

Collisions in Strong Magnetic Fields

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Abstract

Exact cross sections for electron–electron collisions and electron–proton collisions in a superstrong magnetic field are derived using the QED formalism developed by Melrose and Parle. The results are compared with those of Langer who used a different QED formalism. The intended application is to collision processes in the accretion columns above neutron stars where magnetic fields of order 10^9 T are thought to be present. The particular case of electrons initially in their ground states, with one final electron in an excited state is described in detail; this process is thought to be the primary source of photons in X-ray pulsars, through subsequent cyclotron emission.

1. Introduction

Superstrong magnetic fields, up to of order 10^9 T, are known to occur on the surfaces of some neutron stars (Trumper *et al.* 1978; Meszaros 1984; Kirk 1984) and this has stimulated interest in the effect of such fields on physical processes (Kirk and Galloway 1982; Anzer and Borner 1983). The specific processes of electron–electron and electron–proton scattering are of particular interest in models for X-ray pulsars (Langer 1981; Langer and Rappaport 1982; Allen *et al.* 1985). In these models the X rays are generated in an accretion column, where the energy supplied in the form of kinetic energy of the infalling matter is thermalised by inter-particle collisions. Electron–electron and electron–proton ‘collisions’ in such a plasma have qualitatively different properties than in a plasma with a more modest magnetic field. The difference is greatest for collisions involving the electrons because the quantisation of the electron motion perpendicular to $\mathbf{B}$ leads to energy quanta greater than or comparable with the thermal energy. Moreover, gyromagnetic radiation causes rapid transitions so that normally one expects all the electrons to be in their ground state (the ‘lowest Landau orbital’) where their motion is one-dimensional along $\mathbf{B}$.

Scattering from the ground state to the first excited state is of special interest as this leads subsequently to emission of a cyclotron quantum; this process is believed to be the ultimate source of the cyclotron photons in X-ray pulsars with cyclotron lines in emission (Melrose and Kirk 1986).

In this paper the QED formalism for electron–photon (Melrose and Parle 1983*a*, 1983*b*; hereafter MPI and MPIII) and photon–photon (Melrose 1983) interactions in a superstrong magnetic field is extended to treat electron–electron and electron–proton

interactions. In Section 2 this formalism is reviewed briefly, and some changes (to the normalisation of the states) and extensions are made. Electron-electron scattering is treated in Section 3 and electron-proton scattering in Section 5. In Section 4 the formalism used in the present investigation is compared with that used by Langer (1981), and the relation between his results and our results is discussed. In Section 6 we comment on our results.

2. Review and Extension of the Formalism

The formalism developed in MPI and MPIII involves solving Dirac's equation for the exact wavefunctions in the presence of a magnetic field, second quantising the wavefunctions and thereby including the magnetic field exactly in a QED formalism. Here, this formalism is reviewed briefly and the changes and extensions being made to it are discussed. (*Note:* Except where otherwise indicated natural units $\hbar = c = 1$ are used throughout.)

(a) Scattering Operator

The interaction picture is the most convenient for describing operators and wavefunctions when the system consists of several particles interacting with each other. The Hamiltonian of the system is decomposed into two parts; $H = H_0 + H_I$ where H_0 is the Hamiltonian of the system without interaction and H_I is the interaction. Operators in the interaction picture depend on time in the same way as the Heisenberg operator of the system in the absence of the interaction, and the change with time of the wavefunctions is caused entirely by the interaction, i.e.

$$i \partial \psi_I(t) / \partial t = H_I(t) \psi_I(t). \quad (1)$$

Scattering processes are described by a scattering operator $\hat{S}$. The matrix elements of $\hat{S}$ form the scattering matrix which connects the initial state of the system to the final state (cf. Berestetskii *et al.* 1971, §65). If $|i\rangle$ denotes the initial state, the result of the collision can be represented by

$$\sum_f |f\rangle \langle f | \hat{S} | i \rangle,$$

where the summation is taken over the various possible final states $|f\rangle$. The coefficients $S_{fi} = \langle f | \hat{S} | i \rangle$ form the scattering matrix. The squares $|S_{fi}|^2$ are proportional to the probabilities of transitions to particular final states.

(b) Quantised Particles

An exact solution of Dirac's equation in the presence of a magnetic field was presented by Johnson and Lippmann (1949). Their solution has certain undesirable features: the wavefunctions are not eigenvalues of any sensible choice of spin operator, and they exhibit no symmetry between electron and positron states. In MPI a general solution of Dirac's equation was presented and several specific choices of the spin operator and spin eigenfunctions were discussed, with the favoured choice being μ_z which is the z component (i.e. the component in the direction of $\mathbf{B}$) of the magnetic moment operator (Sokolov and Ternov 1968). The importance of this choice is discussed in Section 4.

The normalisation of the wavefunctions chosen here is different from that chosen in MPI. The wavefunctions $\psi_q(x, t)$, where q denotes the wavenumbers collectively, are normalised according to

$$\int_V \psi_q^d(x) \psi_q(x) d^3x = \delta_{q'q}, \quad (2)$$

where ψ^d is the hermitian conjugate of ψ . This leads to the density of states factors discussed below (cf. equation 12). The normalisation in MPI is given by MPI (20). It corresponds to the inclusion of an extra factor of $(eB)^{-\frac{1}{2}}$ on the right-hand side of (2).

In a magnetic field the perpendicular momentum of an electron is quantised. An electron wavefunction in cartesian coordinates and the Landau gauge is characterised by the following quantum numbers: the sign of the energy ϵ , the continuous momentum along the field p_z , the principal quantum number labelling the momentum transverse to the field $n = 0, 1, 2, 3, \dots$, the spin quantum number $s = \mp 1$, and p_y . The total energy is given by ($\hbar \neq c \neq 1$)

$$E_q = (m^2 c^4 + p_z^2 c^2 + 2neB\hbar c^2)^{\frac{1}{2}}. \quad (3)$$

Except for the ground state ($n = 0$), each energy level has a two-fold degeneracy with respect to spin.

When the magnetic field is nonzero, a finite V cannot be easily identified. A single particle can be isolated in the y and z directions within an area $L_y L_z$, but the corresponding length L_x is infinite.

The quantum number p_y is interpreted as specifying the x coordinate of the guiding centre of motion of the particle and it is desirable to average over this quantum number in calculating the transition rate.

The average over x is performed as follows. With $L_x = V/L_y L_z$ we have

$$\begin{aligned} 1 &= \frac{1}{L_x} \int_{-\frac{1}{2}L_x}^{\frac{1}{2}L_x} dx = -\frac{1}{L_x} \int_{-\frac{1}{2}L_x}^{\frac{1}{2}L_x} \frac{dp_y}{eB} \\ &= -\frac{2\pi L_z L_y}{V eB} \int_{-\infty}^{\infty} \frac{dp_y}{2\pi}. \end{aligned} \quad (4)$$

In (4) we have used the result [MPI (63)]

$$\langle x \rangle = \int_{-\infty}^{\infty} d\xi x v_n^2(\xi) = -\epsilon p_y / eB, \quad (5)$$

using the notation of MPI, with $\xi = (eB)^{\frac{1}{2}}(x + \epsilon p_y / eB)$.

The normalisation lengths appearing in (2) as $\delta_{q'q}$ are to be interpreted according to

$$\delta_{q'q} = \delta_{\epsilon'\epsilon} \delta_{s's} (2\pi/L_z) \delta(p'_z - p_z) (2\pi/L_y) \delta(p'_y - p_y). \quad (6)$$

In the Landau gauge our normalised wavefunction is of the form

$$\psi(x, t) = \frac{(eB)^{\frac{1}{4}}}{(L_y L_z)^{\frac{1}{2}}} \exp(-i\epsilon E_q t + i\epsilon p_y y + i\epsilon p_z z) \begin{bmatrix} C_1 v_{n-1}(\xi) \\ C_2 v_n(\xi) \\ C_3 v_{n-1}(\xi) \\ C_4 v_n(\xi) \end{bmatrix}, \quad (7)$$

with

$$v_n(\xi) = H_n(\xi) \exp(-\frac{1}{2}\xi^2)(\pi^{\frac{1}{2}} 2^n n!)^{\frac{1}{2}}, \quad (8)$$

where $H_n(\xi)$ is a Hermite polynomial, and with

$$\sum_{i=1}^4 C_i^{*'} C_i = \delta_{\epsilon'\epsilon} \delta_{s's}. \quad (9)$$

For the choice μ_z of spin operator and for $\epsilon = 1$, $s = -1$, the constants in (7) are given by

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{f}{(4E_q E_q^0 \Xi_q \Lambda_q)^{\frac{1}{2}}} \begin{bmatrix} p_z p_n \\ -i \Xi_q \Lambda_q \\ -p_n \Xi_q \\ i p_z \Lambda_q \end{bmatrix}, \quad (10)$$

where

$$f = \exp\{i\phi(\epsilon, s)\}, \quad \Xi_q = E_q + E_q^0, \quad \Lambda_q = E_q^0 + m;$$

for $\epsilon = 1$, $s = 1$, the constants are given by

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{f}{(4E_q E_q^0 \Xi_q \Lambda_q)^{\frac{1}{2}}} \begin{bmatrix} \Xi_q \Lambda_q \\ -i p_z p_n \\ \Lambda_q p_z \\ i \Xi_q p_n \end{bmatrix}, \quad (11)$$

where $E_q^0 = (m^2 + 2neB)^{\frac{1}{2}}$ and f is an arbitrary phase factor. Equations (10) and (11) correct an error in the denominator of MPI (45).

A consequence of the choice of normalisation made here is that there is no initial density of states factor (i.e. $D_i = 1$) and the final density of states factor is, in place of MPI (34a),

$$D_f = L_z(dp_z/2\pi)L_y(dp_y/2\pi). \quad (12)$$

The Dirac wavefunctions are second quantised and are written

$$\hat{\psi}(x) = \sum_{q,\epsilon} \hat{a}_q^\epsilon \psi_q^\epsilon(x) \exp(-i\epsilon E_q t), \quad (13)$$

$$\hat{\bar{\psi}}(x) = \sum_{q,\epsilon} \hat{a}_q^\epsilon \bar{\psi}_q^\epsilon(x) \exp(i\epsilon E_q t), \quad (14)$$

with $\hat{a}_q^d = \hat{a}_q^+$, $\hat{a}_q = \hat{a}_q^+$ creation and annihilation operators respectively for electrons and $\hat{b}_q^d = \hat{a}_q^-$, $\hat{b}_q = \hat{a}_q^-$ creation and annihilation operators respectively for positrons; the function $\psi = \psi^d \gamma^0$. The creation and annihilation operators satisfy anticommutation relations

$$[\hat{a}_q, \hat{a}_{q'}^d]_+ = [\hat{b}_q, \hat{b}_{q'}^d]_+ = \delta_{q'q}. \quad (15)$$

All other anticommutators are zero.

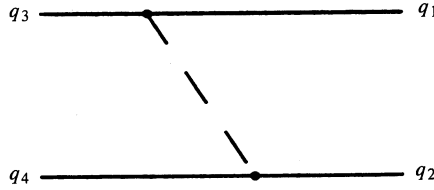


Fig. 1. Feynman diagram for scattering between two electrons in a magnetic field. The set of quantum numbers describing the state of the i th electron is denoted by q_i .

3. Electron-Electron Scattering Calculation

(a) Transition Rate

Electron-electron scattering is represented by the Feynman diagram in Fig. 1. The scattering operator for the interaction shown in Fig. 1 is calculated by constructing the single particle Hamiltonian for each of the initial particles interacting with the exchanged photon and then contracting over the electromagnetic operators.

The single particle interaction Hamiltonian density is given in the notation of MPIII by

$$\hat{H}_I(x) = -e : \hat{\bar{\psi}}(x) \hat{A}_\mu(x) \gamma^\mu \hat{\psi}(x) :. \quad (16)$$

In this expression $\hat{A}_\mu(x)$ is the operator for the radiation field, γ^μ is a Dirac matrix and the colons indicate that the normal order of the products within is to be taken.

The terms of relevance in the expansion of the probability amplitude are as follows. Electron-electron scattering arises from the second-order term in the scattering operator, which can be written

$$\begin{aligned} \hat{S} = - \int d^4 x_1 d^4 x_2 [& -e : \hat{\bar{\psi}}(x_2) \hat{A}_\mu(x_2) \gamma^\mu \\ & \times \hat{\psi}(x_2) :] [-e : \hat{\bar{\psi}}(x_1) \hat{A}_\nu(x_1) \gamma^\nu \hat{\psi}(x_1) :], \end{aligned} \quad (17)$$

where the bar indicates that a contraction over the operators indicated is to be performed.

The contraction over the electromagnetic field operators

$$\hat{A}^\mu(x) = \int \frac{d^4 k}{(2\pi)^4} \exp(-i k x) \hat{A}^\mu(k)$$

is related to the photon propagator $D_{\mu\nu}(k)$ by

$$\overline{\hat{A}^\mu(k)} \hat{A}^\nu(k') = -i(2\pi)^4 \delta^4(k + k') D^{\mu\nu}(k). \quad (18)$$

If, but only if, the dispersion of the plasma is unimportant then one may choose the form

$$D_{\mu\nu}(k) = \epsilon_0^{-1} k^{-2} g_{\mu\nu}, \quad (19)$$

with $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. (Gauge invariance implies that $g_{\mu\nu}$ could be replaced by $g_{\mu\nu} - k_\mu k_\nu / k^2$ without affecting the physical results.) The effect of a medium is neglected in the present paper.

The details of the calculation of the S -matrix elements are presented in Appendix 1. The S -matrix is given by (A5) where the initial electrons are labelled with subscripts 1 and 2 and the final electrons with subscripts 3 and 4.

The transition rate per particle into a unit final momentum interval is related to the matrix element S_{fi} by

$$dw_{fi} = T^{-1} |S_{fi}|^2 \prod D_i \prod D_f, \quad (20)$$

where T represents time, and D_i and D_f are the density of states factors. Equation (A5) depends on the initial positions of the two particles through the functions $d_{q'q}^{++}(k)$. It is desirable to average over the positions of the initial particles and integrate over the positions of the final particles to form $d\bar{w}_{fi}$. One has

$$\begin{aligned} d\bar{w}_{fi} = & T^{-1} L_y \int \frac{dp_{y1}}{2\pi} L_y \int \frac{dp_{y2}}{2\pi} L_y \int \frac{dp_{y3}}{2\pi} L_y \int \frac{dp_{y4}}{2\pi} \\ & \times \left(\frac{2\pi}{L_y L_x eB} \right)^2 |S_{fi}|^2 \frac{L_z dp_{z3}}{2\pi} \frac{L_z dp_{z4}}{2\pi}. \end{aligned} \quad (21)$$

After performing the integrations one has

$$\begin{aligned} d\bar{w}_{fi} = & \frac{e^4}{(2\pi)^2 V} dp_{z3} dp_{z4} \delta(E_1 + E_2 - E_3 - E_4) \delta(p_{z1} + p_{z2} - p_{z3} - p_{z4}) \\ & \times \left\{ \int d^4 k \delta(p_{z1} - p_{z3} + k_z) \delta(E_1 - E_3 + \omega) \right. \\ & \quad \times |[\Gamma_{31}(-k)]^\mu [\Gamma_{42}(k)]^\nu D_{\mu\nu}(k)|^2 \\ & \quad + \int d^4 k \delta(p_{z2} - p_{z3} + k_z) \delta(E_2 - E_3 + \omega) \\ & \quad \times |[\Gamma_{32}(-k)]^\mu [\Gamma_{41}(k)]^\nu D_{\mu\nu}(k)|^2 \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2\pi eB} \int d^4 k \int d^4 k' \delta(p_{z2} - p_{z3} + k_z) \delta(E_2 - E_3 + \omega) \\
& \quad \times \delta(p_{z1} - p_{z3} + k'_z) \delta(E_1 - E_3 + \omega') \\
& \quad \times \left(\exp\{(i/eB)(\mathbf{k} \times \mathbf{k}')_z\} ([\Gamma_{32}(-\mathbf{k})]^\mu [\Gamma_{41}(\mathbf{k})]^\nu D_{\mu\nu}(\mathbf{k}))^* \right. \\
& \quad \times ([\Gamma_{31}(-\mathbf{k}')]^\alpha [\Gamma_{42}(\mathbf{k}')]^\beta D_{\alpha\beta}(\mathbf{k}')) \\
& \quad + \exp\{(-i/eB)(\mathbf{k} \times \mathbf{k}')_z\} ([\Gamma_{32}(-\mathbf{k})]^\alpha [\Gamma_{41}(\mathbf{k})]^\beta D_{\alpha\beta}(\mathbf{k})) \\
& \quad \times ([\Gamma_{31}(-\mathbf{k}')]^\mu [\Gamma_{42}(\mathbf{k}')]^\nu D_{\mu\nu}(\mathbf{k}'))^* \left. \right\}, \quad (22)
\end{aligned}$$

where the integrals extend over all parameter space and the superscripts $++$, indicating that the Γ functions are those appropriate for positive energy particles (electrons), have been omitted for convenience.

(b) Specific Results for Neutron Star Environments

In order to proceed further it is necessary to use explicit expressions for the wavefunction appearing in the transition rate. A major simplification occurs if one need consider only the lowest Landau orbitals, and this is just the case which is relevant for neutron stars.

In a standard model for an X-ray pulsar (see e.g. Meszaros 1984) the magnetic field near the surface of the neutron star is within an order of magnitude of the critical field $B_c = m^2 c^2 / \hbar e = 4.413 \times 10^9$ T. In such a field electrons radiate away any perpendicular momentum they acquire very rapidly, so that one expects all electrons to be in their ground state. Thus the relevant case for scattering is when both electrons are initially in their ground states ($n = 0$). Here we consider transitions such that only one of the final electrons is in an excited state ($n \neq 0$). Electrons in their ground state have spin $s = -1$. An electron can undergo a spin flip during the collision so that in its final state it has $s = 1$ or the electron can jump to a higher state with no spin flip. The transition rates for both possibilities are derived below.

An expression for the function $\Gamma_{ab}(\mathbf{k})$ appearing in (22) in terms of the constants C_i has been given in MPI (50). In deriving the collision cross section we use this expression to write $\Gamma_{ab}(\mathbf{k})$ in terms of the particles' energies and momenta.

The transition probability (22) involves integrals of two types. The first type of integral appears in the first two terms of (22) involving squares of Γ function products. It is of the form

$$\begin{aligned}
I_a = & \int d^4 k \delta(p_{z1} - p_{z3} - k_z) \delta(E_1 - E_3 + \omega) \\
& \times (\epsilon_0^2 k^4)^{-1} (K_1^2)^n \exp(-2K_1^2)(1/ml), \quad (23)
\end{aligned}$$

where $K_1^2 = k_1^2 / 2eB$, which may be reduced to

$$I_a = (2\pi / \epsilon_0^2 n! 4eB) \int_0^\infty dx \frac{x^n \exp(-2x)}{(x + Q^2)^2}, \quad (24)$$

where

$$Q^2 = (1/2eB) \{ (p_{z1} - p_{z3})^2 - (E_3 - E_1)^2 \}. \quad (25)$$

The integral I_a has been discussed by Langer (1981) (see his equation 18) and by Robinson (1986), who expressed it in terms of generalised Dnestrovskii functions $F_{q,r}(z)$ (Dnestrovskii *et al.* 1964). After some minor manipulation Robinson's equation (85) can be written (for $q = n+1$, $r = 1$)

$$\begin{aligned} I_n &= -\Gamma(n+1)F_{n+1,1}(z)2^{1-n} \\ &= \int_0^\infty dx x^n \exp(-2x)(x + \frac{1}{2}z)^{-2}, \end{aligned} \quad (26)$$

where $\Gamma(n)$ is the gamma function. Thus the integral becomes

$$I_n = -n! F_{n+1,1}(z)2^{1-n}; \quad (27)$$

for $z = 2Q^2$ this is of the form I_a . Discussions on and graphs of $F_{q,r}(z)$ were presented by Dnestrovskii *et al.* (1964) and Bornatici *et al.* (1983).

The second type of integral appears in the crossed term in (22) and is of the form

$$\begin{aligned} I_b &= \iint d^4k d^4k' \exp\left(\frac{i}{eB}(k \times k')_z\right) \exp\{-i n(\psi' - \psi)\} \\ &\quad \times (K_1^2)^{n/2} \exp(-K_1^2) \frac{(K_1'^2)^{n/2}}{k^2 k'^2} \exp(-K_1'^2) \\ &\quad \times \delta(p_{z1} - p_{z4} - k_z) \delta(p_{z1} - p_{z3} - k'_z) \delta(E_2 - E_3 + \omega) \delta(E_1 - E_3 + \omega'). \end{aligned} \quad (28)$$

The angle ψ specifies the relative direction of k and the magnetic field B ;

$$\psi = \tan^{-1}(k_y/k_x), \quad k = (k_1 \cos \psi, k_1 \sin \psi, k_z).$$

Using

$$\exp\{(i/eB)(k \times k')_z\} = \exp\{(i/eB)k_1 k'_1 \sin(\psi' - \psi)\} \quad (29)$$

and (Gradshteyn and Ryzhik 1980; 8.411.1)

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-i n\theta + i z \sin \theta) d\theta, \quad (30)$$

(28) becomes

$$I_b = \pi \int dx \int dy J_n(2(xy)^{\frac{1}{2}})(xy)^{n/2} \exp(-x-y)(x+Q^2)^{-1}(y+Q^2)^{-1}, \quad (31)$$

with

$$Q^2 = \frac{(p_{z1} - p_{z4})^2 - (E_3 - E_2)^2}{2eB}.$$

The solution to this integral was discussed by Langer (1981) [cf. his equation (C10)].

(c) Collision Cross Section

A collision cross section is usually defined as the transition rate divided by the incident flux of particles. Here we define the cross section in terms of a uniform beam of incident particles, with the uniform beam formed by averaging over the x coordinate of the guiding centre positions of the initial particles. This average has already been performed to obtain (22) above. Then, with both incident electrons in their ground states and with the particle normalisation adopted above of one particle present in a box of volume V , one has

$$\text{incident flux of particles} = V_r/V,$$

where $V_r = p_{z1}/E_1 - p_{z2}/E_2$ is the relative velocity of approach of the incident electrons (Bjorken and Drell 1964; §7.4). So one obtains the result that the cross section is given by

$$d\bar{\sigma}_{fi} = d\bar{w}_{fi} V/V_r. \quad (32)$$

A problem arises in defining the flux when the particles are not in their ground states. Classically, the perpendicular component of momentum of individually spiralling particles in a uniform beam does not cause any perpendicular component of the momentum or velocity of the beam. Consequently, it is desirable to continue to define the cross section in terms of the relative velocities in the z directions, as for the case $n = 0$.

Using (22), (24) and (31) one can derive the following expression for the cross section (note that $\alpha = e^2/4\pi\epsilon_0$ in the units used here):

$$\begin{aligned} d\bar{\sigma}_{fi} = & \frac{\alpha^2 2\pi}{V_r eB} dp_{z3} dp_{z4} \delta(E_1 + E_2 - E_3 - E_4) \delta(p_{z1} + p_{z2} - p_{z3} - p_{z4}) \\ & \times \frac{E_1 + m}{2E_1} \frac{E_2 + m}{2E_2} \frac{(E_3 + E_3^0)(E_3^0 + m)}{4E_3 E_3^0} \frac{E_4 + m}{2E_4} \frac{1}{n!} \\ & \times \left(\int_0^\infty dx \frac{x^n \exp(-2x)}{(x + Q^2)^2} [R_a]^2 + \int_0^\infty dx \frac{x^n \exp(-2x)}{(x + Q^2)^2} [R_b]^2 \right. \\ & \left. - 2 \int_0^\infty dx \int_0^\infty dy \frac{(xy)^{\frac{1}{2}n} \exp(-x-y)}{(x + Q^2)(y + Q^2)} J_n(2(xy)^{\frac{1}{2}}) [R_a R_b] \right); \quad (33) \end{aligned}$$

$$R_a = P_{31} P_{42} - O_{31} O_{42}, \quad R_b = P_{32} P_{41} - O_{32} O_{41}; \quad (34a, b)$$

$$P_{qq'} = 1 + (p_{zq} p_{zq'} / \Xi_q \Xi_{q'}), \quad O_{qq'} = p_{zq} / \Xi_q + p_{zq'} / \Xi_{q'}. \quad (34c, d)$$

Equations (34a, b) apply when there is no spin flip during the interaction. For the case when a spin flip occurs, they are replaced by

$$R_a = \frac{(2neB)^{\frac{1}{2}}}{\Lambda_3} (O_{31} P_{42} - P_{31} O_{42}), \quad R_b = \frac{(2neB)^{\frac{1}{2}}}{\Lambda_3} (O_{32} P_{41} - P_{32} O_{41}). \quad (35a, b)$$

4. Comparison with Previously Published Results

An expression for the electron-electron cross section has been derived previously by Langer (1981). Referring to the equations in Langer's paper, the cross section is given by (C8), using (C4) and (C5). We note that Langer quotes the cross section integrated over the z momenta of the final particles. There are differences between the results of Langer and the present results brought about by a difference in the choice of the spin operator.

Langer used the wavefunctions of Johnson and Lippmann (1949) which are given by (7) with, for an electron with 'spin down',

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{1}{\{2E_q(E_q + m)\}^{\frac{1}{2}}} \begin{bmatrix} 0 \\ E_q + m \\ -i p_n \\ -p_z \end{bmatrix}, \quad (36)$$

and, for 'spin up',

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \frac{1}{\{2E_q(E_q + m)\}^{\frac{1}{2}}} \begin{bmatrix} E_q + m \\ 0 \\ p_z \\ i p_n \end{bmatrix}. \quad (37)$$

For an electron in the ground state ($n = 0$, $s = -1$) equations (36) and (10) are identical if the arbitrary phase factor is chosen to be $f = i$, consistent with the earlier assertion that all spin operators yield the same expression for the ground-state wavefunction.

On using the Johnson and Lippmann wavefunctions defined by (7), (36) and (37) and our method of treating the interaction, the cross section derived by Langer (1981; equation C8) may be rederived.

The Johnson and Lippmann electron wavefunctions have certain undesirable properties. In particular there is no symmetry between electron and positron wavefunctions, and the spin operator does not commute with the Hamiltonian when the self-energy interaction is included. A further disadvantage is that these wavefunctions are not covariant under Lorentz boosts parallel to the magnetic field and so the spin of a particle is not the same when measured in different reference frames (A. J. Parle, personal communication). The wavefunctions used in the calculations in the present paper are not subject to these limitations. Indeed, only for the spin eigenfunctions chosen here are the spin eigenvalues independent of time (Sokolov and Ternov 1968).

However, a meaningful comparison between the results of Langer and the present results can be made if we sum over the two possible spin states of the final $n \neq 0$ electron. The total cross section

$$d\bar{\sigma}_T = d\bar{\sigma}_{\text{ff}}(s = -1) + d\bar{\sigma}_{\text{ff}}(s = +1)$$

is the same irrespective of the choice of spin operator as any choice of wavefunction must lead to equivalent results when one sums over all spin states. In particular, the sum of the cross sections derived using the Johnson and Lippmann eigenfunctions should be identical to the sum of the cross sections derived here.

Specifically, the quantity

$$F = \sum_{s,s'} [\Gamma_{q'q}(k)]^\mu [\Gamma_{r'r}(k)]^{*\nu} \quad (38)$$

has the same form irrespective of the choice of spin eigenfunctions. Using the results derived above, $d\bar{\sigma}_T$ can be written as sums of products of Γ functions for ground state particles and terms of the form (38). The details of the expansion are outlined in Appendix 2.

Thus it is shown that the sum of the cross sections derived by Langer using the Johnson and Lippmann eigenfunctions is equal to the sum of the cross sections derived above. Of course, it is also possible to derive this particular result by explicitly summing the cross sections derived by Langer and those derived in the present work and comparing the results. A lengthy and tedious calculation leads to the same conclusion as drawn above.

5. Electron-Proton Scattering

(a) Introduction

In the nonrelativistic limit, the splitting between the proton energy levels corresponding to transverse momentum is smaller than the electron energy level splitting $\hbar\Omega_e$ by the electron-to-proton mass ratio m/M . In an accretion column above a highly magnetised neutron star the proton gas can have a significant component of perpendicular momentum—the average value of n can be several hundred. In spite of the high n values, the proton is treated quantum mechanically in the following calculation.

The cross section is calculated for a collision where the electron remains in the ground state and the proton makes a transition upon scattering from n to n' . (We note that if the final electron energy is less than the binding energy of hydrogen in a magnetic field, then the theory is not valid as the final particles may not be free particles.)

(b) Scattering Equations

The S -matrix equation is calculated from the interaction Hamiltonian in a similar manner to that used for electron-electron scattering. One has

$$\begin{aligned} \hat{S} = - \int d^4 x_2 \int d^4 x_1 [& -e : \hat{\psi}(x_2) \hat{A}_\mu(x_2) \gamma^\mu \\ & \times \overline{\hat{\psi}(x_2)} :] [e : \hat{\psi}_p(x_1) \hat{A}_\nu(x_1) \gamma^\nu \hat{\psi}_p(x_1) :], \end{aligned} \quad (39)$$

where the subscript p will be used to indicate a proton wavefunction or other function constructed from proton wavefunctions. The expansion of this function is outlined in Appendix 1. We label the initial and final protons with subscripts 2 and 4 respectively and the initial and final electrons with the subscripts 1 and 3. Using the results

where

$$\Sigma_1'' = \frac{p_{n'}}{\Lambda_4} (P_{31} R_{42}^p - O_{31} P_{42}^p), \quad \Sigma_2'' = \frac{p_n}{\Lambda_2} (-P_{31} O_{42}^p + O_{31} P_{42}^p); \quad (49a, b)$$

$$(iii) \quad d\bar{\sigma}_{fi}(s_2=+1, s_4=+1) = \frac{e^4}{V_r} \frac{1}{\epsilon_0^2} \frac{1}{16\pi} \frac{dp_{z3}}{eB} dP_{z4} E' \Delta \int_0^\infty dx \frac{1}{(x+Q''^2)^2} \\ \times \{J_0^0(x)J_{n'-n}^{n-1}(x)\Sigma_1''' + J_0^0(x)J_{n'-n}^n(x)\Sigma_2'''\}^2, \quad (50)$$

where

$$\Sigma_1''' = P_{31} P_{42}^p - O_{31} O_{42}^p, \quad \Sigma_2''' = \frac{p_{n'}}{\Lambda_4} \frac{p_n}{\Lambda_2} (P_{31} P_{42}^p - O_{31} O_{42}^p). \quad (51a, b)$$

6. Discussion

We have presented calculations for the transition rates or cross sections for Coulomb interactions between electrons and electrons and between electrons and protons in a superstrong magnetic field. The effect of the magnetic field is taken into account exactly, through the use of solutions of the Dirac equation which include the magnetic field.

The results presented above extend the earlier work on electron-proton scattering by Ventura (1973) and Pavlov and Yakovlev (1976). The electron-proton cross section calculated by Ventura (1973) only applies for nonrelativistic velocities and a stationary proton. Pavlov and Yakovlev (1976) included proton motion only in the nonrelativistic limit. Langer (1981) calculated fully relativistic cross sections for both electron-electron and electron-proton Coulomb interactions and our results are compared with those of Langer in Section 4. Our results differ from those of Langer due to our choice of spin eigenfunctions. We choose eigenfunctions of the z component of the magnetic moment operator, called 'transverse polarisation' by Sokolov and Ternov (1968), whereas Langer (1981) used the wavefunctions of Johnson and Lippmann (1949). The advantages of our choice of spin eigenfunctions are discussed in Section 4. On summing over spin states, where appropriate, our expressions and those of Langer give equivalent results.

Our interest is in processes in accretion columns above strongly magnetised neutron stars. One then expects most of the electrons to be in their ground state, and we have considered only processes in which the electrons are initially in their ground state. For electron-electron interactions, if the relative velocity is below the threshold for excitation of one of the electrons to the first excited state then, due to the constraint of the motion to one dimension, the electrons can only either retain or exchange their momenta. In this case electron-electron interactions do not allow the electron distribution to relax, and one can say the electron-electron collisions are strongly suppressed compared with the non-magnetised or weakly magnetised cases. Coulomb interactions with excited electrons are of particular importance in the generation of radiation: a collisionally excited electron has a high probability of relaxing to its ground state through emission of a cyclotron photon. The importance of this two-stage process in generating photons has been emphasised by Melrose and Kirk (1986).

The spacing between energy levels for a proton is less by a factor of m/M than for an electron, and in accretion columns protons are expected to be in moderately high ($n = \text{a few hundred}$) states. Although one could treat such protons classically for many purposes, in this work general results are presented in which the protons are treated quantum mechanically.

As electron–proton interactions are not as suppressed by the magnetic field as electron–electron interactions, electron–proton interactions dominate in tending to thermalise electron and proton distributions. Close to the threshold required for excitation, the electron–electron cross section is suppressed compared with the electron–proton cross section which is relatively flat near threshold. Mathematically, this is due to the ‘crossed’ terms appearing in (22).

The wavefunction for two electrons must be completely antisymmetric under interchange of the electrons. Suppose the electron spins are parallel, so that the spin wavefunction is symmetric and the space wavefunction is antisymmetric. We let $\psi_1(x_1)$ and $\psi_2(x_2)$ be the individual wavefunctions. The total wavefunction for the two-electron system is $\psi = \psi_1(x_1)\psi_2(x_2) - \psi_1(x_2)\psi_2(x_1)$. The Coulomb potential for the two-electron system is given by

$$V = \frac{e^2}{2} \int dx_1 dx_2 [|\psi_1(x_1)\psi_2(x_2)|^2 + |\psi_1(x_2)\psi_2(x_1)|^2 - |\psi_1(x_1)\psi_1^*(x_2)\psi_2(x_2)\psi_2^*(x_1) + \text{complex conjugate}|] / (4\pi\epsilon_0 |x_{1,2}|),$$

where $x_{1,2} = x_1 - x_2$. The final two ‘crossed’ terms inside the square brackets (called interference terms) provide a negative contribution to the potential. The reduction in the Coulomb potential between the two electrons from what it would be in the absence of interference is sometimes ascribed to a repulsive ‘exchange force’.

We have considered transitions with and without a spin flip. For both electron–electron and electron–proton interactions, transitions without a spin flip are more important than those with a spin flip.

One notable limitation on our results is that the use of plane wavefunctions means that our results are invalid at sufficiently low relative velocities. (The cross sections diverge as the relative velocity V_r approaches zero.) Electrons and protons tend to form hydrogen when the relative velocity implies an energy comparable with the binding energy of hydrogen.

We have also neglected ‘shielding’ effects in the plasma due to our use of the vacuum form of the photon propagator. It is known in the classical case that the effect of the ‘shielding’ is to cut off the effect of the Coulomb interactions for impact parameters greater than about one Debye length. It is reasonable to suppose that the effect of ‘shielding’ is similar in the quantum case, but this point has yet to be examined in detail.

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Appendix 1. Calculation of the S-matrix

In the QED formalism developed in MPI and MPIII a vertex function was introduced as follows. Equation (17) involves products of the form $:\hat{\psi}(x)\gamma^\mu\hat{\psi}(x):$. From (13) and (14) we have

$$\begin{aligned} :\hat{\psi}(x)\gamma^\mu\hat{\psi}(x): &= \sum_{\substack{q,\epsilon \\ q',\epsilon'}} \exp(i\epsilon E_q t - i\epsilon' E_{q'} t) : \hat{a}_q^\epsilon \hat{a}_{q'}^{\epsilon'} : \\ &\times [\bar{\psi}_q^\epsilon(x)\gamma^\mu \psi_{q'}^{\epsilon'}(x)]. \end{aligned} \quad (\text{A1})$$

The vertex function $\gamma_{q'q}^{\epsilon'\epsilon}(k)$, defined by

$$[\gamma_{q'q}^{\epsilon'\epsilon}(k)]^\mu = \frac{1}{V} \int d^3x \exp(-i\mathbf{k} \cdot \mathbf{x}) \bar{\psi}_{q'}^{\epsilon'}(x) \gamma^\mu \psi_q^\epsilon(x)$$

[cf. MPI (46)] is used to write

$$\bar{\psi}_{q'}^{\epsilon'}(x) \gamma^\mu \psi_q^\epsilon(x) = \int \frac{d^3k}{(2\pi)^3} \exp(i\mathbf{k} \cdot \mathbf{x}) V[\gamma_{q'q}^{\epsilon'\epsilon}(k)]^\mu,$$

where V is the volume of the system.

The vertex function was shown in MPI to be proportional to a gauge independent quantity $\Gamma_{q'q}^{\epsilon'\epsilon}(k)$, given by [cf. MPI (50)]

$$[\gamma_{q'q}^{\epsilon'\epsilon}(k)]^\mu = d_{q'q}^{\epsilon'\epsilon}(k) [\Gamma_{q'q}^{\epsilon'\epsilon}(k)]^\mu,$$

with $d_{q'q}^{\epsilon'\epsilon}(k)$ given by, with the present wavefunction normalisation,

$$d_{q'q}^{\epsilon'\epsilon}(k) = \frac{1}{V L_z L_y} \exp\left(\frac{i k_x (\epsilon p_y + \epsilon' p'_y)}{2 e B}\right) \times 2\pi \delta(\epsilon p_y - \epsilon' p'_y - k_y) 2\pi \delta(\epsilon p_z - \epsilon' p'_z - k_z). \quad (\text{A2})$$

Then using

$$\exp\{-i t(\epsilon E_q - \epsilon' E_{q'})\} = \int \frac{d\omega}{2\pi} 2\pi \delta(\epsilon E_q - \epsilon' E_{q'} - \omega) \exp(-i \omega t), \quad (\text{A3})$$

one obtains [cf. MPIII (11)]

$$\begin{aligned} :\hat{\psi}(x) \gamma^\mu \hat{\psi}(x): &= \sum_{\substack{q \epsilon \\ q' \epsilon'}} \int \frac{d^4 k}{(2\pi)^4} \exp(-i k x) \\ &\times V : \hat{a}_{q'}^{\epsilon'} \hat{a}_q^\epsilon : d_{q'q}^{\epsilon'\epsilon}(k) 2\pi \delta(\epsilon E_q - \epsilon' E_{q'} - \omega) [\Gamma_{q'q}^{\epsilon'\epsilon}(k)]^\mu. \end{aligned}$$

Thus the scattering operator $\hat{S}$ is given by

$$\begin{aligned} \hat{S} &= i e^2 \left(\sum_{\substack{\epsilon q \\ \epsilon' q'}} \int \frac{d^4 k}{(2\pi)^4} : \hat{a}_{q'}^{\epsilon'} \hat{a}_q^\epsilon : V \right. \\ &\quad \times d_{q'q}^{\epsilon'\epsilon}(-k) \delta(\epsilon E_q - \epsilon' E_{q'} + \omega) [\Gamma_{q'q}^{\epsilon'\epsilon}(-k)]^\mu \Big) \\ &\quad \times \left(\sum_{\substack{\epsilon'' q'' \\ \epsilon''' q'''}} : \hat{a}_{q'''}^{\epsilon'''} \hat{a}_{q''}^{\epsilon''} : V d_{q''q'''}^{\epsilon'''\epsilon''}(k) 2\pi \right. \\ &\quad \times \delta(\epsilon'' E_{q''} - \epsilon''' E_{q'''} - \omega) [\Gamma_{q''q'''}^{\epsilon'''\epsilon''}(k)]^\nu \Big) (2\pi)^4 D_{\mu\nu}(k). \quad (\text{A4}) \end{aligned}$$

Labelling the initial electrons with subscripts 1 and 2 and the final electrons with subscripts 3 and 4, the annihilation and creation operator product required to calculate the scattering matrix element is

$$\langle 0 | \hat{a}_3 \hat{a}_4 : \hat{a}_{q'}^{\epsilon'} \hat{a}_q^\epsilon : \hat{a}_{q'''}^{\epsilon'''} \hat{a}_{q''}^{\epsilon''} : \hat{a}_1^d \hat{a}_2^d | 0 \rangle =$$

$$\langle 0 | \hat{a}_3 \hat{a}_4 : (\hat{a}_{q'}^d + \hat{b}_{q'}) (\hat{a}_q + \hat{b}_q) : (\hat{a}_{q'''}^d + \hat{b}_{q'''})(\hat{a}_{q''} + \hat{b}_{q''}) : \hat{a}_1^d \hat{a}_2^d | 0 \rangle.$$

The contractions which give a nonzero contribution to the scattering operator are

$$\begin{aligned} & \langle 0 | \hat{a}_3 \hat{a}_4 : \hat{a}_{q'''}^d \hat{a}_{q''}^d :: \hat{a}_q^d \hat{a}_q^d : \hat{a}_1^d \hat{a}_2^d | 0 \rangle \\ & - \langle 0 | \hat{a}_3 \hat{a}_4 : \hat{a}_{q'''}^d \hat{a}_{q''}^d :: \hat{a}_q^d \hat{a}_q^d : \hat{a}_1^d \hat{a}_2^d | 0 \rangle. \end{aligned}$$

These nonzero contractions result in the scattering matrix element

$$\begin{aligned} S_{34,12} = & i e^2 \left(\int \frac{d^4 k}{(2\pi)^4} V d_{32}^{++}(-k) 2\pi \right. \\ & \times \delta(E_2 - E_3 + \omega) [\Gamma_{32}^{++}(-k)]^\mu V d_{41}^{++}(k) 2\pi \\ & \times \delta(E_1 - E_4 - \omega) [\Gamma_{41}^{++}(k)]^\nu (2\pi)^4 D_{\mu\nu}(k) \\ & - \int \frac{d^4 k}{(2\pi)^4} V d_{31}^{++}(-k) 2\pi \delta(E_1 - E_3 + \omega) [\Gamma_{31}^{++}(-k)]^\mu \\ & \left. \times V d_{42}^{++}(k) 2\pi \delta(E_2 - E_4 - \omega) [\Gamma_{42}^{++}(k)]^\nu D_{\mu\nu}(k) (2\pi)^4 \right). \quad (A5) \end{aligned}$$

This result has also been derived by A. J. Parle (personal communication). When one of the incident particles is a proton the scattering operator has the form (39). Labelling the initial and final protons with subscripts 2 and 4 respectively and the initial and final electrons with the subscripts 1 and 3 respectively, the annihilation and creation operator products required to calculate the scattering matrix are

$$\langle 0 | \hat{a}_3 \hat{d}_4 : (\hat{d}_q^d + \hat{g}_q)(\hat{d}_q + \hat{g}_q^d) :: (\hat{a}_{q'''}^d + \hat{b}_{q'''})(\hat{a}_{q''}^d + \hat{b}_{q''}^d) : \hat{d}_2^d \hat{a}_1^d | 0 \rangle,$$

where

$$\begin{aligned} \hat{d}_q^d &= \hat{d}_q^+ = \text{proton creation operator,} \\ \hat{d}_q &= \hat{d}_q^+ = \text{proton annihilation operator,} \\ \hat{g}_q^d &= \hat{d}_q^- = \text{antiproton creation operator,} \\ \hat{g}_q &= \hat{d}_q^- = \text{antiproton annihilation operator.} \end{aligned}$$

The contractions which give a nonzero contribution to the scattering operator result in the term in the S -matrix

$$\langle 0 | \hat{a}_3 \hat{d}_4 : \hat{d}_q^d \hat{d}_q^d :: \hat{a}_{q'''}^d \hat{a}_{q''}^d : \hat{d}_2^d \hat{a}_1^d | 0 \rangle.$$

This result gives the expression for the transition probability given by (40).

Appendix 2. Spin-dependence of $d\bar{\sigma}_T$

The cross section (33) can be separated into three terms, proportional to R_a^2 , R_b^2 and $R_a R_b$, respectively. Thus $d\bar{\sigma}_T$ also separates into three terms. The factor of the first term of $d\bar{\sigma}_T$ that depends on the choice of spin eigenfunction is given by

$$C = |\Gamma_{31}^0 \Gamma_{42}^0 - \Gamma_{31}^3 \Gamma_{42}^3|_{s=1}^2 + |\Gamma_{31}^0 \Gamma_{42}^0 - \Gamma_{31}^3 \Gamma_{42}^3|_{s=-1}^2, \quad (A6)$$

where the argument of the Γ functions has been omitted for convenience. Equation (A6) reduces to

$$\begin{aligned}
 C = & (\Gamma_{42}^0)^2 [(\Gamma_{31}^{0u})^2 + (\Gamma_{31}^{0d})^2] \\
 & - 2\Gamma_{42}^0 \Gamma_{42}^3 [\Gamma_{31}^{0u} \Gamma_{31}^{3u} + \Gamma_{31}^{0d} \Gamma_{31}^{3d}] \\
 & + (\Gamma_{42}^3)^2 [(\Gamma_{31}^{3u})^2 + (\Gamma_{31}^{3d})^2],
 \end{aligned} \tag{A7}$$

where

$$\Gamma_{31}^{0u} = [\Gamma_{31}(k)]_{s=1}^0, \quad \Gamma_{31}^{0d} = [\Gamma_{31}(k)]_{s=-1}^0, \quad \text{etc.};$$

Γ_{42}^0 and Γ_{42}^3 are independent of the choice of spin eigenfunction as particles 1, 2 and 4 are in the ground state. The terms in square brackets in (A7) are all of the form (38) and therefore are independent of the choice of spin operator. The result for the second term in the cross section follows easily from the above equations and it is also independent of the choice of spin eigenfunction. The third term in the cross section equation, of the form

$$([\Gamma_{32}(-k)]^\mu [\Gamma_{41}(k)]^\nu D_{\mu\nu}(k))^* ([\Gamma_{31}(-k')]^\alpha [\Gamma_{42}(k')]^\beta D_{\alpha\beta}(k')),$$

when summed over the spin of the final excited electron, can also be reduced to a sum of terms containing products of Γ functions for ground state particles and expressions of the form (38). Thus $d\bar{\sigma}_T$ is independent of the choice of spin operator.

