Aust. J. Phys., 1986, **39**, 189–201

Dependence of the Universal Constants on the Nature of the Maximum of the Mapping Function

R. Delbourgo,[A] *W. Hart*[A] *and B. G. Kenny*[B]

[A] Department of Physics, University of Tasmania,
Hobart, Tas. 7001.
[B] Department of Physics, University of Western Australia,
Nedlands, W.A. 6009.

Abstract

Empirical and analytical evidence is presented for the relation $\delta_N = \{(z-1)/(z-\frac{1}{2})\}\alpha_N^z$ between the universal constants describing the approach to the accumulation points, for cycles undergoing multiplication by period N, in one-dimensional maps possessing an order z maximum.

1. Introduction

One dimensional maps $x \to f(x)$ characterized by a quadratic maximum have been exhaustively studied. It is very apparent that metric universality applies to all cycles undergoing multiplication by a factor N (of a particular structure) and that the universal constants δ and α describing the approach to the limit points satisfy the relation

$$\delta_N \approx c\alpha_N^2 \tag{1}$$

for asymptotic N, where $c = \frac{2}{3}$ for 'rightmost' cycles (Eckmann *et al.* 1984). The startling thing is that the relation $3\delta = 2\alpha^2$ is quite well obeyed even when N is as low as 2 (i.e. ordinary bifurcations).

The existence of a universality relation is rather significant and it can only be established by looking at a sequence of α and δ values, not from an isolated case such as bifurcation for parabolic maxima. In this paper, therefore, as well as finding the dependence on N, we shall exhibit the dependence of the constants on the character of the f maximum, namely its order z, $z = 2$ being the usual quadratic case. In a recent paper, Hu and Satija (1983) did determine δ and α as a function of z for $N = 2, 3$ (duplication, triplication), but our aim here is to extend their work to all N and thereby discover a connection between the constants analogous to (1). In many ways our approach mimics the $1/N$ method for quantum mechanics or field theory, insofar as we will be concerned with large N before extrapolating down to more physical values. We will also take the opportunity to fill in some gaps in a brief earlier report (Delbourgo *et al.* 1985) concerning the case $z = 2$.

Section 2 contains all the empirical evidence for various limiting relations between constants and the associated universal functions for them. The most important facts are stated in equations (13), (13′), (15) and (16) and the numerical evidence for

0004-9506/86/020189$02.00

Table 1. Universal constants and accumulation points for particular cycle structures N and various order maxima z

Blank entries are undetermined values

z	δ	α	μ	δ	α	μ	δ	α	μ
		$N = 2$			$N = 3$			$N = 4b$	
1·001	2·23	226	1·0044316?						
1·01	2·39	37·4	1·02674403						
1·05	2·63	12·3	1·08164904						
1·1	2·83	7·97	1·12549888						
1·2	3·14	5·37	1·18603459	514·5	635	1·65726747	246000	98029	1·87344327
1·5	3·80	3·39	1·29553310	73·1	30·1	1·71354071	2719	323	1·90814094
2	4·67	2·50	1·40115519	55·3	9·28	1·78644026	982	38·8	1·94270435
3	6·08	1·93	1·52187879	67·0	4·37	1·86786595	967	10·6	1·97345649
4	7·29	1·69	1·62227973	86·4	3·16	1·90933547	1276	6·19	1·98550466
5	8·35	1·56	1·66620911	109	2·61	1·93348010	1705	4·52	1·99117621
7	10·2	1·41	1·71270374	154	2·08	1·95932176	2840	3·14	1·99597646
10	12·5	1·29	1·78091188	230	1·73	1·97650061	5164	2·36	1·99831700
11	13·2	1·27	1·79407522	268	1·67	1·97978732	6095	2·22	1·99867422
		$N = 5a$			$N = 5b$			$N = 5c$	
1·2	37100	22400	1·54330726	1·3E6	4·3E5	1·76372265			
1·5	644	129	1·58107396	4984	514	1·81014610	86910	3175	1·97039171
2	256	20·1	1·63192665	1287	45·8	1·86222402	16930	160	1·98553953
3	240	6·72	1·70020473	1106	11·3	1·91829803	14860	26·5	1·99525002
4	292	4·30	1·74335101	1418	6·40	1·94585858	20990	12·5	1·99797402
5	358	3·32	1·77345588	1875	4·63	1·96137177	30520	8·06	1·99898472
7	510	2·46	1·81347920	3080	3·19	1·97733673	59310	4·85	1·99965661
10							1·2E5	3·25	1·99989613
11	840	1·85	1·85797269	6350	2·23	1·98929832	1·5E5	3	1·99992506
		$N = 6a$			$N = 6c$			$N = 6e$	
1·2	3273	4796	1·33497686	1·6E7	2·2E7	1·82897464			
1·5	284	96·8	1·40293619	6·8E4	2788	1·86757002	2·52E6	29640	1·99028865
2	218	20·9	1·48318183	8508	115	1·90750419	2·79E5	648	1·99638325
3	315	7·68	1·58227296	5359	18·9	1·94699527	2·28E5	65·8	1·98628835
4	465	4·94	1·64365927	6491	9·34	1·96529600	3·55E5	25·4	1·99973278
5	650	3·79	1·68654651	8589	6·27	1·97531386	5·76E5	14·5	1·99989060
7	1·1E3	2·77	1·74380080	14840	3·99	1·98544978	1·34E6	7·58	1·99997274
10							3·5E6	4·54	1·99999405
11	2210	2·03	1·80759071	3·5E4	2·6	1·99299488			
		$N = 7a$			$N = 7i$			$N = 11a$	
1·2	2·4E6	7·2E5	1·49386040						
1·5	6400	614	1·52913670	7·2E7	2·7E5	1·99678524	9·0E5	1·7E4	1·49682337
2	1446	49·2	1·57598280	4·5E6	2603	1·99909612	70060	352	1·54792583
3	1145	11·5	1·64218040	3·4E6	162	1·99986264	40020	38·1	1·62000317
4	1413	6·44	1·68718538	6·0E6	51·4	1·99996570	50170	15·8	1·66909911
5	1819	4·63	1·72025512	1·1E7	26·2	1·99998860	69380	9·6	1·70514932
7	2875	3·17	1·76640451	3·2E7	11·9	1·99999792			
10				1·1E8	6·38	1·99999967			
11	5637	2·21	1·82047086						

them is quite solid. Table 1 should prove of great help for those readers wishing to reproduce our results, because the accumulation points are very difficult to locate without some finesse or previous experience. In Section 3 we will show how to derive the relations analytically in terms of the renormalization group equations, with important hints taken from Section 2 on the coefficients of the universal function expansions. In a future paper, we will show how to derive α and δ for these high order cycles directly from the renormalization group equations, without painstaking analysis of individual N^k sequences.

2. Numerical Facts

We shall confine all our remarks to the mapping

$$x_{n+1} = f(x_n) = 1-\mu|x|^z, \tag{2}$$

where the 'order of the mapping' z is defined (at $x = 0$) and assumed to be greater than 1. It is clear that any other endomorphism with the same order maximum possesses the same universal properties that we shall present for map (2) although the precise values of the various accumulation points μ will change with the particular way in which the new f is parametrized.

In Table 1 we list the accumulation points μ_{N^∞} of the various cycle sequences for z values in the range $1{\cdot}001$ to 11 and for many of the kneading sequences (cycle

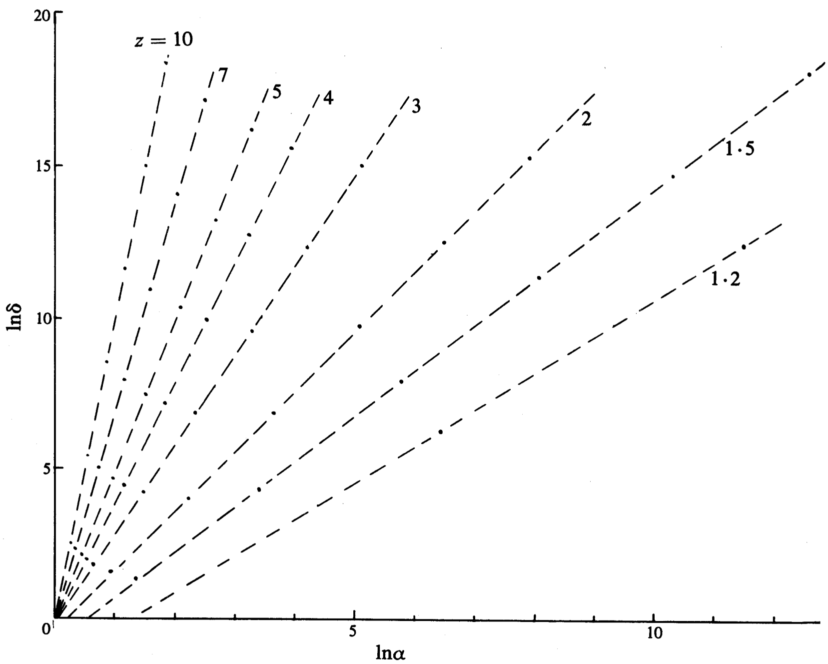

Fig. 1. Plot of $\ln\delta$ against $\ln\alpha$ for the rightmost cycles, for z values ranging from $1{\cdot}2$ to 10. Asymptotic linearity is rapidly achieved and the slope tends to z (and $c \rightarrow 1$) with increasing z.

structures) from $N = 2$ to 11. [Those readers wishing to search for the accumulation points should note that for large N the values may be pinpointed through the formula $\mu_N = 2-(2z)^{-N}e$, where e depends on the cycle structure.] In Table 1 we include the corresponding α and δ, which were arrived at after laborious numerical work. The singularity at $z = 1$ and the extremely high rates of convergence to the accumulation points for large N or z have precluded us from numerically overstepping these ranges, even though our computations have been carried out to 24 significant figures!

When one plots δ against α on a logarithmic scale, as in Fig. 1, it becomes immediately apparent that

$$\delta_N = c(z, N)\,\alpha_N^z \tag{3}$$

agrees very well with the data over the entire range, except down near $N = 2$, which is not surprising as this is hardly an asymptotic number. However, it is quite amazing that $N = 4$ is already in excellent agreement with (3). The constant of proportionality c depends sensitively on z but not on N. In fact for rightmost cycles, as well as other kneading sequences, we find that

$$c = (z-1)/(z-\tfrac{1}{2}) \tag{4}$$

provides an acceptable fit, especially for asymptotic N. Actually the δ–α relation is more than an inspired guess: it is based on an analysis of the renormalization group equations as we shall see.

Universal N-replication functions can be extracted for all z (Delbourgo and Kenny 1985). They arise in the limit

$$g^N(x) = \lim_{n\to\infty} (-\alpha_N)^n [f]^n(\mu_{N\infty}, x/(-\alpha)^n) \tag{5}$$

at the accumulation point of the N sequence. Such functions obey the renormalization group equation

$$g^N(x) = -\alpha_N [g^N]^N(-x/\alpha_N). \tag{6}$$

For the case $z = 2$, Eckmann *et al.* (1984) have argued that

$$g^\infty(x) = \lim_{N\to\infty} g^N(x) = 1-2x^2. \tag{7}$$

It is shown graphically in Fig. 2 that (7) is perfectly correct: for increasing N (and rightmost cycles), successive $g^N(x)$ approach $g^\infty(x)$ more and more closely. In addition to this we see (Fig. 3) that for *fixed* N and over a limited range of x, the several $g^{N_i}(x)$ are reasonably close to $g(x)$, with the goodness of fit improving as one advances through the cycles from left to right. One may reasonably conclude that even for next to rightmost cycles (labelled by a prime) one has

$$\lim_{N\to\infty} g^{N'}(x) = 1-2x^2; \quad z = 2,$$

and so on. This result does not immediately follow from the analysis of Eckmann *et al.* (1984).

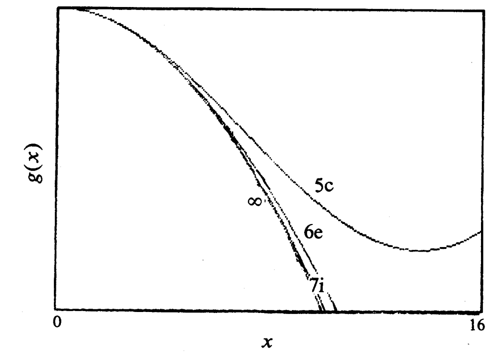

Fig. 2. Universal function $g(x)$ when $z = 2$ for $N =$ 5c, 6e and 7i.

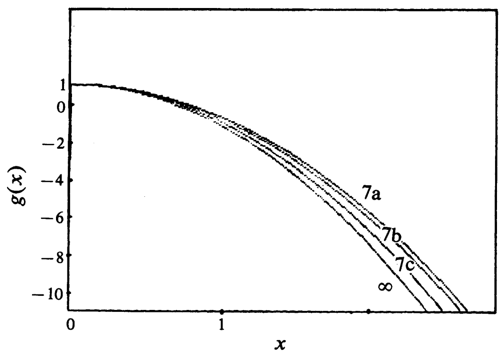

Fig. 3. Universal functions $g(x)$ for cycles $N =$ 7a through to 7c in the (good) aproximation $g^N(x) = -\alpha_N[f]^N(\mu_{N\infty}, -x/\alpha_N)$.

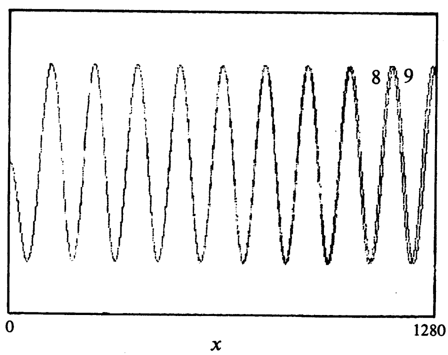

Fig. 4. Comparison of $4g^8(x/2)$ with $g^9(x)$ for rightmost cycles over the range $x = 0$ to 1280, when $z = 2$.

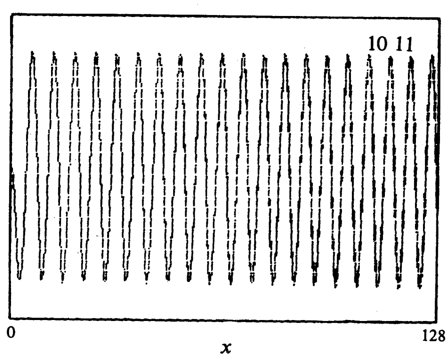

Fig. 5. Comparison of $[f]^{10}(\mu_{10\infty}, x/2^{10})$ with $[f]^{11}(\mu_{11\infty}, x/2^{11})$ for rightmost cycles over the range $x = 0$ to 128, when $z = 2$. A difference is barely discernible, starting at the twentieth oscillation or so.

In systematically exploring these trends, we have observed a more striking result than that contained in (7). By studying successive $g^N(x)$ (for the rightmost cycles)—over an ever-expanding range of x as N increases—it is clear that all $g^N(x)$ deviate from $g^\infty(x)$ for N finite. The striking point is that this deviation occurs in a *very systematic manner*. When x is moderately large and $z = 2$, all $g^N(x) \sim -x^2$ implying that $g^{N+1}(x) \sim 4g^N(\frac{1}{2}x)$. This is totally trivial if g is purely quadratic. However, the fact is that $g(x)$ deviates considerably from a quadratic as we go further out in x; indeed oscillations set in. Nevertheless, it is still true that (see Fig. 4)

$$g^{N+1}(x)-1 \approx 4\{g^N(\tfrac{1}{2}x)-1\}. \tag{8}$$

Now it turns out that for large N a good approximation to $g^N(x)$ over a wide range of x is given by the first term approximation

$$g^N(x) = (-\alpha_N)[f]^N(\mu_{N\infty}, -x/\alpha_N). \tag{9}$$

If we combine (8) and (9) with the result of Eckmann *et al.* (1984) that $\alpha_{N+1}/\alpha_N \approx 4$, then we may reinterpret (8) as

$$[f]^{N+1}(\mu_{N+1\infty}, \tfrac{1}{2}x) = [f]^N(\mu_{N\infty}, x),$$

which in turn lets us conclude that

$$\lim_{N\to\infty} [f]^N(\mu_{N\infty}, x/2^N) \tag{10}$$

converges to a limit function. This is convincingly substantiated in Fig. 5 where we have compared appropriately scaled $[f]^N$ and $[f]^{N+1}$ over an extensive range of x.

Table 2. Relative ratios of successive δ and α values for similar cycle sequences

(Note how closely the relation $(\delta_{N+1}/\delta_N)^{\frac{1}{2}} = \alpha_{N+1}/\alpha_N$ is satisfied, and also the way that the ratios of successive δ approach 16 and the ratios of successive α approach 4

Last cycle	Penultimate cycle	Next to penultimate cycle
$\delta(7i)/\delta(6e) = 16\cdot16$	$\delta(7h)/\delta(6d) = 17\cdot39$	$\delta(7g)/\delta(6b) = 19\cdot65$
$\alpha(7i)/\alpha(6e) = 4\cdot01$	$\alpha(7h)/\alpha(6d) = 4\cdot14$	$\alpha(7g)/\alpha(6b) = 4\cdot38$
$\delta(6e)/\delta(5c) = 16\cdot49$	$\delta(6d)/\delta(5b) = 21\cdot77$	
$\alpha(6e)/\alpha(5c) = 4\cdot05$	$\alpha(6d)/\alpha(5b) = 4\cdot53$	
$\delta(5c)/\delta(4b) = 17\cdot25$		
$\alpha(5c)/\alpha(4b) = 4\cdot12$		

A corresponding exercise for next to last cycles tells a similar story. Again we get

$$g^{N'}(x) \approx 1-2x^2, \qquad g^{N+1'}(x) \approx 4g^{N'}(\tfrac{1}{2}x),$$

and we are similarly led to $\alpha'_{N+1}/\alpha'_N \approx 4$, a new result (supporting numerical evidence is contained in Fig. 6 and Table 2). A consequence of all this is the convergence of

$$\lim_{N\to\infty} [f]^N(\mu_{N'\infty}, x/2^N) \tag{10$'$}$$

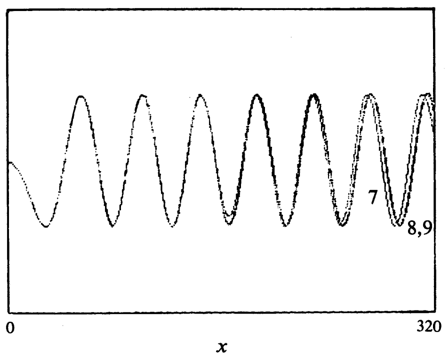

Fig. 6. Comparison of $g^9(x)$, $4g^8(x/2)$ and $16g^7(x/4)$ for next to rightmost cycles, when $z = 2$.

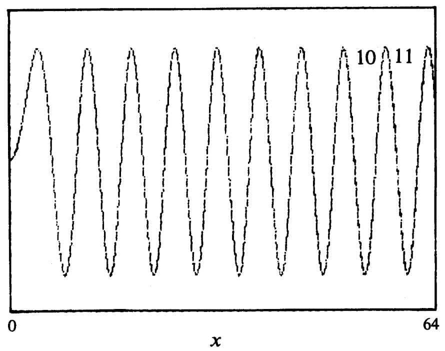

Fig. 7. Comparison between $[f]^{10}(\mu'_{10\infty},\ x/2^{10})$ and $[f]^{11}(\mu'_{11\infty},\ x/2^{11})$ for next to rightmost cycles. The difference between these is imperceptible in the range $x = 0$ to 64.

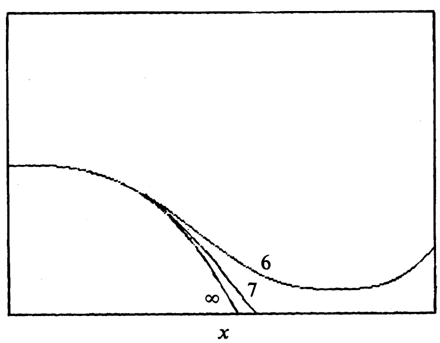

Fig. 8. Plots of g^6, g^7 and $g^\infty = 1-2|x|^3$ when $z = 3$, where the g^N are approximated by the first term of equation (9).

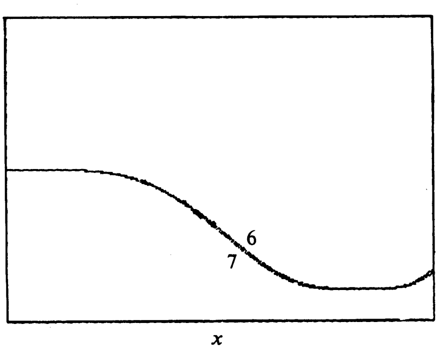

Fig. 9. Plot of $2g^6(x/2^{1/4})$ and $g^7(x)$ when $z = 4$, as a test of equation (13), again in the approximation (9). The difference is barely perceptible.

to a limit function, where $\mu_{N'\infty}$ are the set of accumulation points for next to last cycles. This is confirmed by Fig. 7 and strongly indicates that

$$\alpha = c.4^N \tag{11}$$

for *many kneading sequences*, with c depending on the sequence.

We have also carried out a systematic study for arbitrary z to yield

$$\lim_{N \to \infty} g^N(x) = 1-2|x|^z \tag{12}$$

for a variety of z values (see Fig. 8 for the case $z = 3$). In fact we have been able to establish a more powerful result than this (a generalization of the $z = 2$ case), namely the recurrence property

$$g^{N+1}(x)-1 \approx (2z)^{1/(z-1)}\{g^N(x(2z)^{-1/z(z-1)})-1\} \tag{13}$$

confirmed emphatically in Fig. 9. A direct consequence of the recurrence relation is the asymptotic relation

$$\lim_{N \to \infty} \alpha_{N+1}/\alpha_N = 2z. \tag{14}$$

An even more stringent test of (14) is the evidence for the convergence of

$$\lim_{N \to \infty} [f]^N(\mu_{N\infty}, x(2z)^{-1/z}) \tag{13'}$$

to a limit function for arbitrary positive z. Although our analysis has only been carried out for rightmost cycles, it is very likely that the asymptotic relation $\alpha'_{N+1}/\alpha'_N \approx 2z$ holds good for next to rightmost cycles, etc.

All of these results enable us to predict a generalization of the $z = 2$ result:

$$\alpha(N, z) \approx A(z)(2z)^{(N-1)/(z-1)}, \tag{15}$$

where A does depend on the kneading sequence. This compares very favourably with the data (see Fig. 10). From the relation (3) we may deduce that

$$\delta(N, z) \approx D(z)(2z)^{(N-1)z/(z-1)}. \tag{16}$$

Here too we get direct confirmation in Fig. 11 for rightmost cycles. Indeed when $z = 2$, the numerical data (see Table 2) for low order cycles agree with (16) for the next to rightmost set, etc. Thus, an extension of (16) to arbitrary z and other kneading sequences is more than likely.

In summary we would assert that the main results of Hu and Satija (1983) can be generalized as follows:

$$\delta(N, z) \sim (2z)^{(N-1)z/(z-1)}, \qquad \alpha(N, z) \sim (2z)^{(N-1)/(z-1)},$$

$$g^\infty(x) \sim 1-2|x|^z, \qquad g^{N+1}(x)-1 = (2z)^{1/(z-1)}\{g^N(x(2z)^{1/z(1-z)})-1\}.$$

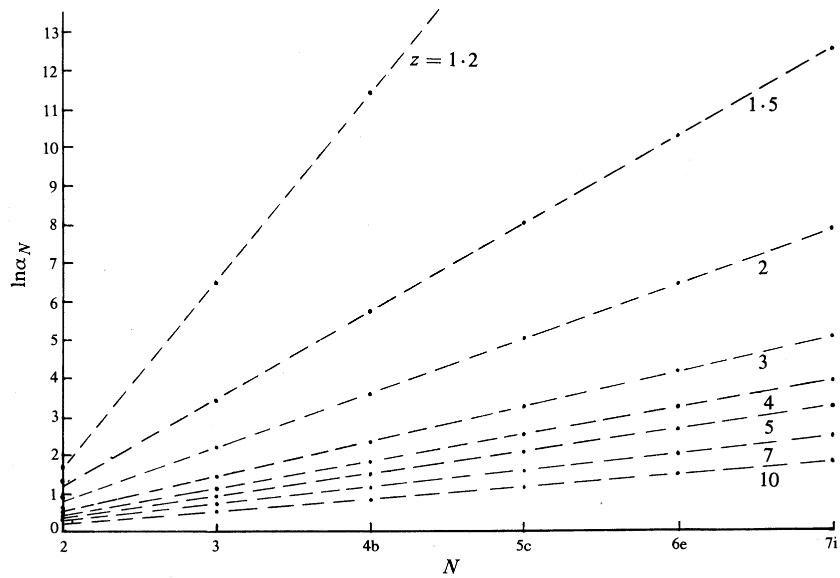

Fig. 10. Plot of $\ln\alpha_N$ versus cycle number N for the rightmost set and the z range $1\cdot2$ to 10.

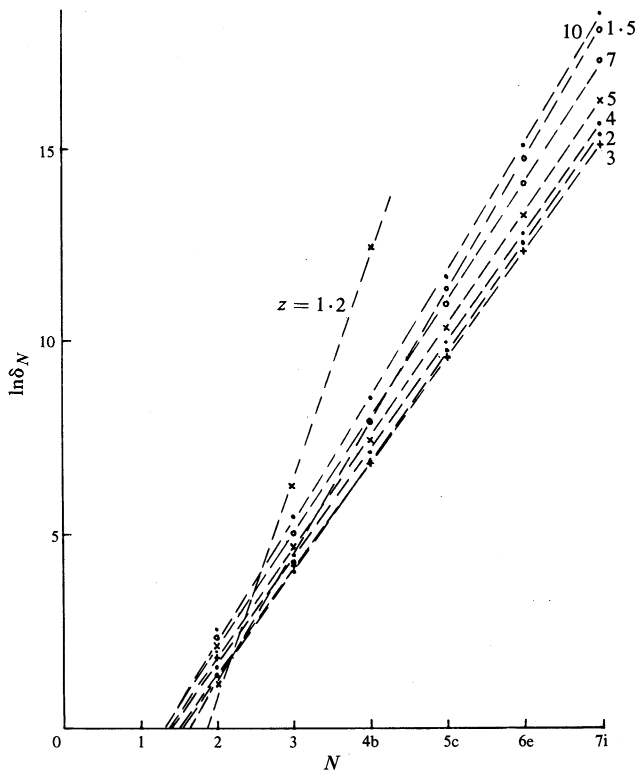

Fig. 11. Plot of $\ln\delta_N$ versus cycle number N for the rightmost set and the z range $1\cdot2$ to 10.

We shall exploit the last of these relations in the next section where we give an analytic derivation of the δ–α connection.

3. Analytical Relations

In Section 2 it was noted that, for $z = 2$,

$$g^{N+1}(x)-1 = 4\{g^N(\tfrac{1}{2}x)-1\} \tag{8}$$

held asymptotically. This means that if we expand $g^N(x)$ in a power series,

$$g^N(x) = \sum_{n=0}^{\infty} g_n^N x^{2n}, \tag{17}$$

starting with $g_0 = 1$, the coefficients obey the recursion relations

$$g_1^{N+1} = g_1^N = -2, \qquad g_2^{N+1} = g_2^N/4, \qquad g_3^{N+1} = g_3^N/16, \quad \ldots, \tag{18}$$

and so on for large N. In other words, we have

$$g_2^N \approx \text{const.}(4)^{1-N}, \qquad g_3^N \approx \text{const.}(16)^{1-N}, \quad \text{etc.} \tag{19}$$

Thus the coefficients of terms of higher order than x^2 *die off very rapidly with N in a systematic fashion.*

For arbitrary z, the power series expansion

$$g^N(x) = \sum_{n=0}^{\infty} g_n^N |x|^{zn}, \tag{20}$$

with $g_0 = 1$, together with the asymptotic result (13), implies that

$$g_1^{N+1} = g_1^N = -2, \qquad g_2^{N+1} = (1/2z)^{1/(z-1)} g_2^N,$$

$$g_3^{N+1} = (1/2z)^{2/(z-1)} g_3^N, \quad \text{etc.} \tag{18$'$}$$

For large N this leads us to

$$g_2^N = \text{const.}(2z)^{(1-N)/(z-1)}, \qquad g_3^N = \text{const.}(2z)^{2(1-N)/(z-1)}, \quad \text{etc.} \tag{19$'$}$$

Here again the coefficients of terms of higher order than $|x|^z$ in the expansion of $g(x)$ must die off very quickly and systematically with N. Making use of equation (15), we conclude that for arbitrary z, the coefficient g_n^N has to die off as

$$g_n^N = \mathrm{O}(1/\alpha_N^{n-1}) \tag{21}$$

for $n > 1$. With these preliminary observations, we are ready to derive an asymptotic analytic form relating α and δ.

Whether or not we are dealing with rightmost cycles, the above trends tell us that to lowest order

$$g(x) = 1-a|x|^z+\epsilon|x|^{2z}, \tag{22}$$

where $a \approx 2$ and ϵ is small (of order $1/\alpha$). [This has been checked numerically over and over again, by direct solution of the functional equation (6), for a variety of N and z and several kneading sequences, and will be reported elsewhere.] The nth iterate of g also may be expanded as

$$[g]^n(x) = -a_n + b_n|x|^z - c_n|x|^z \ldots, \tag{23}$$

where a_n, b_n and c_n are positive for $n > 1$. These coefficients satisfy recurrence relations of the type

$$a_{n+1} = a(a_n)^z - 1, \qquad b_{n+1} = zab_n(a_n)^{z-1}, \tag{24}$$

where we can ignore the term $\epsilon|x|^{2z}$ (and higher order ones) in the expansion of $g(x)$ in the $N \to \infty$ limit where α is large. Now with a universal N-plication function, we know that

$$[g]^N(x) = -a_N + b_N|x|^z + \ldots \tag{25}$$

$$= -(1 - a|\alpha_N x|^z + \ldots)/\alpha_N, \tag{26}$$

upon making use of (6) and (23). Comparison of (25) and (26) shows that

$$a_N = 1/\alpha_N, \qquad b_N = a\alpha_N^{z-1}. \tag{27a, b}$$

Repeated use of the recurrence relation (24) yields

$$b_N = (za)^{N-1} a\left(\prod_{i=1}^{N-1} a_i\right)^{z-1}, \tag{28}$$

from which it follows that (cf. equations 27b and 28)

$$\alpha_N = (za)^{(N-1)/(z-1)} \prod_{i=1}^{N-1} a_i, \tag{29}$$

in agreement with (15). Indeed, we have just extracted the coefficient

$$A(z) = \prod_{i=1}^{N-1} a_i. \tag{30}$$

It is possible to evaluate $A(z)$ for all cycles numerically or analytically (for $z = 2$), in excellent agreement with the 'experimental' data (to be published separately).

With period N-tupling, it is possible to set up the following eigenvalue equation for δ in a similar fashion to that used by Feigenbaum for period doubling:

$$h([g]^N(x)) + h([g]^{N-1}(x))\, g'([g]^N(x)) + h([g]^{N-2}(x))\, [g]^{2\prime}([g]^{N-1}(x)) + \ldots$$

$$+ h(g(x))\, [g]^{N-1\prime}([g]^2(x)) + h(x)\, [g]^{N\prime}(g(x)) = -\delta h(-\alpha x)/\alpha. \tag{31}$$

By expanding $h(x)$ as a power series

$$h(x) = \sum_{n=0}^{m} h_n|x|^{zn}, \tag{32}$$

with $h_0 = 1$, one may obtain a numerical solution of this equation to order $m+1$, in a similar manner to Feigenbaum (this will be exposed elsewhere). As usual, there is a single eigenvalue $\delta > 1$ which is of interest, together with a marginal eigenvalue $\delta = 1$, plus a spectrum of eigenvalues whose absolute values are less than one. The values so obtained for δ are in excellent agreement with those obtained by studying cycle periods undergoing multiplication by N as detailed in Section 2. Indeed, they are far more readily and accurately computed in this fashion, especially for larger values of N! We have found both computationally and analytically that for large N and arbitrary z

$$h_1 = -g_2 = \mathrm{O}(1/\alpha_N). \tag{33}$$

Higher order terms in the expansion of $h(x)$ are comparable with terms we have discarded in the expansion of $g(x)$; for instance, $\mathrm{O}(h_2) = \mathrm{O}(g_3)$ etc.

Since we dropped the term $a_2|x|^{2z} = \mathrm{O}(1/\alpha_N)$ in the expansion of $g(x)$, for our present purposes it is quite appropriate to discard the term $h_1|x|^z = \mathrm{O}(1/\alpha_N)$ in the expansion of $h(x)$. Thus, *to leading order*, we may take $h(x) = 1$, whereupon the eigenvalue equation (31) for δ reads

$$\begin{aligned} &1+g'([g]^N(0)) + [g]^{2\prime}([g]^{N-1}(0)) + \dots \\ &+[g]^{N-1\prime}([g]^2(0)) + [g]^{N\prime}(g(0)) = -\delta/\alpha. \end{aligned} \tag{31$'$}$$

This represents the lowest order approximation to (31) and the solution yields a single eigenvalue δ, the one of physical interest in fact.

The nice thing is that an accurate analytic estimate of the left-hand side of (31′) is now possible. For instance, the last term in the series is

$$\begin{aligned} [g]^{N\prime}(g(0)) &= g'(g(0))\,g'([g]^2(0))\dots g'([g]^N(0)) \\ &= g'(1)\,g'(-a_2)\dots g'(-a_{N-1}) \\ &= -(2z)^N \prod_{i=1}^{N} a_i^{z-1} = -|\alpha|^{z-1} \end{aligned} \tag{34}$$

from equation (29), where we have used $g'(x) = -2z|x|^{z-1}$ for $x > 0$ and $g'(x) = +2z|x|^{z-1}$ for $x < 0$. By treating the second last term in the series of the left-hand side of (31′) in a similar way, and so on, one can readily establish that the sum of the terms has the asymptotic form

$$\alpha^{z-1}(-1+1/2z+1/4z+\dots) = -\alpha^{z-1}(2z-2)/(2z-1). \tag{35}$$

Inserting this into (31′) we finally obtain

$$|\alpha|^z/\delta = (z-\tfrac{1}{2})/(z-1). \tag{36}$$

A more accurate treatment of the above derivation yields a small correction term to the right-hand side, of order $1/\alpha$. However, for large α, the formula (36) is already accurate to considerably better than 1%. It should also be noted that our derivation depends in no way on the particular kneading sequence and will hold for cycles other than the rightmost. Of course for $z = 2$, (36) reduces to $\delta/\alpha^2 = \frac{2}{3}$, as found by Eckmann *et al.* (1984) for the rightmost set and shown numerically to be more generally true by Delbourgo *et al.* (1985). We now have a much better understanding why this should be so and we can foresee generalizations of this approach to other cases, like circle maps or complex or matrix extensions.

References

Delbourgo, R., Hart, W., and Kenny, B. G. (1985). *Phys. Rev.* A **31**, 514.
Delbourgo, R., and Kenny, B. G. (1985). *Aust. J. Phys.* **38**, 1.
Eckmann, J. P., Epstein, H., and Wittwer, P. (1984). *Commun. Math. Phys.* **93**, 495.
Hu, B., and Satija, I. I. (1983). *Phys. Lett.* A **98**, 143.

Manuscript received 2 September, accepted 10 December 1985