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A Bianchi Type V Universe with Stiff Fluid and Electromagnetic Radiation

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Abstract

This paper discusses the possibility of a Bianchi type V universe containing stiff matter and a source-free electromagnetic field which is, of necessity, found to be null. Physical and kinematical consequences of the model have also been considered.

1. Introduction

In a previous paper (Roy and Singh 1983) we examined the possibility of a source-free electromagnetic field coexisting with matter in the form of a viscous fluid and streaming neutrinos in the case of a locally rotationally symmetric (LRS) Bianchi type V universe. Being the natural generalization of FRW models with negative curvature, these open models are favoured by the available evidence for low density universes (Gott *et al.* 1974). The Bianchi type V universes have been considered by a number of workers including Schucking and Heckmann (1958), Ellis (1967), Hawking (1969) and Grishchuk *et al.* (1969). The LRS Bianchi type V space–time models containing stiff matter and an electromagnetic field were first considered by Ftaclas and Cohen (1978). Lorenz (1981) studied LRS Bianchi type tilted models with stiff fluid and an electromagnetic field. The relevance of the stiff equation of state, $p = \epsilon$, to the matter content of the universe in its early stages has been discussed by Barrow (1978).

In the present paper we consider a general (non-LRS) orthogonal Bianchi type V space–time model containing stiff matter and a source-free electromagnetic field. The electromagnetic field is, of necessity, found to be null. We also discuss the physical and kinematical features of the model.

2. Field Equations

The line element describing the Bianchi type V space–time model is taken in the form

$$\mathrm{d}s^2 = A^2(\mathrm{d}x^2 - \mathrm{d}t^2) + B^2\,\mathrm{e}^{2x}\,\mathrm{d}y^2 + C^2\,\mathrm{e}^{2x}\,\mathrm{d}z^2\,, \tag{1}$$

where the metric potentials A, B and C are functions of t alone. The matter content

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with a source-free electromagnetic field is given by the energy–momentum tensor

$$T^j_i = (\epsilon + p) v_i v^j + p g^j_i + E^j_i, \tag{2}$$

where

$$E^j_i = F_{ir} F^{jr} - \tfrac{1}{4} F_{ab} F^{ab} g^j_i. \tag{3}$$

In (2) and (3) p is the isotropic pressure, ϵ the matter density, E^j_i the electromagnetic energy tensor, F_{ij} the electromagnetic field tensor and v^i the flow vector of the fluid. The flow of matter is taken orthogonal to the hyper-surfaces of homogeneity so that $v^1 = v^2 = v^3 = 0$ and $v^4 = A^{-1}$. The field equations are

$$-8\pi G T^j_i = R^j_i - \tfrac{1}{2} R g^j_i + \Lambda g^j_i, \tag{4}$$

and also we have

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0, \tag{5}$$

$$J^i = F^{ij}_{;j} = \bar{\sigma}^i_k F^k_j v^j, \tag{6}$$

where $\bar{\sigma}^i_k$ is the anisotropic conductivity of the fluid, assumed to take the form

$$\bar{\sigma}^i_j = \text{diag.}(\bar{\sigma}^1_1, \bar{\sigma}^2_2, \bar{\sigma}^3_3, \bar{\sigma}^4_4). \tag{7}$$

Equations (4) and (5) show that in this space–time the only non-vanishing components of F_{ij} are F_{12}, F_{13}, F_{24} and F_{34}, satisfying the relations

$$F_{12} = e F_{24}, \qquad F_{13} = e F_{34}; \qquad e = \pm 1. \tag{8}$$

The electromagnetic field is, therefore, null. We choose $e = 1$ in (8) so that it represents an outgoing wave. From (5)–(8) we get

$$J^1 = J^4 = 0, \tag{9}$$

$$\bar{\sigma}^2_2 = -\bar{\sigma}^3_3 = -(C/B)_{,4} B/AC. \tag{10}$$

It can be seen that when J^i is equal to zero, the space–time has to be locally rotationally symmetric. Equations (5) with (9) reduce to the following:

$$F_{24,3} - F_{34,2} = 0, \qquad F_{24,1} + F_{24,4} = 0, \tag{11, 12}$$

$$F_{34,1} + F_{34,4} = 0, \qquad C^2 F_{24,2} + B^2 F_{34,3} = 0. \tag{13, 14}$$

In the above equations, the suffixes 1, 2, 3 and 4 preceded by a comma stand for partial derivatives with respect to x, y, z and t respectively. Equations (4), (11) and (14) show that the components of F_{ij} are functions of x and t. The field equations

(4), along with (5)–(14), give rise to

$$8\pi G(p+\rho)A^2 = 1 - \frac{B_{,44}}{B} - \frac{C_{,44}}{C} - \frac{B_{,4}C_{,4}}{BC} + \frac{A_{,4}}{A}\left(\frac{B_{,4}}{B} + \frac{C_{,4}}{C}\right) - \Lambda A^2, \tag{15}$$

$$8\pi GpA^2 = 1 - \left(\frac{A_{,4}}{A}\right)_{,4} - \frac{C_{,44}}{C} - \Lambda A^2, \tag{16}$$

$$8\pi GpA^2 = 1 - \left(\frac{A_{,4}}{A}\right)_{,4} - \frac{B_{,44}}{B} - \Lambda A^2, \tag{17}$$

$$8\pi G(\epsilon+\rho)A^2 = -3 + \frac{A_{,4}}{A}\left(\frac{B_{,4}}{B} + \frac{C_{,4}}{C}\right) + \frac{B_{,4}C_{,4}}{BC} + \Lambda A^2, \tag{18}$$

$$8\pi G\rho A^2 = \frac{B_{,4}}{B} + \frac{C_{,4}}{C} - 2\frac{A_{,4}}{A}, \tag{19}$$

where ρ is the energy density of electromagnetic radiation given by

$$\rho = \frac{C^2F_{24}^2 + B^2F_{34}^2}{A^2B^2C^2\,e^{2x}}. \tag{20}$$

Equations (15), (16) and (19) yield

$$\frac{2A_{,4}}{A} + \frac{B_{,4}}{B} + \frac{C_{,4}}{C} + \frac{\rho_{,4}}{\rho} + 2 = 0. \tag{21}$$

The conservation equation for the energy–momentum tensor

$$T^j_{i;j} = 0$$

leads to

$$d\epsilon/d\tau + (\epsilon+p)\theta = 0, \tag{22}$$

where τ is the cosmic time given by $\int A\,dt$ and θ is the expansion scalar. Equation (22) shows that the matter density of the universe is decreasing with time during its expansion stage provided the weak energy condition $\epsilon+p > 0$ holds. If p satisfies the barotropic equation of state $p = (\gamma - 1)\epsilon$, where $1 \leqslant \gamma \leqslant 2$, then equation (22) yields

$$\epsilon = \epsilon_0\, R_0^{3\gamma}/R^{3\gamma},$$

where $R^3 = ABC$ and ϵ_0 and R_0 are the present day values of ϵ and R respectively. Equations (15)–(17) and (19) give

$$\sigma_2^2 - \sigma_3^3 = \alpha/R^3, \qquad \sigma_1^1 = \beta\, e^{-2t}/R^3, \tag{23, 24}$$

so that

$$\sigma^2 = \tfrac{1}{4} R^{-6}(3\beta^2 e^{-2t} + \alpha^2), \tag{25}$$

where σ_i^j is the shear tensor, σ its magnitude and α and β are constants. From equations (18) and (23)–(25), we get

$$\theta^2 = 24\pi G(\epsilon + \rho) + 3\sigma^2 + 9/A^2 - 3\Lambda. \tag{26}$$

Equation (26), together with (21)–(25), gives

$$\mathrm{d}\theta/\mathrm{d}\tau = -12\pi G(\epsilon + p) - 16\pi G\rho - 3\sigma^2 - 3/A^2, \tag{27}$$

which is a consequence of the Raychaudhuri equation

$$\mathrm{d}\theta/\mathrm{d}\tau = -4\pi G(\epsilon + 3p + 2\rho) - 2\sigma^2 - \tfrac{1}{3}\theta^2 - \Lambda. \tag{28}$$

From equation (26), we see that when $\Lambda \leqslant 0$

$$0 < \frac{3\sigma^2}{\theta^2} < 1, \qquad 0 < \frac{24\pi G\epsilon}{\theta^2} < 1,$$

assuming $\epsilon > 0$. We also see that the presence of an electromagnetic field lowers the upper limit of anisotropy, as compared with those in the perfect fluid case. Equation (27) shows that the scalar of the expansion θ is a decreasing function of time, provided the weak energy condition $\epsilon + p > 0$ holds for the matter distribution, whereas the Raychaudhuri equation (28) gives the same information when the strong energy condition holds (Hawking and Ellis 1973). Therefore we conclude that the model starts from the big-bang singularity and expands until $\tau = \infty$ for $\Lambda > 0$ while, for $\Lambda > 0$, expansion ceases at $\tau = \tau_0$ and the model may go into the contraction phase to attain the second singularity.

The null electromagnetic field has geodetic and shearing rays defined by the null vector l_i in the axial direction having components $(1, 0, 0, -1)$. The expansion $\hat{\theta}$ and shear $\hat{\sigma}$ for the null ray are given by

$$\hat{\theta} = l^i_{;i} = A^{\frac{1}{2}}\left(2 + \frac{B_{,4}}{B} + \frac{C_{,4}}{C}\right),$$

$$\hat{\sigma} = \frac{1}{2A^2}\left(\frac{B_{,4}}{B} - \frac{C_{,4}}{C}\right).$$

The Raychaudhuri equation for the null geodesic is given by

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}u} = -\frac{8\pi G(\epsilon + p)}{A^2} - 2\hat{\sigma}^2 - \tfrac{1}{2}\hat{\theta}^2,$$

which shows that the null rays converge at the big-bang singularity, assuming the weak energy condition to hold for the matter distribution, where u is an affine parameter along the null geodesic.

If we take the matter content of the universe to be a stiff fluid, we get the metric (1) in the following form due to (15)–(19):

$$\begin{aligned} \mathrm{d}s^2 = {} & T^{-2(2k+1)}(T^4-\tfrac{1}{4}M)^{k+1}(\mathrm{d}X^2 - \mathrm{d}T^2/T^2) \\ & + \mathrm{e}^{2X}\, T^{-2}(T^4 - \tfrac{1}{4}M)\left\{\left(\frac{T^2-\frac{1}{2}M^{\frac{1}{2}}}{T^2+\frac{1}{2}M^{\frac{1}{2}}}\right)^{L} \mathrm{d}Y^2 + \left(\frac{T^2-\frac{1}{2}M^{\frac{1}{2}}}{T^2+\frac{1}{2}M^{\frac{1}{2}}}\right)^{-L}\right\}\mathrm{d}Z^2, \quad (29) \end{aligned}$$

where k, L and M are constants and $M > 0$. The distribution of matter, electromagnetic energy density and kinematical parameters for the model (29) are given as follows:

$$8\pi Gp = 8\pi G\epsilon = \frac{M(2k+3-L^2)\,T^2(2k+3)}{(T^4-\frac{1}{4}M)^{k+3}},$$

$$8\pi G\rho = -\frac{Mk}{(T^4-\frac{1}{4}M)^{k+2}},$$

$$\theta = \frac{(12T^4+2Mk+3M)\,T^{2k+1}}{4(T^4-\frac{1}{4}M)^{\frac{1}{2}(k+3)}},$$

$$\sigma = \frac{(3ML^2T^4+\frac{1}{4}M^2k^2)^{\frac{1}{2}}\,T^{2k+1}}{3^{\frac{1}{2}}(T^4-\frac{1}{4}M)^{\frac{1}{2}(k+3)}}.$$

Magnitudes of the electric and magnetic parts E and H of the free gravitational field are given by

$$E^2 = \frac{T^{4(2k+1)}}{3(T^4-\frac{1}{4}M)^{2(k+3)}}\{4(k+L^2)^2M^2T^8+3ML^2T^4(2T^4+\tfrac{1}{2}M+Mk)^2\},$$

$$H^2 = \frac{ML^2T^{8(k+1)}}{2(T^4-\frac{1}{4}M)^{2(k+2)}}.$$

The expansion $\hat{\theta}$ and shear $\hat{\sigma}$ for the null ray are given by

$$\hat{\theta} = \frac{4T^{2(2k+3)}}{(T^4-\frac{1}{4}M)^{k+2}}, \qquad \hat{\sigma} = \frac{M^{\frac{1}{2}}LT^{4(k+1)}}{(T^4-\frac{1}{4}M)^{k+2}}.$$

3. Discussion and Conclusions

In the model (29) we see that $\rho > 0$ implies $k < 0$ and the reality condition $\epsilon > 0$ requires $2k+3 > L^2$. We also find that $T^4 > \frac{1}{4}M$ in the model. The model starts expanding with a big bang at $T = 2^{-\frac{1}{2}}M^{\frac{1}{4}}$ and continues to expand until $T = \infty$. The singularity at $T = 2^{-\frac{1}{2}}M^{\frac{1}{4}}$ corresponds to proper time $\tau = 0$. The singularity in the model is point type when $k+1 > 0$ and $|L| < 1$, barrel type when $k+1 > 0$ and $L = \pm 1$ or $k = -1$ and $|L| < 1$, cigar type when $k+1 > 0$ and $|L| > 1$ or $k+1 < 0$ and $|L| < 1$, and pancake type when $k = -1$ and $L = \pm 1$. The occurrence of a pancake type singularity in the model is the result of the presence of electromagnetic radiation. We find that when $2k = L^2 - 3$, the metric (29) represents the universe containing only electromagnetic radiation. In this case, when $k = -1$ and $L = \pm 1$, the singularity in the model is of pancake type and the space–time is of Petrov type N. In general, however, the metric is of Petrov type I.

We also see that matter always dominates the electromagnetic field when $3(k+1) \geqslant L^2$, while the electromagnetic field dominates for the time period $T^4 > Mk/4(3k+3 - L^2)$ when $3(k+1) < L^2$. It can be seen that $\sigma/\theta, \epsilon/\theta^2$ and E/θ^2 are finite at the initial singularity which decrease and tend to zero as $T \to \infty$, while H/θ^2 is zero at both limits. The model, therefore, evolves to an isotropic universe. Moreover, we find that E/ϵ tends to a finite constant at the initial singularity while this ratio becomes infinite at late times, which shows that the electric part of the free gravitational field is comparable with the matter field at the big-bang singularity, while the electric part becomes dominant at late times (cf. MacCallum 1971). It is also to be noted that $H/\epsilon \to 0$ as $T^4 \to \frac{1}{4}M$, whereas $H/\epsilon \to \infty$ as $T \to \infty$. Therefore, the matter field is dominant over the magnetic part of the free gravitational field at the initial singularity, while the magnetic part dominates at late times. We also find that at the initial singularity, the electric part is dominant over the magnetic part of the free gravitational field, whereas they are comparable at late times. In the model there exists a particle but not an event horizon along the axial direction. We also see that the magnitude J of the current 4-vector J^i, given by

$$J = \frac{2(-k)^{\frac{1}{2}} M L\, T^{4(k+1)}}{(T^4 - \frac{1}{4}M)^{\frac{1}{2}(2k+5)}},$$

is infinite at the big-bang singularity, whereas it tends to zero as $T \to \infty$. Nonzero components of the conductivity tensor $\bar{\sigma}_i^j$ of the fluid are given by

$$\bar{\sigma}_2^2 = -\bar{\sigma}_3^3 = -\frac{2\, M^{\frac{1}{2}} L\, T^{2k+3}}{(T^4 - \frac{1}{4}M)^{\frac{1}{2}(k+3)}},$$

which are infinitely large at the initial singularity, while they gradually decrease and tend to zero at late times.

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