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Electron Outflow in Axisymmetric Pulsar Magnetospheres

R. R. Burman

Department of Physics, University of Western Australia, Nedlands, W.A. 6009.

Abstract

Mestel *et al.* (1985) have recently introduced an axisymmetric pulsar magnetosphere model in which electrons leave the star with speeds that are non-negligible, but not highly relativistic, and flow with moderate acceleration, and with poloidal motion that is closely tied to the poloidal magnetic field lines, before reaching a limiting surface, near which rapid acceleration occurs. This paper presents an analysis of flows which either encounter the limiting surface beyond the light cylinder or do not meet it at all.

1. Introduction

The canonical pulsar model consists of a steadily rotating neutron star with magnetic axis inclined to the rotation axis. To the extent that dissipative forces can be neglected, the equation of motion of the magnetospheric particles expresses the balance of the Lorentz force by relativistic inertia. In the consequent theoretical description, the inertial effects manifest themselves in two ways: through the existence of a 'non-corotational' electric potential, describing a part of the electric field which adds to that associated with the rotation of the magnetic field structure, and the occurrence of an 'inertial drift' of the flow across magnetic field lines (Mestel *et al.* 1979; Burman and Mestel 1979). The Lorentz factor can become infinite beyond, but not on or inside, the light cylinder—the surface on which the speed of corotation with the star equals c, the vacuum speed of light (Burman 1980).

Mestel, Robertson, Wang and Westfold (1985; henceforth denoted by MRW^2) have recently introduced an axisymmetric pulsar magnetosphere model in which electrons leave the star with speeds that are non-negligible, but not highly relativistic, and flow with moderate acceleration, and with poloidal motion that is almost along poloidal magnetic field lines, before reaching a limiting surface, near which rapid acceleration occurs. They have developed a theoretical treatment of the outflow domain which neglects inertial drift and part of the effect of the non-corotational potential, as well as dissipative forces. Singularities of the resulting Lorentz factor define the limiting surface, signalling failure of this treatment, meaning that inertial or dissipative effects or both have become important.

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In addition to these 'Class I' flows, a second type of moderately accelerated flow is found: 'Class II' flows do not encounter a region of rapid acceleration (Burman 1984). Class I flows may be subdivided into Classes IA, IB and IC, in which the limiting surface indicative of rapid acceleration is met respectively inside, on and outside the light cylinder.

The purpose of the present paper is to study the flows that cross the light cylinder without meeting the limiting surface—that is, flows of Classes IC and II.

2. MRW^2 Formalism

In this section, the basic formalism developed by MRW^2 for the analysis of their model will be outlined. The system is taken to be axisymmetric and steadily rotating at angular frequency Ω. The dimensionless cylindrical radial coordinate X is unity on the light cylinder, which has a radius c/Ω. The unit toroidal vector is denoted by $\boldsymbol{t}$. It follows from Faraday's law and $\nabla.\boldsymbol{B} = 0$ that the electric field can be written as the sum of a part $X\boldsymbol{B}\times\boldsymbol{t}$, associated with the rotation of the magnetic field structure, and a 'non-corotational' part $-\nabla\Phi$, with Φ a gauge-independent potential.

MRW^2 developed their equations in dimensionless form by expressing distances and the flow velocity $\boldsymbol{V}$ in units of c/Ω and c, and normalizing field variables in terms of the equatorial dipolar magnetic field strength at the light cylinder: $B_1 \equiv \frac{1}{2}(\Omega r_{\mathrm{s}}/c)^3 B_0$ where r_{s} is the stellar radius and B_0 is the polar surface magnetic field strength. The magnetic field and the charge density ρ_{e} are expressed in units of B_1 and $B_1/4\pi c$.

The poloidal parts of the magnetic field and electric current density are expressed in terms of Stokes stream functions:

$$\boldsymbol{B}_{\mathrm{p}} = X^{-1}\boldsymbol{t}\times\nabla P, \qquad \boldsymbol{j}_{\mathrm{p}} = X^{-1}\boldsymbol{t}\times\nabla S,$$

with P and S measured in units of $(c/\Omega)^2 B_1$ and $c^2 B_1/4\pi\Omega$ respectively. Charge separation is assumed, so $\boldsymbol{j}_{\mathrm{p}} = \rho_{\mathrm{e}}\boldsymbol{V}_{\mathrm{p}}$. The poloidal part of Ampère's law reduces to $B_\phi = -S/X$. It follows from Gauss's law and the toroidal part of Ampère's law that

$$\nabla^2\Phi + 2B_z = -(1-XV_\phi)\rho_{\mathrm{e}}, \tag{1}$$

with Φ expressed in units of cB_1/Ω. The subscripts ϕ and z denote toroidal and axial components.

In the domain under consideration, the flow is taken to be dissipation-free and inertial drift is neglected, so that the poloidal flow is along the poloidal magnetic field lines, meaning that S is a function of P only. In addition, the rigidly rotating pulsar crust is treated as a perfect conductor, and the particle flow speed at the surface is taken not to be highly relativistic. As a result of these approximations, the flow velocity, reduced by the local velocity of corotation with the star, is parallel to the magnetic field: $\boldsymbol{V}-X\boldsymbol{t} = \kappa\boldsymbol{B}$, with κ a scalar. It follows from the equations $\boldsymbol{V}_{\mathrm{p}} = \kappa\boldsymbol{B}_{\mathrm{p}}$ and $\boldsymbol{j}_{\mathrm{p}} = \rho_{\mathrm{e}}\boldsymbol{V}_{\mathrm{p}}$ that $\rho_{\mathrm{e}}\kappa = \mathrm{d}S/\mathrm{d}P$ (MRW^2), which is constant on the poloidal magnetic field lines, which are also streamlines of the poloidal flow.

MRW^2 wrote $\mathrm{d}S/\mathrm{d}P$ as $-2V_0(P)$, so $\rho_{\mathrm{e}}\kappa = -2V_0$. Near the star, the $\nabla^2\Phi$ term in (1) can be neglected, leaving $\rho_{\mathrm{e}} = -2B_z/(1-XV_\phi)$; hence $V_{\mathrm{p}}/B_{\mathrm{p}} = (1-XV_\phi)V_0/B_z$. But near the star $XV_\phi \ll 1$ and, provided the outflow emanates from a small polar cap, $B_{\mathrm{p}} \approx B_z$; thus MRW^2 identified $V_0(P)$ with the speed at which the electrons, travelling along the lines of constant P, leave the star. This allowance for a significant emission speed is one of the key new features of their work.

The steady rotation constraint implies the existence of a constant of the motion which (for electrons) has the dimensionless form (MRW^2)

$$G = \gamma(1 - X V_\phi) - \Phi/\epsilon; \tag{2}$$

the Lorentz factor is denoted by γ and the small parameter ϵ represents Ω/ω_g, with ω_g denoting the nonrelativistic electron gyrofrequency in the fiducial field B_1. Axisymmetry and neglect of inertial drift imply that G is a function of S only. The perfect conductivity boundary condition on the stellar surface means that the non-corotational electric potential has a constant value there, which can be taken to be zero. Hence, since $X V_\phi \ll 1$ at the star, $G \approx \gamma_0$. When the emission speed is nonrelativistic, the Endean (1972) integral G has a constant value, namely one, throughout the flow.

Following the analysis of MRW^2, elimination of the velocity between $\boldsymbol{V} - X\boldsymbol{t} = \kappa \boldsymbol{B}$ and the definition $\gamma^{-2} \equiv 1 - V^2$ of the Lorentz factor, together with the use of $B_\phi = -S/X$, leads to a quadratic equation for κ, yielding

$$B^2\kappa/S = 1 + \{1 + (1 - X^2 - \gamma^{-2})B^2/S^2\}^{\frac{1}{2}}. \tag{3}$$

Near the emission regions, $V_p \approx V_0$ and $V_\phi \approx X$ so that $1 - X^2 - \gamma^{-2} \approx V_0^2$; hence, for $S > 0$ and outflow, the positive sign before the radical has been taken (MRW^2). The relation $\rho_e \kappa = -2V_0(P)$ now gives

$$-2V_0 B^2/S\rho_e = 1 + \{1 + (1 - X^2 - \gamma^{-2})B^2/S^2\}^{\frac{1}{2}}, \tag{4}$$

while (1) becomes

$$\nabla^2\Phi + 2B_z = 2V_0 SF, \tag{5}$$

where

$$F \equiv 1 + \frac{(1 - X^2)B^2/S^2}{1 + \{1 + (1 - X^2 - \gamma^{-2})B^2/S^2\}^{\frac{1}{2}}}. \tag{6}$$

These are the basic equations developed by MRW^2 for the outflow domain.

There is a minimum value γ_m of γ for the above radical to be real: $\gamma_m^{-2} \equiv 1 - X^2 + S^2/B^2$ (MRW^2). For $X < 1$, γ_m is always real; it is real for $X > 1$ provided $B_\phi^2 \geqslant (X^2 - 1)B_p^2$—the condition obtained by Goldreich and Julian (1969) for their flow to have a real Lorentz factor outside $X = 1$.

The radical can be expressed as $(B/S)(\gamma_m^{-2} - \gamma^{-2})^{\frac{1}{2}}$. Note also that $B/\gamma_m S = F_\infty$, where F_∞ denotes the function F for γ infinite: $F_\infty \equiv \{1 + (1 - X^2)B^2/S^2\}^{\frac{1}{2}}$. Hence the radical is $(1 - \gamma_m^2/\gamma^2)^{\frac{1}{2}} F_\infty$. The MRW^2 formalism expresses flow variables in terms of γ, which is bounded below by γ_m. The basic MRW^2 flow equations above show that, as functions of γ, the variables κ, ρ_e and F are actually functions of γ/γ_m.

GJ flow (Goldreich and Julian 1969) is defined as flow satisfying the equations just listed, together with the additional restriction that the term $\nabla^2\Phi$ in (5) be negligible. Putting $\nabla^2\Phi = 0$ there and writing $\tilde{B}_z$ for $B_z/V_0 S$ yields $\tilde{B}_z = F$, which, after using the definition (6) of F, may be solved for the Lorentz factor:

$$\frac{1}{\gamma^2} = \frac{1-X^2}{\tilde{B}_z-1}\left(\tilde{B}_z+1-\frac{1-X^2}{\tilde{B}_z-1}\frac{B^2}{S^2}\right), \tag{7}$$

which may be written as $\gamma^{-2} = (1-X^2)C/A$, where $A \equiv (\tilde{B}_z-1)^2$ and $C \equiv \tilde{B}_z^2-F_\infty^2$, or as

$$1/\gamma^2 = (1-X^2)(\tilde{B}_z^2-F_\infty^2)/(\tilde{B}_z-1)^2\,. \tag{7$'$}$$

The equations $\rho_e\kappa = -2V_0$, $V_\phi - X = -\kappa S/X$ and $\rho_e = -2B_z/(1-XV_\phi)$ for GJ flow yield, on eliminating ρ_e, V_ϕ and κ in pairs (MRW[2]),

$$\kappa S = (1-X^2)/(\tilde{B}_z-1)\,, \tag{8a}$$

$$\rho_e = -2V_0\,S(\tilde{B}_z-1)/(1-X^2)\,, \tag{8b}$$

$$V_\phi/X = 1-(1-X^2)/(\tilde{B}_z-1)X^2\,. \tag{8c}$$

MRW[2] introduced a surface S_l, defined by putting $\nabla^2\Phi = 0$ and $\gamma = \infty$ in (5), yielding $\tilde{B}_z = F_\infty$. This surface is an outer limit for possible GJ, moderately accelerated, flow; inside it, neglect of $\nabla^2\Phi$ in (5) is consistent with a finite Lorentz factor. In the vicinity of S_l, the GJ flow approximation must fail: the actual flow will be rapidly accelerated there, its Lorentz factor becoming large but remaining finite; dissipation or inertial drift, or both, will quickly become important and most of the above equations will be inapplicable (MRW[2]).

Near the star, taking $X^2 \ll 1$ and $B_\phi^2 \ll B_p^2$ shows that $F_\infty \approx B_p/S$, which is large; near the polar caps $\tilde{B}_z \approx F_\infty/V_0 > F_\infty$. Within the light cylinder, $F_\infty > 1$; on it $F_\infty = 1$; beyond it $F_\infty < 1$ (MRW[2]). The four mathematically distinct types of GJ outflow may be distinguished by considering the behaviour of $\tilde{B}_z$ and F_∞ along an arbitrary poloidal flow line, remembering that $V_0 S$ is constant along each line (Burman 1984). On these lines, which are also poloidal magnetic field lines, $\tilde{B}_z$ may be thought of as behaving qualitatively as it would for a dipole field: it decreases from a large positive value, exceeding F_∞, at the polar caps, eventually passing through zero (on the $B_z = 0$ cones) to reach a maximum negative value on the equatorial plane.

In flows of Classes IC and II, $\tilde{B}_z$ falls to equality with F_∞ on $X = 1$ (where they are both equal to one) so that $C = 0 = A$ there. Provided $(\tilde{B}_z-1)/(1-X^2)$ is finite on $X = 1$, the Lorentz factor is finite on, as well as inside, the light cylinder. For Class IC flow, $\tilde{B}_z$ subsequently intersects either F_∞ or $-F_\infty$ at some point beyond $X = 1$, so C again vanishes and γ is infinite at this second zero of C. In Class II flow, $\tilde{B}_z$ is below F_∞ and above $-F_\infty$ everywhere along the poloidal flow line beyond the light cylinder, so that C vanishes only on the light cylinder: with the proviso that $(\tilde{B}_z-1)/(1-X^2)$ be finite on $X = 1$, the Lorentz factor remains finite all along the flow line.

The MRW[2] treatment of GJ flow does not involve complete neglect of Φ, merely neglect of the $\nabla^2\Phi$ term in the combined Gauss–toroidal Ampère law (1). The Endean integral (2) can be used to calculate Φ in GJ flow, so long as the result is consistent with neglect of the $\nabla^2\Phi$ term in (1).

Inertial drift is neglected in the MRW^2 treatment of GJ flow. It can be incorporated formally by replacing $\boldsymbol{B}$ with the magnetoidal field $\boldsymbol{B}+\epsilon\nabla\times(\gamma\boldsymbol{V})$, denoted by $\boldsymbol{B}^*$ (Burman 1985). Consistency of the neglect of inertial drift with a GJ outflow solution can be checked by making order of magnitude comparisons of the inertial and magnetic terms in $\boldsymbol{B}^*$ (cf. Wright 1978).

3. Class IC/II Flow

In addition to the coordinates used above, Z will represent distance along the star's spin axis in units of the light cylinder radius and R will denote $(X^2+Z^2)^{\frac{1}{2}}$; the angular variables θ and ϕ are the remaining spherical polar coordinates based on the spin axis. The auxiliary variables $\bar{P}$, U and Q, denoting $-P$, $X^{2/3}$ and $\bar{P}^{2/3}$, will be used where convenient.

In a first approximation, the poloidal magnetic field in the domain of Goldreich–Julian outflow can be taken to be dipolar:

$$\bar{P}=X^2/R^3\,,\qquad B_{\mathrm{p}}=(X^2+4Z^2)^{\frac{1}{2}}/R^4\,,\tag{9a, b}$$

$$B_x=3XZ/R^5\,,\qquad B_z=(2Z^2-X^2)/R^5\,.\tag{9c, d}$$

It is useful, in order to follow the variation of quantities along the poloidal field/flow lines, to use X and P, rather than X and Z, as the independent variables. The dipole magnetic field is described by (MRW^2)

$$B_{\mathrm{p}}=(2\bar{P}/X^2)(1-\tfrac{3}{4}QU)^{\frac{1}{2}}\,,\qquad B_z=(2\bar{P}/X^2)(1-\tfrac{3}{2}QU)\,.\tag{10a, b}$$

Its field lines, $P=$ constant, have the equations $\bar{P}R=\sin^2\theta=QU$. Their dimensionless radius of curvature is given by

$$\tfrac{3}{4}Q\rho=U^{\frac{1}{2}}(1-\tfrac{3}{4}QU)^{3/2}/(1-\tfrac{1}{2}QU)\,.\tag{11}$$

It follows that Class IC/II flow, for which $B_z=V_0 S$ on the light cylinder (corresponding to $C=0=A$ there), is described by the relations (MRW^2)

$$S(P)=2\bar{P}(1-\tfrac{9}{8}Q)^{\frac{1}{2}}\,,\qquad V_0(P)=(1-\tfrac{3}{2}Q)/(1-\tfrac{9}{8}Q)^{\frac{1}{2}}\,.\tag{12a, b}$$

The Lorentz factor corresponding to V_0 is given by

$$\gamma_0^2=(8/15Q)(1-\tfrac{9}{8}Q)/(1-\tfrac{6}{5}Q)\,;\tag{13}$$

it is singular on the axis.

For Class IC/II flow, in the dipole approximation, (12a) and (10a) for S and B_{p} show that

$$-B_\phi/XB_{\mathrm{p}}=\{(1-\tfrac{9}{8}Q)/(1-\tfrac{3}{4}QU)\}^{\frac{1}{2}}.\tag{14}$$

The definitions of γ_{m} and F_∞ can be expressed in terms of the ratio B_ϕ/B_{p} by

$$\gamma_{\mathrm{m}}^{-2}\equiv 1-X^2(1+B_\phi^2/B_{\mathrm{p}}^2)^{-1}\,,\qquad X^2F_\infty^2\equiv 1+(1-X^2)B_{\mathrm{p}}^2/B_\phi^2\,.\tag{15, 16}$$

For Class IC/II flow, the poloidal motion is along lines of constant P with Q bounded by zero and $\frac{2}{3}$; these limits correspond to particles emitted from the poles and to particles which cross the light cylinder at $Z^2 = \frac{1}{2}$ (where $B_z = 0$). That is, Class IC/II outflow comes from inner polar cap regions, bounded by colatitudes given by $\sin\theta_1 = (\frac{2}{3})^{3/4}\sin\theta_0$, or $\theta_1 = 0{\cdot}74\theta_0$, where θ_0 denotes the boundary of the GJ polar cap, and symmetrically in the other hemisphere. (The GJ polar cap is bounded by the foot of the dipole magnetic field line which is tangential to the light cylinder, corresponding to $\bar{P} = 1 = Q$.) On Class IC/II flow lines, $B_z > 0$ inside the light cylinder.

The flow analysis used here is based on the relation $\boldsymbol{V} - X\boldsymbol{t} = \kappa\boldsymbol{B}$, the toroidal part of which involves taking the Endean integral to be constant throughout the flow rather than merely constant on each poloidal flow line. According to (12b), which gives the unphysical result $V_0 = 1$ on the axis of symmetry, the present theory of Class IC/II flows is inapplicable to the parts that emanate from the innermost cores of the polar caps.

For convenience and clarity in writing some of the formulas, the following functions will be used:

$$M(Q) = 2/3Q - 1\,, \qquad H(U) = 1/(1+U+U^2)\,. \tag{17a, b}$$

The function $M(Q)$ varies from ∞ to 0 as Q goes from 0 to $\frac{2}{3}$, corresponding to the axis and the outer limit of IC/II flow.

Note from (12) and (17a) that

$$V_0\,S = 3\bar{P}QM(Q)\,. \tag{12$'$}$$

This and (10b) for B_z give

$$\tilde{B}_z = U^{-3}(1-\tfrac{3}{2}QU)/(1-\tfrac{3}{2}Q)\,, \tag{18}$$

from which it follows that

$$(\tilde{B}_z - 1)/(1-X^2) = \{1 + H(U)/M(Q)\}/U^3\,. \tag{19}$$

The ratio on the left-hand side of (19) recurs throughout the MRW^2 theory of GJ flow; it appears in (7) for γ and in (8) for κS, $\rho_{\rm e}$ and V_ϕ; equation (19) provides a helpful separation of its X and P dependences for the case of Class IC/II flow in a dipole magnetic field. In particular, (19) shows that

$$\lim_{X\to 1}\{(\tilde{B}_z - 1)/(1-X^2)\} = (1-Q)/(1-\tfrac{3}{2}Q)\,, \tag{20}$$

which is finite for $Q < \frac{2}{3}$.

On using the decomposition (19), equations (8) can be written as

$$\kappa S = U^3\{1 + H(U)/M(Q)\}^{-1}\,, \tag{21}$$

$$\rho_{\rm e} = -(6\bar{P}Q/U^3)\{M(Q) + H(U)\} \tag{22}$$

$$= -2B_z\{1-(1-H)\tfrac{3}{2}Q\}/(1-\tfrac{3}{2}QU)\,, \tag{22$'$}$$

$$V_\phi/X = \{1 + M(Q)/H(U)\}^{-1}\,; \tag{23}$$

equations (12$'$) for $V_0\,S$ and (10b) for B_z have been used in obtaining (22) and (22$'$).

Substituting (21), together with (10a) for B_p and (12a) for S, into $V_p = \kappa B_p$ gives

$$V_p = \{(1-\tfrac{3}{4}QU)/(1-\tfrac{9}{8}Q)\}^{\frac{1}{2}}\{1+H(U)/M(Q)\}^{-1}, \tag{24}$$

or, on using (12b) for V_0,

$$V_p/V_0 = (1-\tfrac{3}{4}QU)^{\frac{1}{2}}\{1-(1-H)\tfrac{3}{2}Q\}^{-1}. \tag{25}$$

It follows from (18) for $\tilde{B}_z$ that

$$M(Q)\,U^3(\tilde{B}_z+1) = 1-U+(1+U^3)\,M(Q), \tag{26a}$$

$$M(Q)\,U^3(\tilde{B}_z-1) = (1-U)\{1+M(Q)/H(U)\}. \tag{26b}$$

Equation (26b) for $\tilde{B}_z-1$ shows that A, defined as $(\tilde{B}_z-1)^2$, is given by

$$(MU^3)^2A = (1-U)^2\{1+M(Q)/H(U)\}^2. \tag{27a}$$

Equations (26), together with (16) for F_∞, show that C, defined as $B_z^2-F_\infty^2$, is given by

$$(MU^3)^2C = (1-U)\{1-U+2M(Q)+(1-X^2B_p^2/B_\phi^2)M^2(Q)/H(U)\}. \tag{27b}$$

On using (27), together with (14) for $-B_\phi/XB_p$, it follows that

$$\tfrac{3}{2}QH(U)\{1+M(Q)/H(U)\}^2\gamma^{-2} = 2-(1+U)\tfrac{3}{2}Q$$
$$-(1-\tfrac{3}{2}Q)^2(\tfrac{4}{3}-\tfrac{3}{2}Q)^{-1}(1-\tfrac{2}{3}U)/H(U). \tag{28}$$

Thus, the MRW^2 Lorentz factor becomes infinite on $U = 1$ for $Q = \frac{2}{3}$, on $U = \frac{3}{2}$ for $Q = \frac{8}{15}$ and on $U = 2$ for $Q = \frac{1}{2}$; in the first case, γ is infinite where $QU = \frac{2}{3}$, corresponding to $B_z = 0$; in the last case, γ is infinite where $QU = 1$, corresponding to the equatorial plane. The range from $\frac{1}{2}$ to $\frac{2}{3}$ of Q corresponds to Class IC flow; for $0 < Q < \frac{1}{2}$, the flow is of Class II, with the Lorentz factor finite everywhere.

Note from (23) for V_ϕ that

$$1-XV_\phi = (1-\tfrac{3}{2}QU)/\{1-(1-H)\tfrac{3}{2}Q\}. \tag{29}$$

Since $Q < \frac{2}{3}$ and $H > 0$, the denominator is always positive for Class IC/II flow. Therefore, $1-XV_\phi$ changes from positive to negative on the $B_z = 0$ cones beyond the light cylinder. Equation (23) shows that XV_ϕ increases monotonically away from the star along the poloidal field/flow lines.

So long as the flow is only moderately accelerated, appropriate dimensionless length scales for variation of γV_p and $\gamma V_\phi\, t$, in the sense of forming the quantities $\nabla\times(\gamma V_p)$ and $\nabla\times(\gamma V_\phi\, t)$, are ρ and R. Thus, the ratio of inertial to magnetic terms in the toroidal and poloidal parts of B^* may be estimated by the 'magnetic Rossby numbers' ϵ_M^ϕ and ϵ_M^p, defined as $\epsilon\gamma V_p/(-B_\phi)\rho$ and $\epsilon\gamma V_\phi/B_p R$ (cf. Wright 1978).

4. Light Cylinder and Beyond

In this section, the behaviour of Class IC/II flow will be demonstrated by evaluating quantities on certain surfaces, namely the light cylinder, the $B_z = 0$ cones (beyond $X = 1$) and the equatorial plane (beyond $U = 2$). A discussion of inertial effects will be included at the end of the section.

(a) On the Light Cylinder

On $X = 1$, the function H is $\frac{1}{3}$ and Q is $1/(1+Z^2)$. Equation (14) shows that

$$-B_\phi/B_p = \{(1-\tfrac{9}{8}Q)/(1-\tfrac{3}{4}Q)\}^{\frac{1}{2}} = \{(4Z^2-\tfrac{1}{2})/(4Z^2+1)\}^{\frac{1}{2}} \quad \text{on} \quad X = 1. \tag{30}$$

This ratio varies from 1 at $Z = \infty$ to $\surd\frac{1}{2}$ at $Z^2 = \frac{1}{2}$, corresponding to Q values of 0 and $\frac{2}{3}$ respectively: for Class IC/II outflow, the toroidal magnetic field, which is vanishingly small at the star, is, on the light cylinder, equal to between 71% and 100% of the poloidal field.

Equation (20) states that $(\tilde{B}_z-1)/(1-X^2) = Z^2/(Z^2-\frac{1}{2})$ on $X = 1$. Equation (28) yields

$$\begin{aligned}\gamma^2 &= (8/5Q)(1-\tfrac{9}{8}Q)(1-Q)^2/(1-\tfrac{3}{2}Q)(1-\tfrac{9}{10}Q)\\ &= 4Z^4(8Z^2-1)/(10Z^2+1)(2Z^2-1) \quad \text{on} \quad X = 1,\end{aligned} \tag{31}$$

demonstrating the real and finite nature of γ on $X = 1$ for $Z^2 > \frac{1}{2}$, as required for Class IC/II flow. This equation and (13) for γ_0 give

$$(\gamma/\gamma_0)^2 = 3(1-Q)^2(1-\tfrac{6}{5}Q)/(1-\tfrac{3}{2}Q)(1-\tfrac{9}{10}Q) \quad \text{on} \quad X = 1. \tag{32}$$

Thus, on the light cylinder, the ratio γ/γ_0 varies from $\surd 3$ for $Q \to 0$, through $1\cdot48$ for $Q = \frac{1}{2}$, to ∞ for $Q \to \frac{2}{3}$. The values of γ_0 in these three cases are ∞, $1\cdot08$ and 1 respectively.

It follows from (15) and (30) that

$$\gamma_m^2 = (2-\tfrac{15}{8}Q)/(1-\tfrac{9}{8}Q) = (16Z^2+1)/(8Z^2-1) \quad \text{on} \quad X = 1. \tag{33}$$

Equation (22) gives

$$\rho_e = -4\bar{P}(1-Q) = -4Z^2/(1+Z^2)^{5/2} \quad \text{on} \quad X = 1, \tag{34}$$

while (22′) shows that

$$\rho_e/(-2B_z) = (1-Q)/(1-\tfrac{3}{2}Q) = Z^2/(Z^2-\tfrac{1}{2}) \quad \text{on} \quad X = 1. \tag{34′}$$

Equation (25) for V_p/V_0 shows that

$$V_p/V_0 = (1-\tfrac{3}{4}Q)^{\frac{1}{2}}/(1-Q) = (1+\tfrac{5}{4}Z^{-2}+\tfrac{1}{4}Z^{-4})^{\frac{1}{2}} \quad \text{on} \quad X = 1; \tag{35}$$

i.e. the poloidal flow receives net acceleration between the star and the light cylinder by a factor of less than $3/\surd 2$ or $2\cdot12$.

Equations (24) and (23) for V_p and V_ϕ show that

$$V_p = \{(1-\tfrac{3}{4}Q)/(1-\tfrac{9}{8}Q)\}^{\frac{1}{2}}\{(1-\tfrac{3}{2}Q)/(1-Q)\}$$

$$= \{(4Z^2+1)/2(8Z^2-1)\}^{\frac{1}{2}}(2-Z^{-2}) \quad \text{on} \quad X = 1, \tag{36}$$

$$V_\phi = Q/2(1-Q) = \tfrac{1}{2}Z^{-2} \quad \text{on} \quad X = 1. \tag{37}$$

On the light cylinder, V_p varies from 0 at $Z^2 = \frac{1}{2}$ to 1 as $Z \rightarrow \infty$, while the angular speed of the flow normalized to that of the star varies from 1 to 0.

Equation (31) demonstrates that γ on the light cylinder is large at large Z. Equations (37) and (35) for V_ϕ and V_p/V_0 on $X = 1$ show that this is not because of the toroidal flow speed (which is large only near $Z^2 = \frac{1}{2}$) or the acceleration that the poloidal flow has received. Rather, it is because of V_0: the specified emission speed becomes very large as the poles are approached.

Inserting (32) and (37) for γ/γ_0 and V_ϕ on $X = 1$ into the Endean integral (2), with G equated to γ_0, yields

$$\Phi/\epsilon\gamma_0 = 3^{\frac{1}{2}}\{(1-\tfrac{6}{5}Q)(1-\tfrac{3}{2}Q)/(1-\tfrac{9}{10}Q)\}^{\frac{1}{2}}-1 \quad \text{on} \quad X = 1. \tag{38}$$

This is positive for $0 < Q < 0{\cdot}39$ and negative for $0{\cdot}39 < Q < \frac{2}{3}$. For $Q \ll 1$, equation (38) shows that $\Phi/\epsilon\gamma_0 \approx (3^{\frac{1}{2}}-1)$ on $X = 1$.

On $X = 1$, $R = 1/Q^{\frac{1}{2}}$ while the following hold for $Q \ll 1$: $\gamma \approx (8/5Q)^{\frac{1}{2}}$, $V_p \approx 1$, $V_\phi \approx \frac{1}{2}Q$, $B_p \approx 2Q^{3/2} \approx -B_\phi$ and $\rho \approx 4/3Q$. Hence, on flow lines with $Q \ll 1$, $\epsilon_M^\phi \approx 3\epsilon/(2\times10^{\frac{1}{2}}Q)$ and $\epsilon_M^p \approx \epsilon/(10Q)^{\frac{1}{2}}$ on $X = 1$.

(b) On the $B_z = 0$ Cones

All Class IC/II flow lines cross the $B_z = 0$ cones outside the light cylinder. Equation (10b) for B_z shows that $QU = \frac{2}{3}$ on these surfaces; hence $M = U-1$ there. Equation (14) gives

$$-B_\phi/B_p = 2^{\frac{1}{2}}X(1-3/4U)^{\frac{1}{2}} \quad \text{on} \quad B_z = 0. \tag{39}$$

This ratio increases monotonically from $2^{-\frac{1}{2}}$ to ∞ as X goes from 1 to ∞; from $U = 1{\cdot}14$ outward, corresponding to $Q < 0{\cdot}59$, B_ϕ on these cones is of larger magnitude than B_p. Equation (22) gives

$$\rho_e = -6(2/3U)^{5/2}H(U) \quad \text{on} \quad B_z = 0. \tag{40}$$

Equation (25) shows that

$$V_p/V_0 = 2^{-\frac{1}{2}}(1+U^{-1}+U^{-2}) \quad \text{on} \quad B_z = 0. \tag{41}$$

Comparison of (41) with (35) at corresponding values of Q shows that, for $Q < 0{\cdot}41$, the poloidal flow receives net deceleration between the light cylinder and the $B_z = 0$ cones; for $0{\cdot}41 < Q < \frac{2}{3}$, it is very slightly accelerated, by no more than a few per cent. For $U = 1, \frac{4}{3}$ and ∞, corresponding to $Q = \frac{2}{3}, \frac{1}{2}$ and 0, the ratio (41) is $2{\cdot}12$, $1{\cdot}64$ and $0{\cdot}71$; it has fallen to 1 at $U = 3{\cdot}17$, corresponding to $Q = 0{\cdot}21$; for smaller values of Q than this, there is net deceleration of the poloidal flow between the star and these cones.

Inserting (12b) for V_0 into (41) shows that

$$V_{\mathrm{p}} = (1 - U^{-3})/2^{\frac{1}{2}}(1 - 3/4\,U)^{\frac{1}{2}} \quad \text{on} \quad B_z = 0. \tag{42}$$

This varies from zero for $U = 1$ to $\surd\frac{1}{2}$ as $U \to \infty$; for $U = \frac{4}{3}$, corresponding to $Q = \frac{1}{2}$, equation (42) gives $V_{\mathrm{p}} = 0\cdot62$. Equation (23) gives

$$V_{\phi} = 1/X \quad \text{on} \quad B_z = 0. \tag{43}$$

Equations (42) and (43), together with (13) for γ_0, give

$$\gamma^2 = 2(1 - 3/4\,U)/(1 - U^{-3})(1 + U^{-3} - 3/2\,U) \qquad \text{on} \quad B_z = 0, \tag{44}$$

$$(\gamma/\gamma_0)^2 = (5/2\,U)(1 - 4/5\,U)/(1 - U^{-3})(1 + U^{-3} - 3/2\,U) \quad \text{on} \quad B_z = 0. \tag{45}$$

On these surfaces, the ratio γ/γ_0 varies between ∞ at $U = 1$ and zero (because of the divergence of γ_0 when $Q \to 0$) as $U \to \infty$; the Lorentz factor itself approaches $\surd 2$ as $U \to \infty$.

Substituting (39) for $-B_{\phi}/B_{\mathrm{p}}$ on $B_z = 0$ into (15) and (16) shows that

$$\gamma_{\mathrm{m}}^2 = (2 - 3/2\,U + U^{-3})/(1 - 3/2\,U + U^{-3}) \quad \text{on} \quad B_z = 0, \tag{46}$$

$$F_{\infty}^2 = (1 - 3/2\,U + U^{-3})/2\,U^3(1 - 3/4\,U) \qquad \text{on} \quad B_z = 0. \tag{47}$$

Equation (43) for V_{ϕ} shows that the Endean integral (2), with G equated to γ_0, gives $\Phi/\epsilon\gamma_0 = -1$ on these surfaces; comparison with (38) for Φ on $X = 1$ shows that, for $0 < Q < 0\cdot39$, the non-corotational potential changes sign between the light cylinder and the $B_z = 0$ cones.

On these cones, $V_{\phi} = (\frac{3}{2}Q)^{3/2}$, $B_{\mathrm{p}} = \surd 2(\frac{3}{2})^3 Q^{9/2}$, $\rho = \surd\frac{1}{3}\,Q^{-3/2}$ and $R = \frac{2}{3}Q^{-3/2}$, while the following hold for $Q \ll 1$: $\gamma \approx \surd 2$, $V_{\mathrm{p}} \approx \surd\frac{1}{2}$ and $-B_{\phi} \approx 3\surd\frac{3}{2}\,Q^3$. Hence, on flow lines with $Q \ll 1$, $\epsilon_{\mathrm{M}}^{\phi} \approx \surd 2\,\epsilon/3\,Q^{3/2}$ and $\epsilon_{\mathrm{M}}^{\mathrm{p}} \approx \surd\frac{2}{3}\,\epsilon/Q^{3/2}$ on $B_z = 0$.

(c) On the Equatorial Plane

Class II outflow reaches the equatorial plane beyond $U = 2$. Equation (9a) for $\bar{P}$ shows that $QU = 1$ on this plane. Hence, (14) gives

$$-B_{\phi}/B_{\mathrm{p}} = 2X(1 - 9/8\,U)^{\frac{1}{2}} \quad \text{on} \quad Z = 0; \tag{48}$$

this ratio increases monotonically with X from the value $3\cdot7$ on $U = 2$. Equations (22) and (22′) give

$$\rho_{\mathrm{e}} = -(2/X^3)(2 - Q - Q^2)/(1 + Q + Q^2) \qquad \text{on} \quad Z = 0, \tag{49}$$

$$\rho_{\mathrm{e}}/(-2B_z) = -(2 - Q - Q^2)/(1 + Q + Q^2) \qquad \text{on} \quad Z = 0. \tag{49'}$$

Equation (25) shows that

$$V_{\mathrm{p}}/V_0 = (1 + Q + Q^2)/(2 - Q - Q^2) \qquad \text{on} \quad Z = 0. \tag{50}$$

Thus, on the equatorial plane, V_p/V_0 is $\frac{7}{5}$ at $U = 2$, corresponding to $Q = \frac{1}{2}$, and tends to $\frac{1}{2}$ as $U \to \infty$, corresponding to $Q \to 0$. Equations (35), (41) and (50) for V_p/V_0 on $X = 1$, $B_z = 0$ and $Z = 0$ show that the poloidal flow suffers net deceleration both between the light cylinder and the equatorial plane and between the $B_z = 0$ cones and the equatorial plane.

Substituting (12b) for V_0 into (50) gives

$$V_p = \{(1-\tfrac{3}{2}Q)/(1-\tfrac{9}{8}Q)^{\frac{1}{2}}\}(1+Q+Q^2)/(2-Q-Q^2) \quad \text{on} \quad Z = 0. \qquad (51)$$

Equation (23) shows that

$$V_\phi = (3/X)(2-Q-Q^2)^{-1} \quad \text{on} \quad Z = 0. \qquad (52)$$

Equations (51) and (52) show that, on the equatorial plane, V_p varies from 0·53 at $U = 2$ to $\frac{1}{2}$ as $U \to \infty$, while V_ϕ varies from 0·85 to 0; the quantity V_ϕ/X, the angular speed of the flow normalized to that of the star, varies from $\frac{3}{10}$ to 0.

Equations (51) and (52) for V_p and V_ϕ on $Z = 0$ give

$$\gamma^2 = (1-\tfrac{9}{8}Q)(2-Q-Q^2)^2/(1+Q+Q^2)(1-2Q)D \quad \text{on} \quad Z = 0, \qquad (53)$$

where $D \equiv 3-\frac{9}{2}Q+\frac{3}{4}Q^2+\frac{9}{8}Q^3$. Substituting (13) for γ_0 into (53) gives

$$(\gamma/\gamma_0)^2 = \tfrac{15}{8}Q(1-\tfrac{6}{5}Q)(2-Q-Q^2)^2/(1+Q+Q^2)(1-2Q)D \quad \text{on} \quad Z = 0. \qquad (54)$$

The Lorentz factor on $Z = 0$ varies between ∞ on $U = 2$ and $2/\sqrt{3}$ or 1·15 as $U \to \infty$; the former case, corresponding to $Q = \frac{1}{2}$, belongs to Class IC flow. The ratio γ/γ_0 varies from ∞ on $U = 2$ to zero (because of the divergence of γ_0 when $Q \to 0$) as $U \to \infty$.

Substituting (48) for $-B_\phi/B_p$ on $Z = 0$ into (15) and (16) shows that

$$\gamma_m^2 = (4-\tfrac{9}{2}Q-Q^3)/(3-\tfrac{9}{2}Q+Q^3) \quad \text{on} \quad Z = 0, \qquad (55)$$

$$F_\infty^2 = \tfrac{1}{4}Q^3(3-\tfrac{9}{2}Q+Q^3)/(1-\tfrac{9}{8}Q) \quad \text{on} \quad Z = 0. \qquad (56)$$

Equation (52) for V_ϕ on $Z = 0$ shows that $1-XV_\phi < 0$ there; hence, the Endean integral (2), with G replaced by γ_0, implies that $\Phi/\epsilon < 0$; more precisely,

$$-\Phi/\epsilon\gamma_0 = \{\tfrac{15}{8}Q(1-\tfrac{6}{5}Q)(1+Q+Q^2)/(1-2Q)D\}^{\frac{1}{2}}+1 \quad \text{on} \quad Z = 0. \qquad (57)$$

On the equatorial plane, $B_p = Q^{9/2}$, $\rho = \frac{1}{3}Q^{-3/2}$ and $R = Q^{-3/2}$, while the following hold for $Q \ll 1$: $\gamma \approx \frac{4}{3}$, $V_p \approx \frac{1}{2}$, $V_\phi \approx \frac{3}{2}Q^{3/2}$ and $-B_\phi \approx 2Q^3$. Hence, on flow lines with $Q \ll 1$, $\epsilon_M^\phi \approx \epsilon/Q^{3/2}$ and $\epsilon_M^p \approx 2\epsilon/Q^{3/2}$ on $Z = 0$.

(d) Limitations

Examination of the magnetic Rossby numbers calculated above indicates that the inertial and magnetic terms in both the toroidal and poloidal parts of $\boldsymbol{B}^*$ are of the same order, on the $B_z = 0$ cones and the equatorial plane, for poloidal flow lines with $Q = O(\epsilon^{2/3})$. It is interesting to note that, on these lines, the calculated Φ is still very small: it is positive of order $\epsilon^{2/3}$ on the light cylinder and negative of the same order on the $B_z = 0$ cones and the $Z = 0$ plane.

These results suggest that the present theory of Class IC/II flow fails, because of inertial drift generated in the vicinity of the $B_z = 0$ cones, on poloidal field lines having $Q \lesssim \epsilon^{2/3}$ and hence $\gamma_0 \gtrsim \epsilon^{-1/3}$. Since ϵ is roughly in the range from 10^{-6} to 10^{-11} for different pulsars, the corresponding limiting γ_0 is about 10^2–10^4.

This does not imply failure of the MRW^2 treatment of GJ flow: it merely reflects an inconsistency, expressed by the divergence of γ_0 on the axis, between the assumptions of IC/II flow and constancy everywhere of the Endean integral. The conclusion is this: Either outflow from the innermost core of a polar cap is of Class IA/IB, or flow-line dependence of the Endean integral must be incorporated.

5. Concluding Remarks

Flows which cross the light cylinder without encountering the MRW^2 limiting surface are emitted from an inner (Class IC/II) polar cap region, having about $\frac{3}{4}$ of the radius of the standard GJ polar cap. The requirement that their Lorentz factor be finite at the light cylinder, together with the assumption of a dipole form for the poloidal magnetic field, leads to a well defined mathematical description of these flows (MRW^2). Since the Lorentz factor corresponding to the emission speed diverges at the poles, there is a tiny inner core of each polar cap in which the theoretical treatment used here fails. This divergence needs to be removed, either by taking the flow from that core to be of Class IA/IB, or by relaxing the constraint of constancy of the Endean integral throughout the flow: in order to describe Class II flow emanating from the innermost core of a polar cap, the flow-line dependence of this integral will need to be incorporated.

Of the flows studied here, the higher latitude (Class II) flows, emanating from the inner 80% by radius of the IC/II polar cap, do not encounter a limiting surface. The Class IC flows, emanating from the remaining ring of the IC/II polar cap, reach a limiting surface beyond the light cylinder; it extends from where $B_z = 0$ on $X = 1$ out to the equatorial plane at $X = 2^{3/2}$ (MRW^2).

The boundary of the IC/II polar cap is defined by the vanishing of the Class IC/II emission speed. Perhaps emission at infinitesimal speed occurs in the outer ring of the GJ polar cap, external to the IC/II cap. Such flows are subject to rapid acceleration in the vicinity of the $B_z = 0$ surfaces (Jackson 1978; Mestel *et al.* 1979; Burman 1981), which they meet inside the light cylinder. The equation $\tilde{B}_z = F_\infty$ for the MRW^2 limiting surface reduces to $B_z = 0$ if the emission speed is neglected.

These considerations suggest an MRW^2 model in which the limiting surface consists of the northern and southern $B_z = 0$ cones inside the light cylinder, connected by a surface extending from where they meet the light cylinder out to the equatorial plane at about $2^{3/2}$ times the light cylinder radius from the star.

Only a global solution will show whether or not IC and II flows are realized in actual magnetospheres. In any case, their well defined analytical description makes these flows an attractive subject for study and a potentially useful basis for future comparison with other flows.

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