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Quantum Electrodynamics in Strong Magnetic Fields. IV Electron Self-energy

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Abstract

The electron self-energy in a magnetic field is calculated with the effect of the field included exactly. A new representation of the wavefunctions and other quantities is defined, in which the mass operator has a particularly simple form. After renormalisation, the form of the mass operator allows corrections to the Dirac equation, wavefunctions, vertex function and the electron propagator close to the mass shell to be calculated to lowest order in the fine structure constant. The probability for an electron to change spin while remaining in the same Landau level is calculated, and is found to be much less than the probability of cyclotron emission.

1. Introduction

The theory of quantum electrodynamics (QED) in a static homogeneous magnetic field has been developed in recent papers* with the effect of the magnetic field included exactly. Methods were presented for the calculation of amplitudes and transition rates for the lowest order Feynman diagrams for a general process. When higher order diagrams with radiative corrections are included, however, these methods yield divergent results which require renormalisation.

In this paper the electron self-energy is discussed in terms of the mass operator. The motivation for this is twofold: firstly, to renormalise the divergence, and secondly, to find a method for including this kind of radiative correction in lower order diagrams. Previous treatments (Constantinescu 1972*a*, 1972*b*; Ternov *et al.* 1977, 1978) have only considered the self-energy of a free electron and derived corrections to free electron motion. In addition, most work on this subject has used an inappropriate form of the spin wavefunctions of the electron.

The system of natural units $c = 1$, $\hbar = 1$ is used, and the charge of the electron is taken to be $-e$. The notation used below is generally the same as in I, II and III, with modifications as noted below. One addition to the notation is the convention that the quantities $\epsilon p_y = P_y$ (I.6) and s (I.31) will be referred to collectively as positional quantum numbers and denoted by g, and the symbol $P_z = \epsilon p_z$ (the unsigned momentum) is introduced. The two sets of solutions to the Dirac equation in a magnetic field (I.14) and (I.30) are referred to as the P_y and s representations of the wavefunction respectively, apart from the normalisation. In this paper, the

* Melrose and Parle (1983*a*, 1983*b*), Melrose (1983); herein referred to as papers I, III and II respectively. Equations from these papers are referred to as e.g. (I.6).

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normalisation of the wavefunctions is determined by the orthonormality relation

$$\langle q' | q \rangle = \int_V \mathrm{d}^3 x \; \Psi^{\dagger}_{q'}(x) \Psi_q(x) = \delta_{q'q} \tag{1}$$

with an appropriate choice of normalisation volume V. This differs from that used in (I, II, III) by the factor of $\{L_y L_z/(eB)^{\frac{1}{2}}\}^{\frac{1}{2}}$ in the P_y representation, and $(2\pi L_z/eB)^{\frac{1}{2}}$ in the s representation. The symbol q refers to the complete set of quantum numbers $\{\epsilon, \sigma, n, P_z, g\}$; r will in general denote an incomplete set of quantum numbers and these will be noted unless it is clear from the context.

In the remainder of this Section, the self-interaction of the electron is examined in general terms and previous work on this subject is reviewed. The spinor representation, which will be seen to be appropriate for the mass operator, is introduced in Section 2. The mass operator in a magnetic field is discussed in Section 3, and expressions are found to first order in the fine structure constant. The divergent terms are identified at this point. The renormalisation of the mass operator is performed in Section 4, and the modifications to the wavefunctions, vertex functions and electron propagator are found. In Section 5, the weak field limit of the renormalised mass operator is found, and the modified vertex function is used to find the transition rate for spin flip radiation.

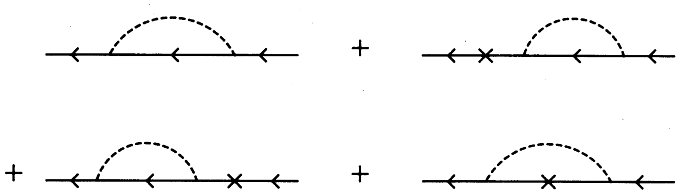

Fig. 1. Perturbation expansion of the exact self-interaction diagram in powers of the magnetic field, to first order. Interactions with the magnetic field are represented by a cross. Note that the vertex correction to the field interaction appears (last diagram).

The electron self-interaction is represented by a subdiagram with two or more vertices which is connected to the rest of the diagram by only two fermion lines. Unless otherwise stated, any such subdiagram referred to below is assumed to be compact: that is, it cannot be separated into two or more self-interaction parts. An alternative method of calculating the self-interaction of a magnetised electron is to use field-free particle electrodynamics and to treat the magnetic field as a perturbation. In the weak-field limit, the two treatments (perturbative and exact) should correspond: in particular, no new divergences should appear. All physically observable effects which arise from the perturbation approach should be retained in the exact treatment, although new effects which depend on the field strength may appear. It should be noted that the lowest order self-interaction diagram in the exact approach includes the lowest order vertex correction to all orders in the external field strength (see Fig. 1). As a consequence, the self-interaction includes the effect of the anomalous magnetic moment to all orders in the external field when the field is treated exactly.

For a free particle, the effect of the self-interaction is to cause a shift in the particle's energy which is inherently unobservable, because the particle cannot be examined with the interaction turned off. For an electron (or positron) in an unmagnetised vacuum, Lorentz covariance required that the energy shift appears as a correction to the rest mass of the particle which is independent of the spin. Similarly, invariance under charge conjugation or time reversal requires that the mass correction is the same for electrons and positrons. Hence, the mass shift is dependent only on the type of particle (electron, muon, etc.) being considered, and not on any quantum number.

If the electron or positron is in a static, homogeneous magnetic field, only Lorentz boosts along the field axis (assumed to lie in the z direction) and rotations around this axis may be performed without changing the physics of the problem. In particular, no rotation around the field axis can change one spin state into the other, and hence in principle the measured energy can be a function of the spin quantum number. Translations can be made in any direction without affecting the energy of the particle (as the field is homogeneous) and this invariance implies that the energy is independent of the positional quantum number. The energy of the particle is thus a function only of the quantum numbers n, σ, P_z. The energy is still the time component of a 4-vector with respect to boosts parallel to the field axis, and so the most general form of the energy is

$$E(n,\sigma,P_z) = [\{f(n,\sigma)\}^2 + P_z^2]^{\frac{1}{2}}, \tag{2}$$

where, if the self-interaction is discounted, one has

$$f_0(n,\sigma) = (m_0^2 + 2neB)^{\frac{1}{2}}. \tag{3}$$

These relations assume that the spin operator commutes with the Lorentz transform operator, and also that the operator describing the self-interaction is diagonal in the chosen representation of spin states. The magnetic moment operator $\hat{\mu}_z$ (I.42) satisfies the first condition, and it will be shown that its eigenvectors, suitably modified, satisfy the second condition. Some previous treatments (Constantinescu 1972*b*) may be criticised because the self-energy has been found using the eigenvectors of an inappropriate spin operator s_z (I.38), which is neither relativistically covariant nor yields a diagonal representation for the spin operator. However, for the special case of $P_z = 0$, the eigenfunctions of the two operators are identical and so the correct value for the self-energy may be obtained by transforming according to (2).

The calculation of $f(n,\sigma)$ for the electron in the ground state has been performed by Demeur (1953) and Jancovici (1969). In this case, the choice of spin operator is irrelevant as the ground state is not spin degenerate. A formula for the case of general n has been presented by Ternov *et al.* (1977, 1978) with the spin operator left indeterminate and the spin dependence of the result in terms of a set of spin coefficients obtained by projecting the desired spin wavefunctions onto the s_z eigenvectors.

Anomalous Magnetic Moment

The Dirac theory of the electron predicted that the electron has a magnetic moment of magnitude $g\mu_B\, s$, with the g-factor exactly equal to 2. Refined experiments (Breit 1947) showed that the measured g-factor for the electron was slightly greater than the Dirac value. Schwinger (1948) calculated the first order vertex correction, separated the finite part, and demonstrated that it produces an additional 'anomalous' magnetic moment which modifies the g-factor by the amount

$$\tfrac{1}{2}(g-2) = \alpha/2\pi. \tag{4}$$

Later refinements of the vertex calculations (Lautrup *et al.* 1972; Calucci 1980*a*, 1980*b*) have extended the theoretical value of $\frac{1}{2}(g-2)$ to seven significant figures and to third order in the fine structure constant α, still in excellent agreement with experiment. In most of these calculations, the external field is treated as a first order perturbation, and hence the interaction energy is linear in the field strength. Perturbative calculations of the anomalous magnetic moment have been extended to the second order in the field strength using the proper time method (Newton 1954; Tsai and Yildiz 1973; Baier *et al.* 1974). The method used in this paper includes the magnetic field exactly and hence in principle all orders of the field strength are included.

In the standard theory, the anomalous magnetic moment is incorporated in the Dirac formalism by an ansatz due to Pauli (1941), who introduced a spin dependent term into the Dirac equation in order to describe the large moments of the proton and the neutron (which do not originate with the vertex correction) yielding the result

$$[\mathrm{i}\,\gamma^{\mu}\partial_{\mu}+e\gamma^{\mu}A_{\mu}(x)-m+\tfrac{1}{2}\mathrm{i}\{\tfrac{1}{2}(g-2)\}\mu_{\mathrm{B}}\,F_{\mu\nu}\,\gamma^{\mu}\gamma^{\nu}]\,\Psi(x)=0\,. \tag{5}$$

Here, the g-factor is treated as another parameter in the theory, which is determined either by experiment or by appealing to Schwinger's argument (for an electron) that part of the vertex correction corresponds to a change in the magnetic moment. The Hamiltonian derived from (5) using the Schwinger value (4) for the anomalous magnetic moment has eigenvalues (Sokolov and Ternov 1968)

$$E=[\{(m^{2}+2neB)^{\frac{1}{2}}+\sigma(\alpha/2\pi)\mu_{\mathrm{B}}\,B\}^{2}+P_{z}^{2}]^{\frac{1}{2}}\,, \tag{6}$$

where σ is the eigenvalue of the magnetic moment operator. This operator commutes with the Hamiltonian, and hence represents a conserved quantity (it is 'stable' in the notation of Sokolov and Ternov) when the anomalous magnetic moment of the electron is included.

In the present paper, an entirely different approach is taken. The self-interaction is incorporated into the Dirac theory from the beginning and, after renormalisation, the Dirac equation takes the form (5) with the quantity $\frac{1}{2}(g-2)$ already defined. In addition, finite corrections to the mass and electric charge are found.

2. Spinor Representation

It is convenient at this point to introduce a new representation for the electron wavefunctions and the quantities which are derived from them such as the electron propagator and the vertex function. The *spinor* representation is loosely analogous to the 'momentum' representation which is used in field-free quantum electrodynamics, and is closely related to the 'E_p representation' used by Ritus (1970, 1972) and others to describe wavefunctions in a crossed field. The spinor representation is related to the coordinate representation by

$$\Psi_{q}(x,t)=\;V(x;\;P_{z},n,g)\,u_{\epsilon\sigma}(P_{z},n)\exp(-\mathrm{i}\,\epsilon\mathscr{E}_{q}\,t)\,, \tag{7}$$

where u is a column spinor and V a matrix. The elements of u are given by the column vector $C^{\epsilon\sigma}$ introduced in (I.14) and u is explicitly independent of the positional quantum number, the space–time coordinates, and the choice of gauge or representation. The time dependence of the wavefunction has been written separately as the free particle energy $\epsilon\mathscr{E}_q$ and is replaced by the energy of a virtual particle E when the wavefunction is continued analytically away from the mass shell. The matrix V is diagonal and is representation and gauge dependent; in the P_y representation it has the form (adopting the notation of I)

$$V(x;\ P_z, n, P_y) = \exp(\mathrm{i}\,P_y\,y + \mathrm{i}\,P_z\,z)\{L_y\,L_z/(eB)^{\frac{1}{2}}\}^{-\frac{1}{2}} \times \operatorname{diag}\{\nu_{n-1}(\xi), \nu_n(\xi), \nu_{n-1}(\xi), \nu_n(\xi)\}\,, \tag{8a}$$

and in the s representation it has the form

$$V(x;\ P_z, n, s) = \exp\{\mathrm{i}\,\phi(n-s) + \mathrm{i}\,P_z\,z\}(2\pi L_z/eB)^{-\frac{1}{2}} \times \operatorname{diag}\{\exp(-\mathrm{i}\,\phi)\mathrm{J}^s_{n-s-1}(\rho),\ \mathrm{J}^s_{n-s}(\rho),\ \exp(-\mathrm{i}\,\phi)\mathrm{J}^s_{n-s-1}(\rho),\ \mathrm{J}^s_{n-s}(\rho)\}\,. \tag{8b}$$

Because V is a diagonal matrix, it commutes with the matrix γ^0, and so the Dirac conjugate wavefunction is given by

$$\bar{\Psi}_q(x, t) = \bar{u}_{\epsilon\sigma}(P_z, n)\ V^{\dagger}(x;\ P_z, n, g)\exp(\mathrm{i}\,\epsilon\mathscr{E}_q\,t)\,. \tag{9}$$

Useful properties of u and V include

$$u^{\dagger}_{\epsilon'\sigma'}(P_z, n)\ u_{\epsilon\sigma}(P_z, n) = \delta_{\epsilon'\epsilon}\,\delta_{\sigma'\sigma}\,, \tag{10a}$$

$$\sum_{\epsilon\sigma} u_{\epsilon\sigma}(P_z, n)\ u^{\dagger}_{\epsilon\sigma}(P_z, n) = \mathrm{I}_4\,, \tag{10b}$$

$$\int \mathrm{d}^3x\ V^{\dagger}(x;\ P'_z, n', g')\ V(x;\ P_z, n, g) = \mathrm{I}_4\,\frac{2\pi}{L_z}\,\delta(P_z - P'_z)\delta_{nn'}\,\delta_{gg'}\,, \tag{11a}$$

$$L_z \int \frac{\mathrm{d}P_z}{2\pi} \sum_{n,g} V(x';\ P_z, n, g)\ V^{\dagger}(x;\ P_z, n, g) = \delta^3(x - x')\,, \tag{11b}$$

where I_4 is the 4th rank unit matrix.

The rules for calculating transition amplitudes in the momentum representation may be readily generalised to allow one to calculate the amplitudes using the spinor representation of the wavefunctions. A vertex function and propagator can be defined, which are matrices rather than scalar functions (as in the coordinate representation), while being independent of the spatial coordinates (as in the momentum representation). Using the symbols $\hat{\gamma}$ and $\mathrm{i}\,\hat{G}$ to indicate the spinor representation, one has

$$\begin{aligned}[\hat{\gamma}_{r'r}(k)]^{\mu} &= \int \mathrm{d}^3x\ V^{\dagger}(x;\ P'_z, n', g')\gamma^{\mu}\ V(x;\ P_z, n, g)\exp(-\mathrm{i}\,k\,.\,x)\\ &= \sum_{\substack{\sigma\sigma'\\ \epsilon\epsilon'}} \gamma^0\{u_{\epsilon'\sigma'}(P'_z, n')\,[\gamma_{q'q}(k)]^{\mu}\bar{u}_{\epsilon\sigma}(P_z, n)\}\gamma^0\,,\end{aligned} \tag{12}$$

where r and r' refer to the sets of quantum numbers $\{n, g, P_z\}$ and $\{n', g', P'_z\}$

respectively, while the electron propagator has the form

$$\mathrm{i}\hat{G}(E, P_z, n) = \mathrm{i}\sum_{\epsilon\sigma} \frac{u_{\epsilon\sigma}(P_z, n)\,\overline{u}_{\epsilon\sigma}(P_z, n)}{E - \epsilon(\mathscr{E}_q - \mathrm{i}0)}$$

$$= \int \frac{\mathrm{d}E'}{2\pi} L_z \int \frac{\mathrm{d}P'_z}{2\pi} \sum_{n'g'} \int \mathrm{d}^4 x \int \mathrm{d}^4 x' \exp\{-\mathrm{i}(Et - E't')\}$$

$$\times\, V^{\dagger}(x';\, P'_z, n', g')\,\mathrm{i}\, G(x', x)\, V(x;\, P_z, n, g)\,. \tag{13}$$

Inverting the relation (13), the propagator in coordinate space has the form

$$\mathrm{i}\, G(x', x) = \int \frac{\mathrm{d}E}{2\pi} L_z \int \frac{\mathrm{d}P_z}{2\pi} \sum_{ng} \exp\{-\mathrm{i}\, E(t' - t)\}$$

$$\times\, V(x';\, P_z, n, g)\,\mathrm{i}\, \hat{G}(E, P_z, n)\, V^{\dagger}(x;\, P_z, n, g)\,, \tag{14}$$

where $\mathrm{i}\,\hat{G}(E, P_z, n)$ has the explicit matrix representation

$$\mathrm{i}\,\hat{G}(E, P_z, n) = \frac{\mathrm{i}}{E^2 - (\mathscr{E}_q - \mathrm{i}0)^2}(E\,\hat{V}^0 - p_n\,\hat{V}^2 - P_z\,\hat{V}^3 + m\hat{S})$$

$$= \frac{\mathrm{i}}{E^2 - (\mathscr{E}_q - \mathrm{i}0)^2} \begin{bmatrix} m+E & 0 & -P_z & \mathrm{i}\,p_n \\ 0 & m+E & -\mathrm{i}\,p_n & P_z \\ P_z & -\mathrm{i}\,p_n & m-E & 0 \\ \mathrm{i}\,p_n & -P_z & 0 & m-E \end{bmatrix}, \tag{15}$$

$$p_n = (2neB)^{\frac{1}{2}}\,, \tag{16}$$

where the Dirac matrices (see Appendix) have been used. Another result used below is

$$\{\mathrm{i}\,\gamma^{\mu}\partial_{\mu} + e\gamma^{\mu} A_{\mu}(x) - m\}\, V(x;\, P_z, n, g)\exp(-\mathrm{i}\, Et)$$

$$= V(x;\, P_z, n, g)\exp(-\mathrm{i}\, Et)(\gamma^{\mu}\Pi_{\mu} - m)\,, \tag{17}$$

where the matrix $\gamma^{\mu}\Pi_{\mu}$ is given by

$$\gamma^{\mu}\Pi_{\mu} = E\,\hat{V}^0 - P_z\,\hat{V}^3 - p_n\,\hat{V}^2 = \begin{bmatrix} E & 0 & -P_z & \mathrm{i}\,p_n \\ 0 & E & -\mathrm{i}\,p_n & P_z \\ P_z & -\mathrm{i}\,p_n & -E & 0 \\ \mathrm{i}\,p_n & -P_z & 0 & -E \end{bmatrix}. \tag{18}$$

Hence, in the spinor representation, the Dirac equation has the form

$$(\gamma^{\mu}\Pi_{\mu} - m)\, u_{\epsilon\sigma}(P_z, n) = 0 \tag{19}$$

with the energy in (18) determined by

$$E = \epsilon \mathscr{E}_q, \tag{20}$$

in order that (19) has non-trivial solutions. It may be noted that the spinor representation of the propagator satisfies

$$(\gamma^\mu \Pi_\mu - m)\, \hat{G}(E, P_z, n) = 1, \tag{21}$$

and hence the electron propagator is equal to

$$\mathrm{i}\,\hat{G}(E, P_z, n) = \mathrm{i}(\gamma^\mu \Pi_\mu - m)^{-1}, \tag{22}$$

which may be verified by inverting (15).

The method of calculating transition amplitudes using the spinor representation is a slight modification of the method used in the momentum representation, as outlined in (III). The propagator (13) only requires summation over the Landau quantum number n, as the summations over ϵ and σ have already been performed explicitly. The contribution of a fermion line is written as a matrix product of vertex functions and propagators, with the order of the factors being in the opposite sense to the fermion line. This matrix product is reduced to a scalar either by taking the trace (for a closed fermion loop) or by pre- and post-multiplying the result by the appropriate spinors $\bar{u}'$ and u representing the outgoing and incoming states respectively.

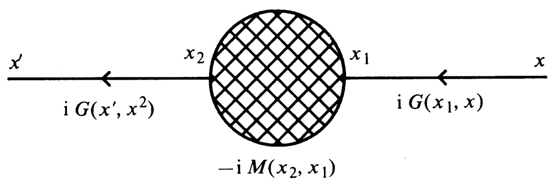

Fig. 2. Feynman subdiagram showing the insertion of the mass operator representing the (compact) self-interaction into the propagator $\mathrm{i}\,G(x', x)$. The phase $-\mathrm{i}\,M$ is determined by the requirement that M represents a positive contribution to the observed fermion mass.

3. Mass Operator

The mass operator is now introduced as a convenient method of describing the effect of the self-interaction in quantum electrodynamics. In coordinate space, the modification of the fermion propagator by the (compact) self-interaction to all orders in the fine structure constant may be represented by Fig. 2, where the hatched circle represents all possible compact self-energy subdiagrams with incoming vertex x_1 and outgoing vertex x_2. The sum of the amplitude factors due to all these subdiagrams is given by the mass operator $-\mathrm{i}\,M(x_2, x_1)$. The term 'mass operator' is used to refer to this amplitude and to the diagram element, in the same manner as the fermion propagator refers both to the quantity (13) and the directed line connecting two vertices. It has been pointed out that 'self-energy' operator is a more precise term in this case, as the operator M contains kinetic as well as mass terms, but 'mass operator' is used here and elsewhere (Ritus 1970) because it corresponds to the mass operator in field-free QED. When radiative corrections to the fermion propagator are taken into account, the 'bare' propagator is to be replaced by the 'dressed' propagator, represented by the double line in Fig. 3.

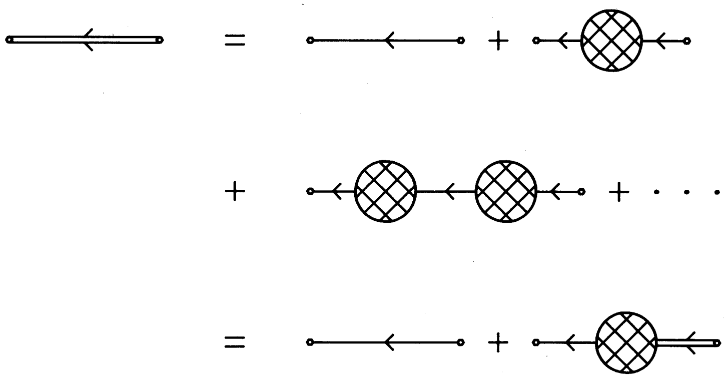

Fig. 3. Expansion of the 'dressed' propagator (left-hand side) in terms of the 'bare' propagator and the mass operator.

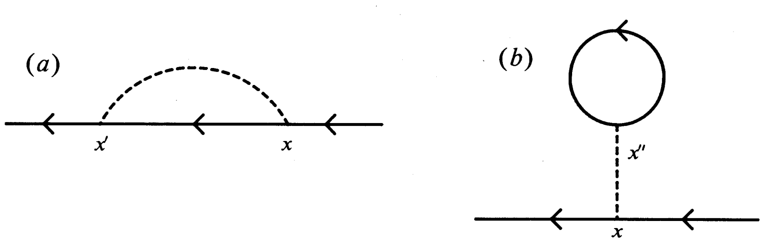

Fig. 4. Subdiagrams contributing to the lowest order mass operator.

To lowest order in the fine structure constant, the mass operator in the coordinate representation is given by (Ritus 1970, 1972)

$$-\mathrm{i}\, M(x', x) = -e^2\gamma^\mu\, G(x', x)\gamma^\nu\, D_{\mu\nu}(x'-x)$$

$$+e^2\delta^4(x-x')\int \mathrm{d}^4x''\; \gamma^\mu\, D_{\mu\nu}(x-x'')\,\mathrm{Tr}\{\gamma^\nu\, G(x'', x'')\}\,, \quad (23)$$

where the two terms refer to the subdiagrams of Figs 4*a* and 4*b*, respectively. Now Fig. 4*b* is the tadpole diagram which in a vacuum or a homogeneous medium makes a zero contribution to the amplitude when the operator representing the interaction of the fermion with the radiation field is normally ordered. This subdiagram may thus be ignored and only Fig. 4*a* and the corresponding term of (23) is considered further. In the spinor representation, this term of the mass operator has the form

$$-\mathrm{i}\, \hat{M}_{r'r}(E', E) = \int \mathrm{d}^4x\, \mathrm{d}^4x'\; \exp\{-\mathrm{i}(Et-E't')\}$$

$$\times\, V^\dagger(x';\, P_z', n', g')\,\{-\mathrm{i}\, M(x', x)\}\; V(x;\, P_z, n, g)\,, \quad (24)$$

where $r \equiv \{P_z, n, g\}$ and so on. This form is particularly useful because of the diagonality properties of the mass operator described below.

A free electron or positron in a magnetic field may be described by the quantum numbers $\epsilon, \sigma, n, P_z, g$. For a particle in a virtual state, another quantum number E is required, denoting the energy of the virtual state (which is in general not equal to $\epsilon \mathscr{E}_q$). The diagonality properties of the mass operator, which is in principle a function of the two sets of quantum numbers associated with the incoming and outgoing fermion edges, are given by the following theorem:

Theorem. The mass operator in a homogeneous, uniformly magnetised medium is diagonal with respect to the quantum numbers n, P_z, g and E, to all orders in the fine structure constant.

Corollary. The mass operator in the spinor representation may be written in terms of a reduced mass operator

$$-\mathrm{i}\,\hat{M}_{r'r}(E', E) = -\mathrm{i}\,\hat{M}(E, P_z, n)\,2\pi\delta(E-E')(2\pi/L_z)\delta(P_z-P_z')\delta_{nn'}\,\delta_{gg'}\,. \tag{25}$$

The only non-trivial part of this theorem is diagonality with respect to n. This is proven by separating out that part of the mass operator in the momentum representation which depends on the azimuthal angles of the virtual photons and appealing to rotational invariance around the axis of the magnetic field. This yields a factor of the form

$$\exp\{-\mathrm{i}(n'-n)\Psi\}\,, \tag{26}$$

and the required result follows when one integrates over the azimuthal angle of the photon.

As the mass operator is diagonal in E, P_z, n and g while being otherwise independent of the positional quantum number, only the reduced mass operator (defined by 25) is of further interest. The spinor representation is thus particularly appropriate because it is explicitly a function only of the first three parameters. Unless otherwise specified, from this point it is assumed that the mass operator is in the reduced form.

Modification to the Propagator

As noted above, in the spinor representation a fermion edge connecting two vertices of a Feynman diagram contributes a factor of $\mathrm{i}\,\hat{G}(E, P_z, n)$ to the amplitude, where E and P_z are the energy and parallel momentum of the virtual state and one sums over the Landau quantum number n. When the self-interaction is to be included (cf. Fig. 2 and Fig. 3), the propagator is modified to become

$$\begin{aligned}\mathrm{i}\,\hat{G}'(E, P_z, n) &= \mathrm{i}\,\hat{G}(E, P_z, n) + \mathrm{i}\,\hat{G}(E, P_z, n)\{-\mathrm{i}\,\hat{M}(E, P_z, n)\}\,\mathrm{i}\,\hat{G}(E, P_z, n)\\ &\quad + \mathrm{i}\,\hat{G}(E, P_z, n)\{-\mathrm{i}\,\hat{M}(E, P_z, n)\}\,\mathrm{i}\,\hat{G}(E, P_z, n)\\ &\quad \times\{-\mathrm{i}\,\hat{M}(E, P_z, n)\}\,\mathrm{i}\,\hat{G}(E, P_z, n) + \dots\\ &= [\{\mathrm{i}\,\hat{G}(E, P_z, n)\}^{-1} + \mathrm{i}\,\hat{M}(E, P_z, n)]^{-1}\,,\end{aligned} \tag{27}$$

where the result of the last line is exact. On substituting (22) into the last line, one obtains the modified exact electron propagator in the spinor representation

$$\mathrm{i}\,\hat{G}'(E, P_z, n) = \mathrm{i}\{\gamma^{\mu}\Pi_{\mu} - m - \hat{M}(E, P_z, n)\}^{-1}. \tag{28}$$

The 'dressed' propagator is thus in the same form as the 'bare' propagator, with an additional term which corresponds to a mass correction. It would appear that the 'dressed' wavefunction should satisfy a Dirac equation of the form

$$\{\gamma^{\mu}\Pi_{\mu} - m - \hat{M}(E, P_z, n)\}\, w_{\epsilon\sigma}(P_z, n) = 0, \tag{29}$$

where $E = \epsilon\mathscr{E}_q'$ is the modified energy, but in fact $w_{\epsilon\sigma}$ does not represent a wavefunction. It will be seen in Section 3 how a further transformation is required to obtain the correct wavefunction with the self-interaction included exactly.

Evaluation of the Mass Operator

The results derived up to this point are valid, not only in a vacuum, but also in homogeneous media. From this point, the effect of the medium is neglected. On performing the explicit calculation for conditions in a vacuum, the mass operator in the spinor representation is found to have the general form

$$\begin{aligned}\hat{M}(E, P_z, n) = (\alpha/2\pi)\{\, & m\hat{S}I_1 + p_n\,\hat{V}^2 I_2 + (P_z\,\hat{V}^3 - E\,\hat{V}^0)I_3 \\ & + m\hat{T}^{12}I_4 - (P_z\,\hat{A}^0 - E\hat{A}^3)I_5\},\end{aligned} \tag{30}$$

where the coefficients I_m are scalar functions only of n and of the Lorentz invariant distance between the particle energy and the mass shell $E^2 - \mathscr{E}_q^2$. The first four terms in (30) contain the same Dirac matrices which appear in the electron propagator (18), while the last three terms involve the matrices

$$m\hat{T}^{12} = (m/2\,U^{\frac{1}{2}})F_{\mu\nu}\,\hat{T}^{\mu\nu}, \tag{31a}$$

$$P_z\,\hat{A}^0 - E\hat{A}^3 = (1/2\,U^{\frac{1}{2}})F_{\mu\nu}\,\hat{T}^{\mu\nu}(E\,\hat{V}^0 - P_z\,\hat{V}^3), \tag{31b}$$

which are Lorentz invariant under boosts parallel to the magnetic field. In (31), U is the Lorentz invariant describing the field strength $B^2 - E^2$. It seems likely that if higher order terms in the fine structure constant are included in the mass operator, then (30) would be modified only by changes to the coefficients I_m, but this has not yet been proven. To lowest order in the fine structure constant, however, the problem of calculating the mass operator reduces to finding the coefficients I_m.

The mass operator is now calculated using the electron propagator in the Géhéniau–Demeur form (I.82). The (unreduced) mass operator is given by

$$\begin{aligned}\hat{M}_{r'r}(E', E) = -\mathrm{i}\,e^2 \int & \mathrm{d}^4x\,\mathrm{d}^4x'\,\exp\{-\mathrm{i}(Et - E't')\}\,D_{\mu\nu}(x' - x) \\ & \times V^{\dagger}(x';\, P_z', n', g')\,\gamma^{\mu}\,G(x', x)\,\gamma^{\nu}\,V(x;\, P_z, n, g).\end{aligned} \tag{32}$$

The photon propagator is chosen in the form

$$D^{\lambda\nu}(x) = -\frac{g^{\lambda\nu}}{8\pi^2\epsilon_0}\int_0^{\infty} \mathrm{d}\mu\,\exp(-\tfrac{1}{2}\mathrm{i}\,\mu x^2). \tag{33}$$

By writing

$$L = eB/m^2 \tag{34}$$

as the ratio of the field strength to the critical field, and changing the variables of integration to

$$w = m^2/2\lambda, \qquad u = \mu/(\mu+\lambda), \tag{35a, b}$$

where w is assumed to have a small negative imaginary part in order that the integral converges, i.e. $w = |w|(1-\mathrm{i}0)$, the space–time integrations in (32) can be performed. The coefficients defining the mass operator are given by

$$I_m = L\int_0^\infty \mathrm{d}w \int_0^1 \mathrm{d}u\ \exp[-\mathrm{i}\,w\{u-(1-u)(E^2-\mathscr{E}_q^2)/m^2\}]\frac{1}{a^2+b^2}$$

$$\times\left(\frac{a-\mathrm{i}\,b}{a+\mathrm{i}\,b}\exp\{2\mathrm{i}\,Lw(1-u)\}\right)^n K_m; \tag{36}$$

$$K_1 = 2a, \qquad K_2 = b\sec Lw \operatorname{cosec} Lw, \tag{37a, b}$$

$$K_3 = (1-u)(a-b\tan Lw), \tag{37c}$$

$$K_4 = 2\mathrm{i}\,b, \qquad K_5 = \mathrm{i}(1-u)(b+a\tan Lw), \tag{37d, e}$$

with

$$a = u\tan Lw + Lw(1-u), \tag{38a}$$

$$b = Lw(1-u)\tan Lw. \tag{38b}$$

This form of the mass operator, when written in the momentum representation, generalises the result of Constantinescu (1972*b*), who derived the matrix element of the operator for the special case $\epsilon' = \epsilon$, $E = \mathscr{E}_q$, using the Johnson–Lippmann spin wavefunctions for $\sigma = \sigma'$. As these wavefunctions are the same as the magnetic moment wavefunctions only in the frame $P_z = 0$, (36) reproduces Constantinescu's result only in the rest frame.

The convergence or otherwise of the integrals in equation (36) has been examined by Constantinescu (1972*b*). In his analysis, the integrand is expanded in powers of u and w, so the mass operator coefficients may be written in the form

$$I_m = \sum_{x=0}^{\infty}\ \sum_{y=\max[0,x-1]}^{\infty}\int_0^1 \mathrm{d}u\int_0^\infty \frac{\mathrm{d}w}{w}\ u^x(\mathrm{i}\,w)^y\exp(-\mathrm{i}\,uw)I_m^{x,y}. \tag{39}$$

There are three distinct classes of terms:

(i) Terms with $y = 0$ which lead to a divergent integral in w, and require renormalisation. These are the 'zero field' terms in the notation of Constantinescu.

(ii) Terms with $x-y \geqslant 0$, with $y \neq 0$, giving a finite contribution to the mass operator.

(iii) Terms with $x-y < 0$. Members of this class have infrared divergences when integrated with respect to u. These divergences are not a characteristic of the mass operator, but of the method of expansion in powers of u and w. For a fixed order of u, the sum of all orders of w leads to a finite result.

Hence only terms of class (i), with $y = 0$, lead to real divergences. [A referee has suggested that this is a special case of the general result proved by 't Hooft (1971).]

Classical Limit

In the classical limit the perpendicular momentum $p_\perp = (2neB)^{\frac{1}{2}}$ is kept constant and the limit as $n \to \infty$ is taken. The coefficients are found to first order in L. All divergent terms of class (i) are retained by this procedure, as they are of zero order in L. This is termed the classical limit because it is under the conditions of weak fields and large n that classical physics may be expected to remain a valid approximation. The result is

$$I_m = L \int_0^\infty \mathrm{d}w \int_0^1 \mathrm{d}u \, \exp[-\mathrm{i}\, w\{u-(1-u)(E^2-\mathscr{E}_q^2)/m^2\}]K'_m; \tag{40}$$

$$K'_1 = 2, \qquad K'_2 = (1-u), \qquad K'_3 = (1-u), \tag{41a–c}$$

$$K'_4 = 2\mathrm{i}\, Lw(1-u), \qquad K'_5 = \mathrm{i}\, Lw(2-u). \tag{41d, e}$$

By inspection $K'_{1,2,3}$ belong to class (i), while $K'_{4,5}$ belong to classes (ii) and (iii). Hence to first order in the fine structure constant, terms in $\hat{S}$, $\hat{V}^0$, $\hat{V}^2$ and $\hat{V}^3$ are divergent, while the terms in $\hat{T}^{12}$, $\hat{A}^3$ and $\hat{A}^0$ are finite once the infrared divergences are regularised. The renormalisation of the former terms and the interpretation and regularisation of the latter is discussed in Section 4.

Momentum Representation of the Mass Operator

The mass operator in the momentum representation is related to the spinor representation by

$$-\mathrm{i}\, M_{q'q}(E', E) = \bar{u}_{\epsilon'\sigma'}(P'_z, n')\{-\mathrm{i}\, \hat{M}_{r'r}(E', E)\}\, u_{\epsilon\sigma}(P_z, n). \tag{42}$$

Using (25), (36) and (37), the reduced form is found to be

$$\begin{aligned}
M_{\epsilon'\sigma',\epsilon\sigma}(E, P_z, n) &= \bar{u}_{\epsilon'\sigma'}(P_z, n)\, \hat{M}(E, P_z, n)\, u_{\epsilon\sigma}(P_z, n) \\
&= \frac{\alpha}{2\pi}\, \frac{1}{\mathscr{E}_q \mathscr{E}_q^0}\Big(\delta_{\epsilon\epsilon'}(\delta_{\sigma\sigma'}\, \epsilon[\mathscr{E}_q^0\{m^2 I_1 + p_n^2 I_2 + (P_z^2 - \epsilon E\mathscr{E}_q)I_3\} \\
&\qquad + m\sigma\{(\mathscr{E}_q^0)^2 I_4 + (P_z^2 - \epsilon E\mathscr{E}_q)I_5\}] \\
&\qquad + \delta_{\sigma-\sigma'}\, \epsilon\sigma\, \mathrm{i}\, p_n\, P_z(\mathscr{E}_q - \epsilon E)I_5) \\
&\qquad + \delta_{\epsilon-\epsilon'}(\delta_{\sigma\sigma'}\, P_z[m\mathscr{E}_q^0(I_5 - I_4) \\
&\qquad + \sigma\{(\mathscr{E}_q^0)^2 I_3 - m^2 I_1 - p_n^2 I_2\}] \\
&\qquad + \delta_{\sigma-\sigma'}\, \mathrm{i}\, p_n\{\epsilon E\mathscr{E}_q^0\, I_5 - m(I_1 - I_2)\})\Big),
\end{aligned} \tag{43}$$

where $\mathscr{E}_q^0 = (m^2 + 2neB)^{\frac{1}{2}}$. Now let us consider the effect of the self-interaction on a free particle. In this case, which is also the one considered almost exclusively by other authors, we have $\epsilon' = \epsilon$, $E = \epsilon\mathscr{E}_q$. The amplitude for the self-energy diagram, which is just the matrix element of the mass operator to within a phase factor, is then

$$M_{\epsilon\sigma'\epsilon\sigma}(\epsilon\mathscr{E}_q,\, P_z,\, n) = \frac{\alpha}{2\pi}\,\frac{\epsilon}{\mathscr{E}_q}\,\delta_{\sigma\sigma'}$$

$$\times\{\, m^2(I_1 - I_3) + p_n^2(I_2 - I_3) + \sigma m\mathscr{E}_q^0(I_4 - I_5)\}\,. \tag{44}$$

We note that the amplitude for the particle to change from one spin state to another is zero. This property is unique to the magnetic moment eigenfunctions, and was used by Herold *et al.* (1982) to derive these wavefunctions independently of the operator. Even for applications where the self-interaction is only of passing interest, the magnetic moment spin wavefunctions are the most appropriate ones to use. In the following section, a more sophisticated treatment is presented which incorporates into the theory all terms of the mass operator, including the $\delta_{\epsilon-\epsilon'}$ terms in (43) which have not yet been dealt with.

4. Renormalisation

In this section, the renormalisation of the mass operator is considered, and the Dirac equation including the mass operator is written down in a form which is free of ultraviolet divergences. This leads to finite corrections to the mass and perpendicular momentum of the magnetised electron, and to modifications to the wavefunctions, propagator and the vertex function. The dressed forms of these quantities are derived in terms of the coefficients I_m which were defined in the previous section.

'Bare' quantum electrodynamics is based on the Dirac equation. When the mass operator is included, one obtains a Dirac-like relation given by (29) in the spinor representation. Substituting the momentum operator (18) and using the general form for the mass operator in a magnetic field (30), one obtains the result

$$\bigg\{(E\hat{V}^0 - P_z\,\hat{V}^3)\Big(1+\frac{\alpha I_3}{2\pi}\Big) - m\hat{S}\Big(1+\frac{\alpha I_1}{2\pi}\Big) - p_n\,\hat{V}^2\Big(1+\frac{\alpha I_2}{2\pi}\Big)$$

$$- m\hat{T}^{12}\,\frac{\alpha I_4}{2\pi} - (E\hat{A}^3 - P_z\,\hat{A}^0)\frac{\alpha I_5}{2\pi}\bigg\}\, w_{\epsilon\sigma}(P_z,\, n) = 0\,. \tag{45}$$

Let us consider the terms I_1, I_2, I_3, which are the only ones containing ultraviolet divergences. These divergences may be removed by renormalisation of the electron mass, the electronic charge, and the normalisation of the electron wavefunction respectively. The divergent parts are present in the zero field limit, and using the expansion (39) have $y = 0$, which means that the renormalised quantities are the same in the magnetised and unmagnetised vacuum. This procedure is thus equivalent to the more familiar zero field renormalisation, and the divergent terms may be dropped and the field-free 'dressed' values for the mass and charge should be substituted.

Once the divergent parts are removed, $I_{1,2,3}$ contain finite parts (denoted by $I'_{1,2,3}$) which are zero in the limit of zero field. According to equation (40), these parts are at least second order in the magnetic field parameter L, and they represent finite corrections to the measured electron mass, charge and wavefunction. The terms

$I_{4,5}$, on the other hand, represent finite corrections to the Dirac equation which are first order in L. Using the renormalised mass and charge, the relation (45) may be rewritten in the form

$$[E\hat{V}^0 - P_z\hat{V}^3 - mf_1\hat{S} - p_n f_2 \hat{V}^2 + (F_{\mu\nu}\hat{T}^{\mu\nu}/2U^{\frac{1}{2}})\{(E\hat{V}^0 - P_z\hat{V}^3)f_5 - m\hat{S}f_4\}]w_{\epsilon\sigma}(P_z, n) = \hat{X}w_{\epsilon\sigma}(P_z, n) = 0, \tag{46}$$

where

$$f_{1,2} = \frac{1+\alpha I'_{1,2}/2\pi}{1+\alpha I'_3/2\pi}, \qquad f_{4,5} = \frac{\alpha I_{4,5}/2\pi}{1+\alpha I'_3/2\pi}. \tag{47a, b}$$

Now equation (46) is manifestly covariant under Lorentz boosts parallel to the field axis, but is not in the form of a Dirac equation because the energy appears in the additional term. This can be corrected by writing

$$\hat{Z}u'_{\epsilon\sigma}(P_z, n) = 0 \tag{48}$$

as the modified Dirac equation, with

$$\hat{Z} = \hat{Y}\hat{X}\hat{Y}, \qquad u'_{\epsilon\sigma}(P_z, n) = \hat{Y}^{-1}w_{\epsilon\sigma}(P_z, n), \tag{49a, b}$$

$$\hat{Y} = \mathrm{diag}\{(1+f_5)^{-\frac{1}{2}}, (1-f_5)^{-\frac{1}{2}}, (1+f_5)^{-\frac{1}{2}}, (1-f_5)^{-\frac{1}{2}}\}. \tag{50}$$

Then the matrix $\hat{Z}$ has the form

$$\hat{Z} = E\hat{V}^0 - P_z\hat{V}^3 - m'\hat{S} - p'_n\hat{V}^2 - \Delta\hat{T}^{12}, \tag{51}$$

with the new parameters given by

$$m' = \tfrac{1}{2}m\left(\frac{f_1+f_4}{1+f_5} + \frac{f_1-f_4}{1-f_5}\right), \tag{52}$$

$$\Delta = \tfrac{1}{2}m\left(\frac{f_1+f_4}{1+f_5} - \frac{f_1-f_4}{1-f_5}\right), \tag{53}$$

$$p'_n = p_n f_2(1-f_5^2)^{-\frac{1}{2}}. \tag{54}$$

In the limit where the self-interaction is zero, this reproduces the normal Dirac equation. For the modified Dirac equation (48) to have non-trivial solutions, one requires $\det\hat{Z} = 0$, i.e.

$$\{E^2 - P_z^2 - p_n'^2 - (m'+\Delta)^2\}\{E^2 - P_z^2 - p_n'^2 - (m'-\Delta)^2\} - 4p_n'^2\Delta^2 = 0. \tag{55}$$

The solutions for the free particle energy may then be written in the form

$$E = \epsilon\mathscr{E}'_q, \qquad \mathscr{E}'_q = \{(\mathscr{E}'^0_q)^2 + P_z^2\}^{\frac{1}{2}}; \tag{56a, b}$$

$$\mathscr{E}'^0_q = \mathscr{E}_n \pm \Delta, \qquad \mathscr{E}_n = (m'^2 + p_n'^2)^{\frac{1}{2}}. \tag{56c, d}$$

We note that the particle energy satisfies the relation (2), as required by relativistic covariance. The connection between the sign of Δ and the spin quantum number σ has not yet been established, and this will be performed below.

Wavefunctions

The four independent solutions of (48) are non-degenerate as there are four different energies (56). Hence the solutions of the modified Dirac equation and the eigenfunctions of the corresponding Hamiltonian are completely specified. The Hamiltonian operator corresponding to (48) is given by

$$\hat{H} = E - \gamma^0 \hat{Z}, \tag{57}$$

which determines the Schrödinger equation

$$\hat{H}\, u'_{\epsilon\sigma}(P_z, n) = \epsilon \mathscr{E}'_q\, u'_{\epsilon\sigma}(P_z, n)\,. \tag{58}$$

As the theory is covariant under Lorentz boosts parallel to the magnetic field, it is convenient to first seek solutions in the frame $P_z = 0$, and then Lorentz transform to generate the general solutions. Writing u'_i for the elements of $u'_{\epsilon\sigma}(P_z, n)$, (48) yields the relations

$$\{\pm(E-\Delta) - m'\}\, u'_{1,4} + \mathrm{i}\, p'_n\, u'_{4,1} = 0\,, \tag{59a}$$

$$\{\pm(E+\Delta) - m'\}\, u'_{2,3} - \mathrm{i}\, p'_n\, u'_{3,2} = 0\,. \tag{59b}$$

Substituting for the energy (56), one may write down the relations between the u'_i for $\epsilon = \pm 1$ as

$$u'_{4,1} = \frac{\mathrm{i}\, p'_n}{\mathscr{E}'^0_q + m' \mp \Delta}\, u'_{1,4}\,, \qquad u'_{3,2} = \frac{-\mathrm{i}\, p'_n}{\mathscr{E}'^0_q + m' \pm \Delta}\, u'_{2,3}\,. \tag{60a, b}$$

In order to determine the spin of the wavefunctions, we recall that in the limit of the self-interaction tending to zero, the magnetic moment eigenfunctions (I.45) should be reproduced. In particular, the ground state should have $\sigma = -1$. The spinors $u'_{\epsilon\sigma}(0, n)$ then have the form

$$u'_{++} \approx \begin{bmatrix} \mathscr{E}'^0_q + m' - \Delta \\ 0 \\ 0 \\ \mathrm{i}\, p'_n \end{bmatrix}, \qquad u'_{+-} \approx \begin{bmatrix} 0 \\ \mathscr{E}'^0_q + m' + \Delta \\ -\mathrm{i}\, p'_n \\ 0 \end{bmatrix}, \tag{61a, b}$$

$$u'_{-+} \approx \begin{bmatrix} 0 \\ -\mathrm{i}\, p'_n \\ \mathscr{E}'^0_q + m' - \Delta \\ 0 \end{bmatrix}, \qquad u'_{--} \approx \begin{bmatrix} \mathrm{i}\, p'_n \\ 0 \\ 0 \\ \mathscr{E}'^0_q + m' + \Delta \end{bmatrix}. \tag{61c, d}$$

Substituting (61) into (48) and using the expressions (56) for the energy of the particle, one obtains the relation

$$\mathscr{E}_q'^0 = \mathscr{E}_n + \sigma\Delta . \tag{62}$$

The spinors (61) may now be Lorentz transformed in order to obtain the solutions of (48) for arbitrary parallel momentum. Writing the results in terms of the signed momentum $p_z = \epsilon P_z$ which is in the same sense as the velocity, the elements of $u'_{\epsilon\sigma}(\epsilon p_z, n)$ are

$$u'_{++} = W_q'^{-\frac{1}{2}} \begin{bmatrix} (\mathscr{E}_q'+\mathscr{E}_q'^0)(\mathscr{E}_n+m') \\ -\mathrm{i}\,p_n' p_z \\ p_z(\mathscr{E}_n+m') \\ \mathrm{i}\,p_n'(\mathscr{E}_q'+\mathscr{E}_q'^0) \end{bmatrix}, \qquad u'_{+-} = W_q'^{-\frac{1}{2}} \begin{bmatrix} -\mathrm{i}\,p_n' p_z \\ (\mathscr{E}_q'+\mathscr{E}_q'^0)(\mathscr{E}_n+m') \\ -\mathrm{i}\,p_n'(\mathscr{E}_q'+\mathscr{E}_q'^0) \\ -p_z(\mathscr{E}_n+m') \end{bmatrix}, \tag{63a, b}$$

$$u'_{-+} = W_q'^{-\frac{1}{2}} \begin{bmatrix} p_z(\mathscr{E}_n+m') \\ -\mathrm{i}\,p_n'(\mathscr{E}_q'+\mathscr{E}_q'^0) \\ (\mathscr{E}_q'+\mathscr{E}_q'^0)(\mathscr{E}_n+m') \\ \mathrm{i}\,p_n' p_z \end{bmatrix}, \qquad u'_{--} = W_q'^{-\frac{1}{2}} \begin{bmatrix} \mathrm{i}\,p_n'(\mathscr{E}_q'+\mathscr{E}_q'^0) \\ -p_z(\mathscr{E}_n+m') \\ \mathrm{i}\,p_n' p_z \\ (\mathscr{E}_q'+\mathscr{E}_q'^0)(\mathscr{E}_n+m') \end{bmatrix}; \tag{63c, d}$$

$$W_q' = 4\mathscr{E}_q'\mathscr{E}_n(\mathscr{E}_q'+\mathscr{E}_q'^0)(\mathscr{E}_n+m') . \tag{63e}$$

In the weak field limit, when the effect of the self-interaction tends to zero, these yield the magnetic moment spin wavefunctions. The spin operator is obtained from the magnetic moment operator by making the substitutions $m \to m'$, $p_n \to p_n'$ into the defining relation (I.42)

$$\hat{\mu}_z = \Sigma_z + (1/m')\rho_2(\boldsymbol{\Sigma}\times\boldsymbol{\Pi}')_z . \tag{64}$$

Electron Propagator

When the self-interaction is neglected, the electron propagator may be calculated from the Dirac equation in the spinor representation using the relation (21). If the self-interaction is included, this relation takes the form

$$\hat{Z}\,G'(E, P_z, n) = (\gamma^\mu \Pi_\mu' - m' - \Sigma_z\,\Delta)\,G'(E, P_z, n) = 1 , \tag{65}$$

where the replacement $p_n \to p_n'$ has been made in the construction of Π^μ. Inverting the operator $\hat{Z}$, and writing $\mathscr{E}'_q = \mathscr{E}'_{r,\sigma}$, one obtains

$$G'(E, P_z, n) = (E^2 - \mathscr{E}'^2_{r,+} + \mathrm{i}\,0)^{-1}(E^2 - \mathscr{E}'^2_{r,-} + \mathrm{i}\,0)^{-1}$$

$$\times \begin{bmatrix} A^{++} & -B & -C^{+} & D^{--} \\ B & A^{-+} & D^{+-} & C^{-} \\ C^{+} & D^{-+} & A^{+-} & -B \\ D^{++} & -C^{-} & B & A^{--} \end{bmatrix}; \tag{66}$$

$$A^{ab} = (E^2 - P_z^2 + \Delta^2 - \mathscr{E}_n^2)m' + a\Delta(E^2 - P_z^2 - \Delta^2 + \mathscr{E}_n^2)$$

$$+ bE(E^2 - P_z^2 - \Delta^2 - \mathscr{E}_n^2 + 2a\Delta m'), \tag{67a}$$

$$B = -\mathrm{i}\, p'_n P_z 2\Delta, \tag{67b}$$

$$C^a = P_z(E^2 - P_z^2 - \Delta^2 - \mathscr{E}_n^2 + 2a\Delta m'), \tag{67c}$$

$$D^{ab} = \mathrm{i}\, p'_n\{2\Delta E + ab(E^2 - P_z^2 + \Delta^2 - E_n^2)\}, \tag{67d}$$

where the imaginary part of the denominator is determined by Feynman's rule for avoiding the poles. The result (66) may also be obtained by using the result (13) with the modified wavefunctions:

$$G'(E, P_z, n) = \sum_{\epsilon\sigma} \frac{u'_{\epsilon\sigma}(P_z, n)\,\bar{u}'_{\epsilon\sigma}(P_z, n)}{E - \epsilon(\mathscr{E}'_{r,\sigma} - \mathrm{i}\,0)}. \tag{68}$$

Vertex Function

The vertex function is unaltered by the self-interaction in the spinor representation. In the momentum representation, the modified vertex function may be found by substituting the elements of $u'_{\epsilon\sigma}$ into (I.46). Expanding the vertex function in terms of the basis vectors

$$T^\mu = (1,0,0,0), \qquad B^\mu = (0,0,0,1), \qquad E^\mu_\pm = (0,1,\pm\mathrm{i},0)\exp(\mp\mathrm{i}\,\phi), \tag{69}$$

the modified vertex function is given by

$$[\Gamma'_{r'r}(k)]^\mu = \left(\frac{(\mathscr{E}'_q + \mathscr{E}'^0_q)}{2\mathscr{E}'_q}\,\frac{(\mathscr{E}_n + m')}{2\mathscr{E}_n}\,\frac{(\mathscr{E}'_{q'} + \mathscr{E}'^0_{q'})}{2\mathscr{E}'_{q'}}\,\frac{(\mathscr{E}_{n'} + m')}{2\mathscr{E}_{n'}}\right)^{\frac{1}{2}} \{\mathrm{i}\exp(\mathrm{i}\,\phi)\}^{-\lambda}$$

$$\times \big(\delta_{\sigma\sigma'}[\{\delta_{\epsilon\epsilon'}(T^\mu a^+ + \epsilon B^\mu b^+) - \sigma\delta_{\epsilon-\epsilon'}(\epsilon T^\mu b^- - B^\mu a^-)\}(\mathrm{J}^l_\lambda + \rho_n\rho_{n'}\mathrm{J}^{l+\sigma}_\lambda)$$

$$+ (-\epsilon\delta_{\epsilon\epsilon'}\, a^- + \sigma\delta_{\epsilon-\epsilon'}\, b^+)(E^\mu_\sigma \rho_{n'} \mathrm{J}^l_{\lambda+\sigma} + E^\mu_{-\sigma}\rho_n \mathrm{J}^{l+\sigma}_{\lambda-\sigma})]$$

$$- \mathrm{i}\,\delta_{\sigma-\sigma'}[\{\delta_{\epsilon\epsilon'}(T^\mu b^+ + \epsilon B^\mu a^+) - \sigma\delta_{\epsilon-\epsilon'}(\epsilon T^\mu a^- - B^\mu b^-)\}$$

$$\times (\rho_n \mathrm{J}^{l+\sigma}_\lambda - \rho_{n'}\mathrm{J}^l_\lambda) + (-\epsilon\delta_{\epsilon\epsilon'}\, b^- + \sigma\delta_{\epsilon-\epsilon'}\, a^+)$$

$$\times (E^\mu_\sigma \mathrm{J}^l_{\lambda+\sigma} - E^\mu_{-\sigma}\rho_n\rho_{n'}\mathrm{J}^{l+\sigma}_{\lambda-\sigma})]\big); \tag{70}$$

$$\lambda = n' - n, \qquad l = n - \tfrac{1}{2}(\sigma + 1), \tag{71a, b}$$

$$a^{\pm} = 1 \pm \frac{P_z}{(\mathscr{E}_q' + \mathscr{E}_q'^0)} \frac{P_z'}{(\mathscr{E}_{q'}' + \mathscr{E}_{q'}'^0)}, \tag{71c}$$

$$b^{\pm} = \frac{P_z'}{(\mathscr{E}_{q'}' + \mathscr{E}_{q'}'^0)} \pm \frac{P_z}{(\mathscr{E}_q' + \mathscr{E}_q'^0)}, \tag{71d}$$

$$\rho_n = \frac{p_n'}{\mathscr{E}_n + m'}, \qquad \rho_{n'} = \frac{p_{n'}'}{\mathscr{E}_{n'} + m'}, \tag{71e, f}$$

where the argument of all the J functions is $k_\perp^2/2eB$. Transition amplitudes taking the self-interaction into account can be calculated in the momentum representation by using the rules given in (III) with the following modifications:

(i) The modified vertex function $[\Gamma'_{r_2 r_1}(k)]^\mu$ given by (70) is to be used instead of the 'bare' vertex function (I.57).

(ii) The energy $\mathscr{E}_q'$ including the self-energy is to be used in the denominator of the electron propagator and in the energy conservation relation for each open fermion line.

We note that the vertex function will be further modified by the inclusion of the radiative vertex correction. This correction in a magnetic field has not yet been calculated.

5. Weak Field Limit

Some useful results may be obtained by retaining only those terms in the mass operator which are linear in the external field strength L. According to the analysis in the previous section, one has

$$f_{1,2} \approx 1 + \mathrm{O}(L^2), \qquad f_{4,5} \approx \mathrm{O}(L), \tag{72a, b}$$

and hence to first order in L the quantities derived above are given by

$$m' \approx m, \qquad p_n' \approx p_n, \qquad \Delta \approx (\alpha m/2\pi)(I_4 - I_5), \tag{73a–c}$$

$$\mathscr{E}_n \approx \mathscr{E}_q^0, \qquad \mathscr{E}_q'^0 \approx \mathscr{E}_q^0 + \sigma\Delta, \qquad \mathscr{E}_q' \approx \mathscr{E}_q + \sigma\Delta(\mathscr{E}_q^0/\mathscr{E}_q), \tag{73d–f}$$

and hence all corrections to the wavefunctions, propagators, and so on may be expressed in terms of the parameter Δ. In particular, the finite self-interaction correction to the electron rest mass and perpendicular momentum are second order in L, which is important as any first order corrections would be large enough to be detectable experimentally (yet they have not been detected). In this approximation, Δ is given by

$$\Delta = \mathrm{i}\frac{\alpha m}{2\pi} L \int_0^1 \mathrm{d}u \int_0^\infty \mathrm{d}w \, \exp(-\mathrm{i}\, uw) u(1-u) = \frac{\alpha m}{4\pi} L = \frac{\alpha \Omega_{\mathrm{c}}}{4\pi}, \tag{74}$$

where Ω_c is the cyclotron frequency. We note that to first order in L, Δ is independent of the Landau quantum number n. If this value of Δ is inserted into the expression (56) for the energy of a free electron, the value (6) for the energy found by Sokolov and Ternov (1968) is reproduced. Then Δ represents an 'anomalous' correction to the magnetic moment of

$$\mu' = \mu + \mu_{\text{anom}} = (e/2m)(1 + \alpha/2\pi), \tag{75}$$

which agrees with the accepted value.

Spin Flip Radiation

In almost all cases, the self-interaction modification makes only small corrections to the calculated amplitude. One process which is allowed only when the self-interaction is included involves an electron changing its spin while remaining in the same Landau level, accompanied by the emission or absorption of a photon. It is instructive to consider this process in detail.

Suppose the electron is in an initial state with quantum numbers $n, p_z = 0$, $\sigma = +1$. In principle, by the emission of a photon, the electron can change to a final state with $n' = n$, $\sigma = -1$. Owing to the anomalous magnetic moment, the final state has a lower energy. Calculating the rate of transition using the modified vertex function, one obtains

$$\begin{aligned}\overline{w}_{\text{fi}} &= \int \mathrm{d}^3 k \, \mathrm{d}p'_z \sum_M \frac{\alpha}{2\pi\omega} \, \delta(E_{\text{f}} - E_{\text{i}})\delta(k_z - p'_z)\,|[\Gamma'_{r'r}(k)]^\mu [e^*_M(k)]_\mu|^2 \\ &= \alpha \int \omega \, \mathrm{d}\omega \, \mathrm{d}(\cos\theta) \, \frac{\mathscr{E}'_{q'}}{\mathscr{E}'_q - \omega \sin^2\theta} \, \frac{\mathscr{E}'_{q'} + \mathscr{E}'^0_{q'}}{2\mathscr{E}'_{q'}} \, \frac{1}{4\mathscr{E}^2_n} \{\mathrm{J}_1^{n-1}(k_\perp^2/2eB)\}^2 \\ &\quad \times \left(\omega^2 \sin^4\theta - \frac{4\omega^2 \mathscr{E}_n}{\mathscr{E}'_{q'} + \mathscr{E}'^0_{q'}} \cos^2\theta \sin^2\theta \right. \\ &\quad \left. + \frac{4\omega^2}{(\mathscr{E}'_{q'} + \mathscr{E}'^0_{q'})^2} (\mathscr{E}^2_n \cos^4\theta + m'^2 \cos^2\theta) \right) \delta(\omega - \omega_0),\end{aligned} \tag{76}$$

where

$$\omega_0 = (1/\sin^2\theta)[\mathscr{E}'^0_q - \{(\mathscr{E}'^0_q)^2 - 4\mathscr{E}_n \Delta \sin^2\theta\}] \tag{77}$$

is the frequency of the emitted radiation. Now ω_0 varies from $2\Delta(1 - \Delta/\mathscr{E}'^0_q)$ at $\theta = 0, \pi$ to 2Δ at $\theta = \frac{1}{2}\pi$, and so may be approximated by the latter value throughout the range of emission values. Thus the spin flip photon is emitted at a frequency of around $10^{-3}\,\Omega_c$. The recoil of the electron may be neglected and the J function is evaluated at the limit of small argument. One obtains the result

$$\overline{w}_{\text{spin flip}} \approx \frac{1}{8} \frac{\alpha^6}{(2\pi)^5} \left(\frac{B}{B_{\text{crit}}} \right)^3 m \frac{p_n^2}{\mathscr{E}^2_q}, \tag{78}$$

which is very small. For comparison, the rate of the cyclotron transition from the state $n = 1$, $\sigma = +1$ to the ground state is approximately

$$\overline{w}_{1 \to 0,\,\mathrm{flip}} \approx 0 \cdot 6\alpha(B/B_{\mathrm{crit}})^3 m, \tag{79}$$

which is larger by a factor of $>10^{15}$. Thus the spin flip transition at constant n, although allowed, does not occur before the electron radiates away its energy via the usual cyclotron process.

In conclusion, the purpose of this paper has been to extend the theory of QED in magnetic fields presented in papers (I, II, III) to include the electron self-energy radiative correction to lowest order in the fine structure constant. The mass operator was renormalised and the remaining finite corrections to the properties of electrons and positrons were incorporated into the propagator and vertex functions. This reproduced the standard lowest order correction to the magnetic moment of the electron, and also allowed the calculation of the transition rate for the spin flip transition, which could not be performed using the original version of the theory.

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Appendix. Dirac Matrices

The 16 Dirac matrices γ^A are a complete set of independent 4×4 matrices derived from the γ^μ matrices. They are:

$\hat{S} = I_4$ which is a Lorentz scalar;

$\hat{V}^\mu = \gamma^\mu$ which form a 4-vector;

$\hat{T}^{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma^\mu, \gamma^\nu]$ which form an anti-symmetric 4-tensor;

$\hat{A}^\mu = \gamma^5\gamma^\mu$ which form an axial 4-vector;

$\hat{P} = \gamma^5$ which is a Lorentz pseudo-scalar, with

$$\gamma^5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{A1}$$

The standard representation of the Dirac matrices may be written in terms of the 2×2 Pauli matrices and the 2×2 unit (1′) and zero (0′) matrices in the following way:

$$\hat{S} = \begin{bmatrix} 1' & 0' \\ 0' & 1' \end{bmatrix}, \qquad \hat{P} = \begin{bmatrix} 0' & 1' \\ 1' & 0' \end{bmatrix}, \tag{A2a, b}$$

$$\hat{V}^0 = \begin{bmatrix} 1' & 0' \\ 0' & -1' \end{bmatrix}, \qquad \hat{V}^i = \begin{bmatrix} 0' & \sigma^i \\ -\sigma_i & 0' \end{bmatrix}, \tag{A2c, d}$$

$$\hat{T}^{0j} = \begin{bmatrix} 0' & \mathrm{i}\,\sigma_j \\ \mathrm{i}\,\sigma_j & 0' \end{bmatrix}, \qquad \hat{T}^{ij} = \epsilon_{ijk}\begin{bmatrix} \sigma_k & 0' \\ 0' & \sigma_k \end{bmatrix}, \tag{A2e, f}$$

$$\hat{A}^0 = \begin{bmatrix} 0' & -1' \\ 1' & 0' \end{bmatrix}, \qquad \hat{A}^i = \begin{bmatrix} -\sigma_i & 0' \\ 0' & \sigma_i \end{bmatrix}. \tag{A2g, h}$$

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