Aust. J. Phys., 1986, **39**, 421–32

Relativistic Electron Injection into Axisymmetric Pulsar Magnetospheres

R. R. Burman

Department of Physics, University of Western Australia,
Nedlands, W.A. 6009.

Abstract

A formalism introduced by Mestel, Robertson, Wang and Westfold for the description of electron outflow in axisymmetric pulsar magnetospheres, following injection with non-negligible speeds from the stellar surface, is extended here to incorporate emission with relativistic speeds. The formalism is then used to study the possible kinds of outflow. They are organized into five types, corresponding to differences in the emission speed and its variation with latitude, and into two classes according to whether or not they reach a region of rapid acceleration.

1. Introduction

Mestel, Robertson, Wang and Westfold (1985; henceforth denoted by MRW^2) recently introduced an axisymmetric pulsar magnetosphere model in which electrons leave the star with speeds that are non-negligible, but not very relativistic, and flow with moderate acceleration, and with poloidal motion that is closely tied to poloidal magnetic field lines, before reaching S_l, a limiting surface near which rapid acceleration occurs. The formalism they introduced to describe these flows can be interpreted in terms of a plasma drift across the magnetic field, following injection along it (Burman 1985*a*). I presented an analysis of such moderately accelerated outflows (Burman 1984), showing that there is a second class of flows—ones which do not encounter a region of rapid acceleration.

I shall now extend the basic MRW^2 formalism, and my earlier analysis, so as to incorporate emission from the stellar surface with relativistic speeds. The need for this extension has arisen as the result of study (Burman 1985*b*) of the solutions which represent flows that either encounter S_l beyond the light cylinder (the surface on which the speed of corotation with the star equals c, the vacuum speed of light) or not at all: I found that outflow from tiny inner cores of the polar caps is either not of this kind, or, if it is, then the formalism to be developed here will be needed in order to treat it.

2. MRW^2 Formalism

In this section, the basic formalism developed by MRW^2 for the analysis of their model will be extended so as to incorporate relativistic emission. The system is taken to be axisymmetric and steadily rotating at angular frequency Ω. The dimensionless cylindrical radial coordinate X is unity on the light cylinder, which has radius c/Ω.

0004-9506/86/030421$02.00

The unit toroidal vector is denoted by $\boldsymbol{t}$. It follows from Faraday's law and $\nabla.\boldsymbol{B} = 0$ that the electric field can be written as the sum of a part $X\boldsymbol{B}\times\boldsymbol{t}$, associated with rotation of the magnetic field structure, and a non-corotational part $-\nabla\Phi$, with Φ a gauge-independent potential.

MRW[2] developed their equations in dimensionless form by expressing distances and the flow velocity $\boldsymbol{V}$ in units of c/Ω and c, and normalizing field variables in terms of the equatorial dipolar magnetic field strength at the light cylinder: $B_1 \equiv \frac{1}{2}(\Omega r_s/c)^3 B_0$ where r_s is the stellar radius and B_0 is the polar surface magnetic field strength. The magnetic field and the charge density ρ_e are expressed in units of B_1 and $B_1/4\pi c$.

The poloidal parts of the magnetic field and electric current density are expressed in terms of Stokes stream functions: $\boldsymbol{B}_p = X^{-1}\boldsymbol{t}\times\nabla P$ and $\boldsymbol{j}_p = X^{-1}\boldsymbol{t}\times\nabla S$, with P and S measured in units of $(c/\Omega)^2 B_1$ and $c^2 B_1/4\pi\Omega$ respectively. The electric field is $\nabla(P-\Phi)$. Charge separation is assumed, so $\boldsymbol{j}_p = \rho_e \boldsymbol{V}_p$. The poloidal part of Ampère's law reduces to $B_\phi = -S/X$. It follows from Gauss's law and the toroidal part of Ampère's law that (Mestel *et al.* 1979; eq. 2.8)

$$\nabla^2\Phi + 2B_z = -(1 - XV_\phi)\rho_e, \tag{1}$$

with Φ expressed in units of cB_1/Ω. The subscripts ϕ and z denote toroidal and axial components.

In the domain under consideration, the flow is taken to be dissipation-free: the equation of motion represents balance of the Lorentz force by relativistic inertia. The inertial effects manifest themselves in two ways: through the existence of the non-corotational electric potential Φ and through inertial drift of the flow across magnetic field lines.

The steady rotation constraint implies the existence of an integral of the motion (Endean 1972) which (for electrons) has the dimensionless form (MRW[2])

$$G = \gamma(1 - XV_\phi) - \Phi/\epsilon; \tag{2}$$

the Lorentz factor is denoted by γ and the small parameter ϵ represents Ω/ω_g, with ω_g denoting the nonrelativistic electron gyrofrequency in the fiducial field B_1.

In this domain, inertial drift is neglected, so the poloidal flow is along the poloidal magnetic field lines, meaning that S is a function of P only. The electrons' equation of motion (cf. Burman and Mestel 1978) can now be written as

$$(\boldsymbol{V} - X\boldsymbol{t})\times\boldsymbol{B} = -\epsilon\nabla G. \tag{3}$$

Axisymmetry and neglect of inertial drift imply that G is a function of S, or P, only. Taking the vector product of equation (3) with $\boldsymbol{t}$ and using $\boldsymbol{B}_p = X^{-1}\boldsymbol{t}\times\nabla P$ shows that the flow velocity is related to the magnetic field by

$$\boldsymbol{V} = \kappa\boldsymbol{B} + K(P)X\boldsymbol{t}, \tag{3'}$$

where κ is a scalar and K denotes $1 + \epsilon G'(P)$, with G' representing $\mathrm{d}G/\mathrm{d}P$; this is equation (2.27) of Mestel *et al.* (1979), with inertial drift neglected and with dimensionless variables. It follows from $\boldsymbol{V}_p = \kappa\boldsymbol{B}_p$ and $\boldsymbol{j}_p = \rho_e\boldsymbol{V}_p$ that $\rho_e\kappa = \mathrm{d}S/\mathrm{d}P$ (MRW[2]), which is constant on the poloidal magnetic field lines, which are also streamlines of the poloidal flow.

MRW[2] wrote $\mathrm{d}S/\mathrm{d}P$ as $-2V_0(P)$, so $\rho_e\kappa = -2V_0$. At least near the star, the $\nabla^2\Phi$ contribution to (1) can be neglected, leaving $\rho_e = -2B_z/(1-XV_\phi)$; hence $V_p/B_p = (1-XV_\phi)V_0/B_z$. But near the star, $XV_\phi \ll 1$ and, provided the outflow emanates from a small polar cap, $B_p \approx B_z$; thus MRW[2] identified $V_0(P)$ with the speed at which the electrons, travelling along the lines of constant P, leave the star. This allowance for a significant emission speed is one of the key new features of their work.

The perfect conductivity boundary condition on the stellar surface means that Φ has there a constant value, which can be taken to be zero. Hence, since $XV_\phi \ll 1$, $V_p \approx V_0$ and $V_\phi^2 \ll 1 - V_0^2$ at the polar cap, $G(S)$ may be approximated very closely by $\gamma_0(P)$, the Lorentz factor corresponding to V_0. Because there is a degree of approximation, though slight, in the identification of $G(S)$ with the more directly physical quantity $\gamma_0(P)$, I shall continue to write the general theory in terms of $G(S)$.

When the emission speed is nonrelativistic, G has a constant value, namely one, across the flow, and equation (3′) reduces to the statement that the flow velocity, reduced by the local velocity of corotation with the star, is parallel to the magnetic field: $V - Xt = \kappa B$. Earlier calculations (MRW[2]; Burman 1984) based on this relation will be extended here, by using the fuller relation (3′), to incorporate the possibility of relativistic injection.

Following the procedure of MRW[2], elimination of the velocity between (3′) and the definition of the Lorentz factor, together with use of $B_\phi = -S/X$, leads to a quadratic equation for κ, yielding

$$\kappa = (D+K)S/B^2\,, \tag{4}$$

where

$$D \equiv (\gamma_m^{-2} - \gamma^{-2})^{\frac{1}{2}} B/S\,, \tag{5}$$

with

$$\gamma_m^{-2} \equiv 1 - K^2X^2 + K^2S^2/B^2 = 1 - K^2X^2B_p^2/B^2\,. \tag{6, 6′}$$

Near the emission regions, $V_p \approx V_0$ and $V_\phi \lesssim X$ so $D \approx V_0 B/S$ so long as $V_0^2 \gg X^2\max(K^2, 1)$; hence, for $S > 0$ and outflow, the positive sign before the radical has been taken (MRW[2]).

The quantity γ_m is the minimum value of γ for the radical to be real (cf. MRW[2]). For $|K|X \leqslant 1$, γ_m is always real; it is real for $|K|X > 1$ provided $B_\phi^2 \geqslant (K^2X^2-1)B_p^2$ [cf. the condition $B_\phi^2 \geqslant (X^2-1)B_p^2$ obtained by Goldreich and Julian (1969) for their flow to have a real Lorentz factor outside $X = 1$].

On using (4), the relation $\rho_e\kappa = -2V_0$ takes the form

$$-2V_0/\rho_e = (D+K)S/B^2\,. \tag{7}$$

On substituting κ from (4) into V_ϕ obtained from the toroidal part of (3′), replacing B_ϕ by $-S/X$ and using (7) for ρ_e, equation (1) becomes

$$\nabla^2\Phi + 2B_z = 2V_0SF\,, \tag{8}$$

where

$$F \equiv 1 + (B^2/S^2)(1-KX^2)/(D+K)\,. \tag{9}$$

Since D_∞, the quantity D for γ infinite, is $B/\gamma_m S$, it follows that

$$D_\infty = \{K^2 + (1-K^2X^2)B^2/S^2\}^{\frac{1}{2}} = (1-K^2X^2B_p^2/B^2)^{\frac{1}{2}} B/S. \quad (10, 10')$$

Hence D itself is $(1-\gamma_m^2/\gamma^2)^{\frac{1}{2}} D_\infty$. The MRW^2 formalism expresses flow variables in terms of γ, which is bounded below by γ_m. The basic MRW^2 flow equations above show that, as functions of γ, the variables κ, ρ_e and F are actually functions of γ/γ_m.

On using (10) for D_∞, the definition (9) of F can be rearranged into the form

$$F = 1 + \{(D_\infty^2 - K^2)/(D+K)\}\{(1-KX^2)/(1-K^2X^2)\}. \quad (9')$$

Hence the function F for γ infinite is given by

$$F_\infty = 1 - (K-D_\infty)(1-KX^2)/(1-K^2X^2). \quad (11)$$

Equations (4)–(11) are the fundamental equations of the MRW^2 formalism for a domain with relativistic injection.

Goldreich–Julian (GJ) flow is now defined as flow satisfying the equations of the MRW^2 formalism, subject to the additional restriction that the term $\nabla^2\Phi$ in (8)—the MRW^2 version of the Gauss–toroidal Ampère law—be negligible. Putting $\nabla^2\Phi = 0$ there and writing $\tilde{B}_z$ for $B_z/V_0 S$ yields $\tilde{B}_z = F$ which, after using the definition (9) of F, together with (5) for D and (6) for γ_m, may be solved for the Lorentz factor:

$$\gamma^{-2} = 1 - K^2X^2 + \frac{1-KX^2}{\tilde{B}_z - 1}\left(2K - \frac{1-KX^2}{\tilde{B}_z-1}\frac{B^2}{S^2}\right), \quad (12)$$

which may be expressed as

$$\gamma^{-2} = (1 - K^2X^2)C/A, \quad (12')$$

with

$$A \equiv (\tilde{B}_z - 1)^2, \qquad C \equiv (\tilde{B}_z - f^+)(\tilde{B}_z - f^-), \quad (13a, b)$$

where

$$f^\pm \equiv 1 - (K \mp D_\infty)(1-KX^2)/(1-K^2X^2), \quad (14)$$

in which (10) for D_∞ has been used; comparison of (14) with (11) shows that f^+ is in fact F_∞. Equation (12′) may be written as

$$\gamma^{-2} = (1-K^2X^2)(\tilde{B}_z - f^+)(\tilde{B}_z - f^-)/(\tilde{B}_z - 1)^2. \quad (12'')$$

The equations $\rho_e \kappa = -2V_0$, $V_\phi - KX = -\kappa S/X$ and $\rho_e = -2B_z/(1-XV_\phi)$ for GJ flow yield, on eliminating ρ_e, V_ϕ and κ in pairs,

$$\kappa S = (1-KX^2)/(\tilde{B}_z - 1), \qquad \rho_e = -2V_0 S(\tilde{B}_z - 1)/(1-KX^2), \quad (15a, b)$$

$$V_\phi = KX - (1-KX^2)/X(\tilde{B}_z - 1). \quad (15c)$$

MRW^2 introduced the surface S_l, defined by putting $\nabla^2\Phi = 0$ and $\gamma = \infty$ in (8), yielding $\tilde{B}_z = F_\infty$, as an outer limit for possible GJ, moderately accelerated, flow:

inside S_l, neglect of $\nabla^2\Phi$ in (8) is consistent with a finite Lorentz factor. In the vicinity of S_l, the GJ flow approximation must fail: the actual flow will be rapidly accelerated there, its Lorentz factor becoming large but remaining finite; dissipation or inertial effects, or both, will quickly become important and the above equations, except for (1) and possibly (2), will be inapplicable.

3. The Functions $f^{\pm}$

As a preliminary step to obtaining information on the qualitative behaviour of GJ outflow, the functions $f^{\pm}$ will now be evaluated at a number of locations, with a view to following the behaviour of the GJ Lorentz factor, given by equation (12″), as it varies along a poloidal field/flow line. The locations include V and W, two surfaces which arise naturally from the extended MRW[2] formalism in a way that the light cylinder does not; that surface appears now merely as the common limit of V and W as $G' \to 0$.

Near the Star

Taking $X^2\max(K^2, |K|) \ll 1$ and $B_\phi^2 \ll B_\mathrm{p}^2$ in (10′) and (14) shows that $D_\infty \approx B_\mathrm{p}/S$ and $f^{\pm} \approx \pm B_\mathrm{p}/S - \epsilon G' \approx \pm B_\mathrm{p}/S$ near the star; B_p/S is very large there. Equation (12″) shows that $\gamma^{-2} \approx (1-V_\mathrm{p}^2)\tilde{B}_z^2/(\tilde{B}_z-1)^2$, with $V_\mathrm{p} \approx V_0\, B_\mathrm{p}/B_z$, near a polar cap; we note that $V_0(P)$ must be less than B_z/B_p evaluated near the polar cap, which has a value slightly below one.

On the Light Cylinder

Equation (10) becomes

$$D_\infty = \{K^2 + (1-K^2)B^2/S^2\}^{\frac{1}{2}} \quad \text{on} \quad X = 1. \tag{16a}$$

For $K \neq \pm 1$ (i.e. $\epsilon G' \neq 0$ or -2), equation (14) shows that

$$f^{\pm} = (1 \pm D_\infty)/(1+K), \quad f^+ - f^- = 2D_\infty/(1+K) \quad \text{on} \quad X = 1. \tag{16b, c}$$

For $K > 1$ (i.e. $\epsilon G' > 0$), it follows that $0 \leqslant D_\infty < K$ on $X = 1$ and hence that

$$1/(1+K) \leqslant f^+ < 1, \quad -(K-1)/(1+K) < f^- \leqslant 1/(1+K) \quad \text{on} \quad X = 1, \tag{17a, b}$$

with

$$0 \leqslant f^+ - f^- < 2K/(1+K) \quad \text{on} \quad X = 1; \tag{17c}$$

in particular, $0 < f^+ < 1$ and $f^- \leqslant f^+$ on $X = 1$.

For $0 < K < 1$ (i.e. $-1 < \epsilon G' < 0$), equations (16) show that $D_\infty > K$ on $X = 1$ and hence that

$$f^+ > 1, \quad f^- < (1-K)/(1+K) \quad \text{on} \quad X = 1, \tag{18a, b}$$

with

$$f^+ - f^- > 2K/(1+K) \quad \text{on} \quad X = 1; \tag{18c}$$

in particular, $f^- < f^+$ on $X = 1$.

For $-1 < K < 0\,(-2 < \epsilon G' < -1)$, it follows from equations (16) that $D_\infty > |K|$ on $X = 1$ and hence that

$$f^+ > (1+|K|)/(1-|K|), \quad f^- < 1 \quad \text{on} \quad X = 1 \tag{19a, b}$$

with

$$f^+ - f^- > 2|K|/(1-|K|) \quad \text{on} \quad X = 1; \tag{19c}$$

in particular, $f^- < f^+$ on $X = 1$.

For $K < -1$ ($\epsilon G' < -2$), equations (16) show that $0 \leqslant D_\infty < |K|$ on $X = 1$ and hence that

$$-(|K|+1)/(|K|-1) < f^+ \leqslant -1/(|K|-1),$$

$$-1/(|K|-1) \leqslant f^- < 1 \quad \text{on} \quad X = 1, \tag{20a, b}$$

with

$$0 \leqslant f^- - f^+ < 2|K|/(|K|-1) \quad \text{on} \quad X = 1; \tag{20c}$$

in particular, $f^+ < 0$ and $f^- \geqslant f^+$ on $X = 1$.

For $K = 1$ ($G' = 0$), equation (14) shows that $f^\pm = \pm D_\infty$ everywhere, with, from (11) and (10), $D_\infty = F_\infty = \{1+(1-X^2)B^2/S^2\}^{\frac{1}{2}}$; in particular, $f^+ = +1$ and $f^- = -1$ on $X = 1$. For $K = 0$ ($\epsilon G' = -1$), equations (10) and (14) show that $f^\pm = 1 \pm B/S$ everywhere. For $K = -1$ ($\epsilon G' = -2$), equation (10) becomes $D_\infty = \{1+(1-X^2)B^2/S^2\}^{\frac{1}{2}}$, so (14) shows that $(1-X^2)f^+ = 4$ and $f^- = -B_p^2/B_\phi^2$ on $X = 1$: although f^+ diverges on the light cylinder, the quantity $(1-X^2)C$, containing vanishing and infinite factors, remains finite, and (12′) yields $\gamma^{-2} = -4(\tilde{B}_z + B_p^2/B_\phi^2)/(\tilde{B}_z - 1)^2$ on $X = 1$.

On W: $|K|X = 1$

For $K \neq 0$ ($\epsilon G' \neq -1$), we let W denote the surface $|K|X = 1$. Examination of the definition (14) of $f^\pm$, together with (10) for D_∞, shows that for $K > 0$ ($\epsilon G' > -1$),

$$f^+ = 1 + \tfrac{1}{2}\epsilon G' B^2/B_\phi^2, \quad (1-K^2X^2)f^- = -2\epsilon G' \quad \text{on} \quad W. \tag{21a, b}$$

So f^+ is finite on W, but f^- is singular there when $G' \neq 0$. Substituting (21b) into (12″) shows that, for $K > 0$,

$$\gamma^{-2} = 2\epsilon G'(\tilde{B}_z - f^+)/(\tilde{B}_z - 1)^2 \quad \text{on} \quad W. \tag{22}$$

[The quantity $(1-K^2X^2)C$ on W contains vanishing and infinite factors, resulting in $2\epsilon G'(\tilde{B}_z - f^+)$.] Equation (21b), and hence (22), are not applicable for $G' = 0$, when W coincides with the light cylinder, on which $f^\pm = \pm 1$. For $K < 0$ ($\epsilon G' < -1$), the expressions (21a, b) for f^+ and f^- are interchanged and f^+ in (22) is replaced by f^-.

On V: $K^{\frac{1}{2}}X = 1$

For $K > 0$ ($\epsilon G' > -1$), we let V denote the surface $K^{\frac{1}{2}}X = 1$. It is readily seen that, for $K \neq 1$ ($G' \neq 0$), $f^+ = 1 = f^-$ on V, and that neither function can be unity anywhere else. This means that C and A can vanish together on V only. Since f^+ is large and positive near the star, remains finite on W and is unity on V only, $f^+ > 1$ inside V and $f^+ < 1$ outside V. For $G' = 0$, the surfaces V and W coincide on the light cylinder, where $f^\pm = \pm 1$. If $\tilde{B}_z \neq 1$ on V, then $C = A \neq 0$ there for $G' \neq 0$, so (12′) shows that $\gamma^{-2} = -\epsilon G'$ on V; for $G' = 0$, the ratio C/A is $(\tilde{B}_z + 1)/(\tilde{B}_z - 1)$ on $X = 1$, so (12′) shows that $\gamma^{-2} = 0$ there, vanishing because of the $(1-X^2)$ factor. If $\tilde{B}_z = 1$ on V, then $A = 0 = C$ there (whether G' is zero or not) and a limiting process must be applied in order to evaluate γ on V. If L denotes the limiting value of $(1-KX^2)/(\tilde{B}_z-1)$ as V is approached along a flow line, then (12) shows that

$$\gamma^{-2} = L(2K - LB^2/S^2) - \epsilon G' \qquad \text{on} \quad V, \tag{23}$$

covering all cases in which V exists. This may be written as

$$\gamma^{-2} = (B^2/S^2)(L^+ - L)(L - L^-) \quad \text{on} \quad V, \tag{23'}$$

where $B^2 L^\pm = B_\phi^2 \pm S(B_\phi^2 - \epsilon G' B_p^2)^{\frac{1}{2}}$; we note that $B_\phi^2 = KS^2$ on V. For $K > 1$ ($\epsilon G' > 0$), the easily-met condition $B_\phi^2/B_p^2 > \epsilon G'$ on V ensures that $L^\pm$ are real; for $0 < K \leqslant 1$ ($-1 < \epsilon G' \leqslant 0$), they are real, with values $2S^2/B^2$ and 0 when $G' = 0$.

Crossover

Let us consider further the conditions for equality of f^+ and f^-. For $K > 0$ but $\neq 1$ ($\epsilon G' > -1$ but $\neq 0$), $f^+ = 1 = f^-$ on V and $f^+ = f^- \neq 1$ if $D_\infty = 0$ somewhere. For $K = 0$ ($\epsilon G' = -1$), equality cannot occur. For $K < 0$ ($\epsilon G' < -1$), equality is possible only with $f^+ = f^- \neq 1$, occurring if $D_\infty = 0$. For $G' = 0$, $f^\pm = \pm D_\infty$, so f^+ and f^- can be equal only if they vanish, the condition being $D_\infty = 0$; this condition is $\gamma_m = \infty$, corresponding to $B_\phi^2 = (K^2X^2 - 1)B_p^2$, and cannot be satisfied inside W.

Zeros

The conditions for $f^\pm$ to vanish are $D_\infty(1-KX^2) = \pm\epsilon G'$ not on W. Hence, for $K > 1$ ($\epsilon G' > 0$), f^+ cannot vanish outside, nor f^- inside, V. For $0 < K < 1$ ($-1 < \epsilon G' < 0$), f^+ cannot vanish inside, nor f^- outside, V. For $K \leqslant 0$ ($\epsilon G' \leqslant -1$), f^+ cannot vanish; the condition for f^- to vanish is $D_\infty(1+|K|X^2) = -\epsilon G'$ not on W; for $K = 0$ ($\epsilon G' = -1$), f^- vanishes where $B/S = 1$, corresponding to $B_p^2 = (X^2-1)B_\phi^2$, which cannot occur inside or on the light cylinder. For $G' = 0$, the condition for $f^\pm$ to vanish is $D_\infty = 0$, which is $\gamma_m = \infty$ or $B_\phi^2 = (X^2-1)B_p^2$, and cannot be satisfied inside the light cylinder; nor do $f^\pm$ vanish on the light cylinder—in this case, $f^\pm = \pm 1$ there.

4. Flow Classes

The qualitative behaviour of GJ outflow will now be studied by considering the equations for its Lorentz factor, taken together with the information on the functions $f^\pm$ deduced in the last section. This will be done by examining the behaviour of $\tilde{B}_z$, f^+ and f^- along an arbitrary poloidal flow line, remembering that $V_0 S$ is constant on each line. On these lines, which are also poloidal magnetic field lines, $\tilde{B}_z$ may be thought of as behaving qualitatively as it would for a dipole field.

As pointed out near the beginning of the last section, $f^+ \approx B_p/S$ near the star, a very large quantity, and $V_0(P)$ must be less than B_z/B_p evaluated near the polar cap. So $\tilde{B}_z$ decreases from a very large positive value, exceeding f^+, at the polar caps, and eventually passes through zero (on the $B_z = 0$ cones) to reach a maximum negative value on the equatorial plane.

The flows will be grouped into five types, labelled by values, or ranges of values, of K, corresponding to differences in their mathematical description. They will also be divided into Class I flows, for which the Lorentz factor given by equations (12) becomes infinite at some point on the flow line, and Class II flows, which reach the equatorial plane without encountering such a singularity.

Type (> 1) Flows: $K > 1$; $\epsilon G' > 0$

In this case, the surfaces V and W both exist, with W lying inside V, which is inside the light cylinder. The function f^+ is large and positive near the star; $f^+ > 1$ inside V, $f^+ = 1$ on V and, since it cannot vanish outside V, $1 > f^+ > 0$ there. The function f^-, which is large and negative near the star, diverges on W, tending to $-\infty$ as W is approached from the inside and decreasing from $+\infty$ beyond W; since it cannot vanish inside V, $f^- < 0$ inside W; $f^- > 1$, $=1$ and <1 between W and V, on V and beyond V.

If $\tilde{B}_z$ decreases to equality with f^+ at some point inside W, then $C = 0$ and $\gamma = \infty$ at that point: GJ outflow ends in a pole of its Lorentz factor inside W. If $\tilde{B}_z$ falls to equality with f^+ on W, where f^- diverges, then the product $(1-K^2X^2)C$ is not only finite but zero there: as shown by (22), $\gamma = \infty$ on W. If $\tilde{B}_z$ is still above f^+ on W, then γ is real and finite inside and, as shown by (22), on W. It continues that way beyond W so long as $\tilde{B}_z$ remains above f^+ and below f^-: either $\tilde{B}_z$ intersects one of these functions between W and V, at which point $C = 0$ and $\gamma = \infty$, or $\tilde{B}_z$ falls to unity on V so $\tilde{B}_z = f^+ = f^- = 1$ and $C = 0 = A$ there. (It is clear that A cannot vanish inside V without $\gamma = \infty$ occurring closer to the star.)

In the case in which $\tilde{B}_z$ falls to unity on V, since γ^{-2} cannot jump from positive to negative values without passing through zero, $\gamma^{-2} \geqslant 0$ on V; equation (23′) shows that γ is real and finite on V if $L^- < L < L^+$ there and is infinite if $L = L^-$ or L^+. If γ is real and finite on V, then, by continuity, $\gamma^{-2} > 0$ immediately beyond V, so $\tilde{B}_z$ must lie above f^- and below f^+ infinitesimally beyond V: $\tilde{B}_z$ must cross both f^+ and f^- on V; it cannot cross one and just touch the other. Outside V, either $\tilde{B}_z$ intersects one of those functions, at which point $C = 0$ and $\gamma = \infty$, or the flow continues on with $f^- < \tilde{B}_z < f^+$ to reach the equatorial plane with γ still finite.

Type 1 Flows: $K = 1$; $G' = 0$

In this case, the surfaces V and W coincide on the light cylinder. The function f^+ decreases from its large positive value at the polar caps, passing through unity on the light cylinder; $f^+ > 1$, $=1$ and <1 inside, on and outside $X = 1$. The function f^- is just the negative of f^+.

If $\tilde{B}_z$ falls to equality with f^+ at some point inside the light cylinder, then $C = 0$ and $\gamma = \infty$ at that point. If $\tilde{B}_z$ is still above f^+ on $X = 1$, then $\gamma = \infty$ on the light cylinder because of the $(1-X^2)$ factor in (12″) for γ^{-2}. (It is clear that A cannot vanish inside $X = 1$ without $\gamma = \infty$ occurring closer to the star.)

If $\tilde{B}_z$ decreases to unity on $X = 1$, so that $\tilde{B}_z = 1 = f^+$ and $C = 0 = A$ there, then, since γ^{-2} cannot jump from positive to negative values without passing through zero, $\gamma^{-2} \geqslant 0$ on $X = 1$; equation (23) shows that γ is real and finite if $0 < L < 2S^2/B^2$ and is infinite if $L = 0$ or $2S^2/B^2$. In that case, beyond $X = 1$, either $\tilde{B}_z$ intersects one of the functions $f^{\pm}$, at which point $C = 0$ and $\gamma = \infty$, or the flow continues on with $f^- < \tilde{B}_z < f^+$ to reach the equatorial plane with γ still finite.

Type (0, 1) Flows: $0 < K < 1$; $-1 < \epsilon G' < 0$

In this case, the surfaces V and W both exist, with W lying outside V, which is outside the light cylinder. The function f^+ decreases from its large positive value at the polar caps to unity on V; $f^+ > 1$, $=1$ and <1 inside, on and beyond V. The function f^- increases from its large negative value at the polar caps to $+1$ on V, and goes on to diverge to $+\infty$ as W is approached from the inside, beyond which it increases from $-\infty$; $f^- < 1$, $=1$ and >1 inside V, on V and between V and W; since f^- cannot vanish outside V, $f^- < 0$ beyond W.

If $\tilde{B}_z$ decreases to equality with f^+ inside V, then $C = 0$ and $\gamma = \infty$ at that point. If $\tilde{B}_z$ is still above f^+ on V (so $\gamma^{-2} = -\epsilon G'$ there), then $\tilde{B}_z$ must reach equality with f^- somewhere between V and W; at that point, $C = 0$ and $\gamma = \infty$. (Since $f^+ > 1$ inside V and $f^- > 1$ between V and W, the quantity A cannot vanish inside V unless $\gamma = \infty$ occurs closer to the star.)

If $\tilde{B}_z$ falls to equality with f^+ on V, so $\tilde{B}_z = f^+ = f^- = 1$ and $C = 0 = A$ there, then, by continuity, $\gamma^{-2} \geqslant 0$ on V; equation (23′) shows that γ is real and finite there provided $L^- < L < L^+$ and is infinite if $L = L^-$ or L^+. If γ is real and finite on V, then, by continuity, $\gamma^{-2} > 0$ immediately beyond V; so $\tilde{B}_z$ must lie below f^+, which is below f^-, infinitesimally beyond V: $\tilde{B}_z$ must cross f^+ (as well as f^-) on V, not just touch it. If $\tilde{B}_z$ intersects f^+ again between V and W, then $C = 0$ and $\gamma = \infty$ there. But, if $\tilde{B}_z$ is still below f^+ on W, then γ is real and finite inside and, as shown by (22), on W. It continues that way beyond W so long as $\tilde{B}_z$ lies below f^+ and above f^- (which is negative there): either $\tilde{B}_z$ intersects one of these functions beyond W, at which point $C = 0$ and $\gamma = \infty$, or the flow continues on with $f^- < \tilde{B}_z < f^+$ to reach the equatorial plane with γ still finite.

Type 0 Flows: $K = 0$; $\epsilon G' = -1$

In this case, there are no surfaces V and W; the functions $f^\pm$ are $1 \pm B/S$ everywhere and $\gamma^{-2} = C/A$. Since f^+ is always above unity, while $\tilde{B}_z$ along a poloidal flow line must eventually pass through zero, $\tilde{B}_z$ must decrease somewhere to equality with f^+, at which point $C = 0$ and $\gamma = \infty$. (Clearly, A cannot vanish without $\gamma = \infty$ occurring closer to the star.) There are no Class II flows of Type 0: in fact, GJ flows of Type 0 do not reach the $B_z = 0$ cones, but terminate with $\tilde{B}_z > 1$.

Type (<0) Flows: $K < 0$; $\epsilon G' > -1$

In this case, there is no surface V, so neither f^+ nor f^- can equal unity. The surface W is outside, on and inside the light cylinder in the subcases $-1 < K < 0$ ($-2 < \epsilon G' < -1$), $K = -1$ ($\epsilon G' = -2$) and $K < -1$ ($\epsilon G' < -2$). The functions $f^\pm$ cannot cross each other inside W. The function f^+ cannot vanish; it diverges on W, tending to $+\infty$ as W is approached from the inside and increasing from $-\infty$ beyond W; thus $f^+ > 1$ inside W and $f^+ < 0$ outside W. Since it remains finite, $f^- < 1$ everywhere.

Clearly, $\tilde{B}_z$ must intersect f^+ somewhere inside W, at which point $C = 0$ and $\gamma = \infty$. There are no Class II flows of Type (<0): as with Type 0, GJ flows of Type (<0) do not reach the $B_z = 0$ cones, but terminate with $\tilde{B}_z > 1$.

IA, IB and IC Flows

The following subdivision of Class I, according to where the GJ flow terminates in the singularity of its Lorentz factor, is convenient in distinguishing flows having different mathematical descriptions: IA and IC flows are those for which the infinity arises from the vanishing of C at some point where $\tilde{B}_z > 1$ and $\tilde{B}_z < 1$ respectively; IB flows are those with the infinity on V.

Class IA flows are confined within the outermost of V and W. For $K \leqslant 0$ ($\epsilon G' \leqslant -1$), there is no surface V, and all GJ flow is confined within W (considered as being at infinity when $K = 0$)—*all* flows are of Class IA when $K \leqslant 0$. Class IC flows terminate outside V. We note that, for $0 < K < 1$ ($-1 < \epsilon G' < 0$), the zone between V and W can see the termination of both IA and IC flows.

For $G' \neq 0$, Class IB flows all have $\tilde{B}_z = 1$ on V, with $L = L^-$ or L^+. For IB flows when $G' = 0$, either the same is true or $\tilde{B}_z > 1$ (so $C \neq 0$) on $X = 1$.

IC and II Flows of Type 1

When the Endean integral is constant throughout the region concerned, rather than merely constant on each poloidal field/flow line, flows of Classes IC and II—which cross the light cylinder without encountering S_l—are described, for a dipolar poloidal magnetic field, by

$$S(P) = 2\bar{P}(1-\tfrac{9}{8}Q)^{\frac{1}{2}}, \quad V_0(P) = (1-\tfrac{3}{2}Q)/(1-\tfrac{9}{8}Q)^{\frac{1}{2}}, \tag{24a, b}$$

where $\bar{P} \equiv -P$ and $Q \equiv \bar{P}^{2/3}$. (For the dipole field, $\bar{P} = X^2/R^3$ with R denoting the spherical polar radial coordinate normalized to unity on the light cylinder in the equatorial plane.) These relations satisfy $V_0 = \frac{1}{2}\mathrm{d}S/\mathrm{d}\bar{P}$ and the condition that $\tilde{B}_z = 1$ on the light cylinder. They were introduced by MRW[2], who regarded them as describing Class IA flow in the limiting case in which S_l is just inside the light cylinder, and have been used in a detailed study of IC/II flows (Burman 1985*b*). The Lorentz factor corresponding to this $V_0(P)$ is given by

$$\gamma_0^2(P) = (8/15Q)(1-\tfrac{9}{8}Q)/(1-\tfrac{6}{5}Q). \tag{25}$$

The relations (24) for $S(P)$ and $V_0(P)$ contain the unphysical limit $V_0(0) = 1$, so it is necessary in this particular case to check whether or not the condition $\gamma_0^2\, V_\phi^2 \ll 1$ at the polar cap, assumed in the identification of $G(S)$ with $\gamma_0(P)$, remains satisfied as $P \to 0$. Equation (25) for γ_0 together with $V_0/X \approx \frac{3}{2}Q$, which is valid near a polar cap for flows described by (24), show that this condition is essentially $QX^2 \ll 1$, which is satisfied very well.

Since G is very closely equal to γ_0, it follows that

$$G'(P) = (\tfrac{2}{15})^{\frac{1}{2}}(2V_0/3Q^2)(1-\tfrac{9}{10}Q)/(1-\tfrac{6}{5}Q)^{3/2}. \tag{26}$$

This vanishes at the edge of the zone of GJ outflow (where $V_0 = 0$ and $Q = \frac{2}{3}$) and diverges on the axis; it is $0\cdot800$ on the poloidal field line $Q = \frac{1}{2}$ separating IC and II flows; it is $1\cdot00$, $2\cdot00$ and $3\cdot00$ on $Q = 0\cdot46$, $0\cdot34$ and $0\cdot28$ respectively; it is already $0\cdot11$ on $Q = 0\cdot65$, just inside the edge of the zone. It is interesting to see that, although γ_0 does not vary much across Class IC flow (from 1 at $Q = \frac{2}{3}$ to $1\cdot080$ at $Q = \frac{1}{2}$), $G'(P)$ varies from 0 to $0\cdot800$. (This is caused largely by the $1/Q^2$ factor, along with the V_0 factor, which varies from 0 at $Q = \frac{2}{3}$ to $0\cdot378$ at $Q = \frac{1}{2}$.)

In the paper on Class IC/II flow (Burman 1985*b*), some order-of-magnitude estimates were made which indicated that the theory based on the relations (24) fails, because of inertial drift generated in the vicinity of the $B_z = 0$ cones, on poloidal field/flow lines having $Q \lesssim \epsilon^{2/3}$ and hence $\gamma_0 \gtrsim \epsilon^{-1/3}$. Since ϵ is roughly in the range from 10^{-6} to 10^{-11} for different pulsars, the corresponding limiting γ_0 is from about 10^2 to 10^4.

The extended MRW^2 formalism developed in Section 2 shows that neglect of the variation of the Endean integral is invalid if $\epsilon G'(P)$ is a significant fraction of one. Equation (26) for G' shows that the theory based on the relations (24) fails for this reason on poloidal field/flow lines with $Q \lesssim \epsilon^{\frac{1}{2}}$ and hence $\gamma_0 \gtrsim \epsilon^{-\frac{1}{4}}$; the corresponding limiting γ_0 for different pulsars is from about 30 to 10^3. This limitation is a bit more stringent than that obtained from examining inertial drift.

These considerations do not imply failure of the MRW^2 approach to GJ flow: they merely reflect an inconsistency between the assumptions of IC/II flow and constancy everywhere of the Endean integral. The conclusion is this: either outflow from the innermost core of a polar cap is of Class IA/B, or poloidal flow-line dependence of the Endean integral must be incorporated.

5. Concluding Remarks

When $G' \neq 0$, the light cylinder plays no special part in the mathematical description of GJ flows: rather, it is the surfaces V and W that do so, particularly the former. It is only flows with $\tilde{B}_z = 1$ on V that have the possibility of reaching values of $\tilde{B}_z$ below one (Classes IC and II flows), and therefore the possibility of continuing to the $B_z = 0$ cones and beyond to the equatorial plane.

The surface W exists unless $K = 0$ ($\epsilon G' = -1$), for which it has gone off to $X = \infty$; this forms the simplest case mathematically, in that there is no special surface at all. For all cases with no V, namely those with $K \leqslant 0$ ($\epsilon G' \leqslant -1$), not only are there no IA or IB flows, but there are no Class II flows either: the GJ flows all terminate in an infinite Lorentz factor with $\tilde{B}_z > 1$, and hence cannot reach the $B_z = 0$ cones. For all cases with a surface V, namely those with $K > 0$ ($\epsilon G' > -1$), both Class I and Class II flows, with all three subdivisions of the former, are possible.

Since G is closely equal to the Lorentz factor γ_0 corresponding to V_0, which is equal to $\frac{1}{2}\mathrm{d}S/\mathrm{d}\bar{P}$, it follows that

$$G'(P) = -\gamma_0^3 V_0 \,\mathrm{d} V_0/\mathrm{d}\bar{P} = -\tfrac{1}{2}\gamma_0^3 V_0 \,\mathrm{d}^2 S/\mathrm{d}\bar{P}^2 .$$

Hence, $G' > 0$ and $G' < 0$ correspond to emission speeds that respectively decrease and increase away from the poles. So, of the various flow types, Type (>1) with K just slightly above unity is likely to be the one of most physical interest in the treatment of electron emission. Other types may also be required, for example in treating the outflow of positively charged particles.

Acknowledgment

I thank the referee for a helpful review and for noticing a small, but potentially significant, error.

References

Burman, R. R. (1984). *Proc. Astron. Soc. Aust.* **5**, 467.
Burman, R. R. (1985*a*). *Aust. J. Phys.* **38**, 97.
Burman, R. R. (1985*b*). *Aust. J. Phys.* **38**, 749.
Burman, R. R., and Mestel, L. (1978). *Aust. J. Phys.* **31**, 455.
Endean, V. G. (1972). *Nature Phys. Sci.* **237**, 72.
Goldreich, P., and Julian, W. H. (1969). *Astrophys. J.* **157**, 869.
Mestel, L., Phillips, P., and Wang, Y.-M. (1979). *Mon. Not. R. Astron. Soc.* **188**, 385.
Mestel, L., Robertson, J. A., Wang, Y.-M., and Westfold, K. C. (1985). *Mon. Not. R. Astron. Soc.* **217**, 443.

Manuscript received 5 August 1985, accepted 8 January 1986