulses arriving at the two ends of the counter. In this case, the arrival of a particle at any position can render the counter dead, for a short time, to the recording of particles arriving at any position whatsoever. On intuitive grounds, it might be expected that the dead time correction is now a function of the total rate of arrival of particles from all positions rather than the rates for particular positions and, therefore, that the low count rates are not enhanced relative to the peak rates; the rates at all positions require correction by the same factor.

* Dedicated to Dr A. McL. Mathieson on the occasion of his 65th birthday.

0004-9506/85/030329$02.00

Some experimenters have expressed doubts about the validity of their intuition and these doubts are reinforced when it is realized that otherwise independent streams of events become dependent when they are subjected to a selection criterion imposed by dead time. Therefore, the purpose of the present paper is to set these doubts at rest by providing a theoretical justification of the above result. The theory developed also extends to give some less intuitively clear results for variances.

The next section introduces some statistical notation and gives some relevant information concerning two sorts of idealized counters which have frequently been used to model real counters. Section 3 proves the required result in fairly general terms under the sole assumption that the time behaviour of the incoming particles is statistically independent of their spatial behaviour. Section 4 introduces generating functions and uses them to show that, in general, the registered counts in different position channels are not statistically independent. In the final section it is shown that uncertainty in the allocation of a particular registration to its correct position does not significantly distort the count rate, provided that this uncertainty is symmetric about each and every position channel.

2. Single Counter Statistics

A counter is supposed to register 'random events' which may be X-ray photons or particles such as electrons. A common assumption is that these events constitute a Poisson process characterized by the natural conditions that in any sufficiently small interval δt (i) the probability of more than one event occurring is asymptotically negligible, and (ii) the probability of just one event occurring is asymptotically $r\delta t$ (where r is the rate), independent of any previous events. For such a process the probability $P(T_0 < t)$ that the time T_0 from an arbitrary moment to the next event will not exceed t is given by

$$P(T_0 < t) = 1 - \exp(-rt). \tag{1}$$

It also follows that the probability for exactly n events to occur during any time interval of length t is given by the Poisson law

$$P\{N(t) = n\} = (rt)^n \exp(-rt)/n!. \tag{2}$$

The mean number of such events $\mathrm{E}\{N(t)\}$ and the square of the standard deviation or variance $\mathrm{Var}\{N(t)\}$ are given by

$$\mathrm{E}\{N(t)\} = \mathrm{Var}\{N(t)\} = rt.$$

Due to 'dead time' a counter is unable to register all events and Feller (1948) introduced two ideal cases. Type I is a counter with non-extending dead time and the counter is locked and dead for a constant time d after each registration: an event is registered if, and only if, no registration has occurred during the interval d preceding it. Type II is a counter with an extending dead time and the counter registers if, and only if, no event has occurred during the interval d preceding it: an event occurring while the counter is dead prolongs the dead time by a further interval d and the counter can, in theory, remain dead indefinitely.

For practical counters, the overall dead time must include not only that associated with the actual counter, but also that associated with the subsequent electronics. Thus, real counters are a compromise between these two ideal types and various authors, such as Albert and Nelson (1953) and Ramakrishnan (1954), have considered more complicated cases. Since, in practice, the theoretical results do not differ much for the two types, attention is usually confined to the mathematically simpler Type I case.

Feller (1948) has given exact expressions for the mean and variance of the number of registrations $N(t)$ in time t for both types of counter. More usefully, he has given approximations which are asymptotically valid as t tends to infinity. In this limit, for Type I,

$$\mathrm{E}\{N(t)\} = rt/(1+rd), \tag{3}$$

$$\mathrm{Var}\{N(t)\} = rt/(1+rd)^3. \tag{4}$$

For Type II, the approximations are

$$\mathrm{E}\{N(t)\} = rt\exp(-rd), \tag{5}$$

$$\mathrm{Var}\{N(t)\} = rt\exp(-rd)\{1-2rd\exp(-rd)\}. \tag{6}$$

In the same limit, the actual number of counts has a normal (gaussian) distribution with these means and variances.

3. Single Channel Statistics

For a position sensitive counter, each event of the incoming Poisson stream is labelled with a linear position in space. Events with labels corresponding to some small range of positions are all counted into the same accumulator or channel. Now the dead time behaviour of the counter is controlled by the effect of all the incoming events together and the question arises as to how this dead time affects the statistics of the individual channels.

The critical assumption is that the statistics of the spatial arrangement of events are completely independent of their statistics in time. As will be shown below, it now follows, independently of the counter type, that the relative values of the expectations (mean values) of the counts in the various channels are unaffected by dead time: all channels need correction by the same factor determined solely by the combined rate for all channels. The variance of the channel counts is also quite simply computed.

Let us consider one particular channel with the label zero. We let $p = 1-q$ be the proportion of incoming events with this label and, for an interval of length t, we let $N(t)$ and $N_0(t)$ be respectively the number of registrations in the combined stream and those in the 'diluted' stream with the label zero. Then using an indicator variable X_i we have

$$N_0 = \sum_{i=1}^{N(t)} X_i, \tag{7}$$

where X_i has the value 1 if the ith registration has the label zero, and is zero otherwise. Now the association of a registration with the label zero is independent of $N(t)$

and so the calculation of the mean value of $N_0(t)$ can be carried out in two steps. Firstly, conditional on some fixed but arbitrary value of $N(t)$, the spatial average over X_i is calculated and, finally, this result is averaged for all relevant values of $N(t)$. Conditional on a fixed value of $N(t)$, the random variable ΣX_i has a binomial distribution and $\mathrm{E}\{\Sigma X_i | N(t)\}$ is a common generic notation for the mean value of this sort of conditional distribution. Thus, we have

$$\mathrm{E}\left(\sum_{i=1}^{N(t)} X_i\right) = \mathrm{E}_N\left\{\mathrm{E}_X\left(\sum_{i=1}^{N(t)} X_i | N(t)\right)\right\}$$
$$= \mathrm{E}\{pN(t)\}, \tag{8}$$

whence

$$\mathrm{E}\{N_0(t)\} = p\mathrm{E}\{N(t)\}. \tag{9}$$

For a Type I counter, equation (3) gives

$$\mathrm{E}\{N_0(t)\} \approx prt/(1+rd), \tag{10}$$

where r refers to the rate of the whole incoming stream.

The variance is slightly more complicated to calculate and depends on the general result (Feller 1966, p. 164, Ex. 18)

$$\mathrm{Var}\left(\sum_{i=1}^{N(t)} X_i\right) = \mathrm{E}_N\left\{\mathrm{Var}_X\left(\sum_{i=1}^{N(t)} X_i | N(t)\right)\right\} + \mathrm{Var}_N\left\{\mathrm{E}_X\left(\sum_{i=1}^{N(t)} X_i | N(t)\right)\right\}$$
$$= \mathrm{E}\{N(t)pq\} + \mathrm{Var}\{pN(t)\},$$

whence

$$\mathrm{Var}\{N_0(t)\} = pq\mathrm{E}\{N(t)\} + p^2\mathrm{Var}\{N(t)\}. \tag{11}$$

For a Type I counter, equations (3) and (4) give

$$\mathrm{Var}\{N_0(t)\} \approx pqrt/(1+rd) + p^2 rt/(1+rd)^3. \tag{12}$$

4. Generating Functions

Study of random variables such as N which take only integral values $n = 0, 1, 2, \ldots$ is facilitated by the powerful method of generating functions as outlined by Feller (1968, Ch. XI). We let $p_n = P(N=n)$ and define the generating function

$$G(s) = \sum_{n=0}^{\infty} p_n s^n. \tag{13}$$

For a proper distribution we have

$$G(1) = \sum_{n=0}^{\infty} p_n = 1, \tag{14}$$

and so the series converges absolutely and represents an analytic function in the unit disc. If the mean μ and variance σ^2 exist, differentiation of (13) shows that

$$G'(1) = \sum_{n=0}^{\infty} np_n = \mu\,, \tag{15}$$

$$G''(1) = \sum_{n=0}^{\infty} n(n-1)p_n = \sigma^2+\mu^2-\mu\,, \tag{16}$$

which is the second factorial moment and, moreover, the Taylor series expansion about $s = 1$ is

$$G(s) = 1+\mu(s-1)+\tfrac{1}{2}(\sigma^2+\mu^2-\mu)(s-1)^2+\dots. \tag{17}$$

For the Poisson process with rate r, the number of events at time t has the generating function

$$G_{\mathrm{P}}(s) = \sum_{n=0}^{\infty} (rt)^n \exp(-rt)\, s^n/n! = \exp\{-rt(s-1)\}\,, \tag{18}$$

while for the binomial distribution with parameter p

$$G_{\mathrm{B}}(s) = \sum_{n=0}^{m} \binom{m}{n} p^n q^{m-n} s^n = (ps+q)^m\,. \tag{19}$$

A diluted stream can be constructed by selection from the events of an incoming stream, the selections being made independently with probability p. Then, for the resulting diluted stream, we have

$$P(N_{\mathrm{dil}}=n) = \sum_{m=n}^{\infty} \binom{m}{n} p^n q^{m-n}\, P(N=m)\,,$$

and so

$$G_{\mathrm{dil}}(s) = \sum_{n=0}^{\infty} s^n \sum_{m=n}^{\infty} \binom{m}{n} p^n q^{m-n}\, P(N=m)\,,$$

which, on interchanging the order of summations, becomes

$$\begin{aligned} G_{\mathrm{dil}}(s) &= \sum_{m=0}^{\infty} P(N=m) \sum_{n=0}^{m} \binom{m}{n} p^n s^n q^{m-n} \\ &= \sum_{m=0}^{\infty} P(N=m)\,(ps+q)^m\,, \end{aligned}$$

so that

$$G_{\mathrm{dil}}(s) = G(ps+q)\,. \tag{20}$$

This result is a special case of a general theorem (Feller 1968, p. 287) concerning the generating function for a sum, such as (7), of a random number of random variables. From equation (18) it follows that a diluted Poisson process is another Poisson process with rate rp.

The results of the previous section are recovered by regarding the registrations in a particular channel (with label zero) as obtained by dilution of the whole stream with probability p. If we let $\mu = \mathrm{E}\{N(t)\}$ and $\sigma^2 = \mathrm{Var}\{N(t)\}$ be the mean and variance of the registrations due to the entire total stream, then the generating function for the combined stream is given by (17), so that the generating function for the diluted stream is

$$\begin{aligned} G(ps+q) &= G\{p(s-1)+1\} \\ &= 1+p\mu(s-1)+\tfrac{1}{2}p^2(s-1)^2(\mu^2-\mu+\sigma^2)+\dots . \end{aligned}$$

Since the coefficient of $\frac{1}{2}(s-1)^2$ can be written in the form

$$(p\mu)^2-p\mu+pq\mu+p^2\sigma^2,$$

the results (9) and (11) are recovered immediately on comparison with (17). Of course, the generating function contains information about the whole distribution and not just about the first and second moments.

The bivariate generalization of (13) is the two variable generating function

$$H(s_1, s_2) = \sum_{n,m} P(N_1=n, N_2=m)\, s_1^n\, s_2^m, \tag{21}$$

and the covariance is

$$\mathrm{Cov}(N_1, N_2) = \partial^2 H/\partial s_1\, \partial s_2 - (\partial H/\partial s_1)(\partial H/\partial s_2), \tag{22}$$

when the partial derivatives are evaluated at $s_1 = s_2 = 1$. Moreover, by following the model of the calculation leading to (20), the joint generating function for a pair of streams diluted independently with probabilities p_1 and p_2 from a common parent stream is

$$H_{\mathrm{dil}}(s_1, s_2) = G(p_1 s_1 + p_2 s_2 + 1 - p_1 - p_2). \tag{23}$$

It now follows that the covariance of these two diluted streams is

$$\mathrm{Cov}(N_1, N_2) = p_1 p_2(\sigma^2-\mu). \tag{24}$$

Thus, in general, the two streams are correlated and so are not independent: the only exception arises when $\sigma^2 = \mu$, in which case all streams are Poisson.

5. Effect of Uncertainty in Channel Assignment

In practice, the assignment of a registration to its proper channel is not always correctly made and, moreover, every channel is contaminated with registrations from neighbouring channels. Thus, a registered particle which actually arrives in channel i ($i = \pm 1, \pm 2, \dots$) will be misassigned to channel 0 with some probability π_i and, if it arrives in channel 0, it will be correctly assigned with probability π_0. The effect of this on the preceding analysis is quite simple: the stream of registrations in channel 0 is still a dilution of the entire stream of registrations, but the probability of a particular event being registered in channel 0 is altered. Provided that misassignment occurs independently of the spatial and temporal arrangement of particles, this probability is

$$p = \sum \rho_i \pi_i,$$

where ρ_i is the probability that a particle arrives in channel i, and, both here and below, the summation without limits is for i from $-\infty$ to $+\infty$.

The formulas (9)–(12) and (20) continue to apply with the altered interpretation for p. To see the effect of misassignment on the true arrival stream, we let $\mu_i = \rho_i \mathrm{E}\{N(t)\}$ be the mean number of particles, at some fixed time t, which should have been registered in channel i, and σ_i^2 be the corresponding variances. Then, from (9), the mean count in channel 0 is given by

$$m_0 = \sum \pi_i \mu_i = p\mu, \tag{25}$$

and from (11) the variance of this count is

$$s_0^2 = \sum \pi_i \sigma_i^2 + (\sigma^2 - \mu)(p^2 - \sum \pi_i \rho_i^2), \tag{26}$$

where $\mu = \mathrm{E}\{N(t)\}$ and $\sigma^2 = \mathrm{Var}\{N(t)\}$ are the mean and variance of the overall total number of registrations in time t. Here use has been made of the relation

$$\sigma_i^2 = \rho_i \mu + \rho_i^2(\sigma^2 - \mu),$$

which is just (11) for the individual streams. If there were no dead time, the second term would disappear, as the registration streams would all be Poisson. In most practical cases it should make only a small contribution and so the variance of the count should be crudely equal to $\sum \pi_i \mu_i$.

To see what effect this sort of uncertainty in the assignment of the registrations to their correct channels may have, we consider the case of symmetric misassignment, $\pi_i = \pi_{-i}$, and a locally linear variation of mean counts with channel number or $\mu_i = \lambda + i\delta$. Then, we have

$$\sum \pi_i \mu_i = \lambda$$

and there is no effect of misassignment in this first approximation.

Acknowledgment

The authors thank Dr A. McL. Mathieson of the Division of Chemical Physics for discussions and for bringing this problem to their attention in the first place. This work was carried out while one of us (T.C.B.) was a member of the Department of Mathematics, Monash University.

References

Albert, G. E., and Nelson, L. (1953). *Ann. Math. Stat.* **24**, 9–22.

Feller, W. (1948). *In* 'Studies and Essays Presented to R. Courant on his 60th Birthday, Jan. 8 1948', pp. 105–15 (Interscience: New York).

Feller, W. (1966). 'An Introduction to Probability Theory and Its Applications', Vol. 2, 1st edn (Wiley: New York).

Feller, W. (1968). 'An Introduction to Probability Theory and Its Applications', Vol. 1, 3rd edn (Wiley: New York).

Ramakrishnan, A. (1954). *Philos. Mag.* **45**, 1050–2.

Manuscript received 26 April, accepted 18 October 1984