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The Cubic Response Tensor for a Magnetised Plasma: A Covariant Forward-scattering Approach

D. B. Melrose

School of Physics, University of Sydney,
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Abstract

The quadratic and cubic nonlinear response tensors are calculated for an arbitrary magnetised plasma using a covariant forward-scattering method. Some relevant approximate forms for the response tensors are derived, and translated into 3-tensor notation in the Appendix. The general form of the tensors is used to re-enforce an argument for the non-existence of turbulent bremsstrahlung.

1. Introduction

A hierarchy of response tensors for a plasma is defined by expanding the Fourier transform (in space and time) of the response in powers of the Fourier transform of the disturbance. Different choices of the response and of the disturbance lead to different tensors, such as the conductivity and susceptibility tensors; these are related to each other so that the choice made is unimportant from a formal viewpoint. The choice made here is to expand the 4-current in powers of the 4-potential, defining a hierarchy of polarisation 4-tensors. In practice only the linear, quadratic and cubic response tensors are of interest. The quadratic and cubic nonlinear response tensors are used in treating various nonlinear wave–particle and wave–wave interactions (see e.g. Tsytovich 1967, 1970; Melrose 1980, 1986*a*). For most purposes approximate forms of the tensors are required, and relevant approximate forms correspond to the cold-plasma and longitudinal approximations. The general forms for the tensors may be obtained using Vlasov theory, which also needs to be used (in simplified form) to treat the longitudinal case. The quadratic and cubic response tensors for an unmagnetised plasma have been calculated in this way (e.g. Al'tshul' and Karpman 1965; Tsytovich 1967) as has the quadratic response tensor for a magnetised plasma (Tsytovich and Shvartsburg 1966; Melrose and Sy 1972*a*). The extension of this method to the cubic response tensor for a magnetised plasma leads to a very cumbersome expression which has not been written down explicitly. Nevertheless, the cubic response tensor is required for some formal purposes, three of which are mentioned below, and it is desirable to have an explicit form for it.

In this paper the cubic polarisation 4-tensor for a magnetised plasma is calculated using a covariant forward-scattering approach. The method is manifestly covariant and gauge invariant in the sense discussed by Melrose (1982). In 'forward-scattering' the perturbations are calculated in the orbits of individual particles, rather than in the

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distribution function; the forward-scattering component is that for which the initial phase is preserved, and then the contributions from all particles add in phase. In terms of a description in phase space, the forward-scattering method corresponds to a Lagrangian approach, in which the perturbations are in the trajectories of the phase-space points, as opposed to the Vlasov method which corresponds to an Eulerian approach, in which the perturbations are in the distribution function of phase-space points along their unperturbed trajectories. This method was used by Melrose (1983, cf. Appendix 4) in calculating the quadratic and cubic polarisation 4-tensors for an unmagnetised plasma and the quadratic polarisation 4-tensor for a magnetised tensor, but the method of calculation was not explained in that earlier paper. A non-covariant form of the forward-scattering method has been discussed by Melrose (1986*a*, cf. Ch. 5). The covariant forward-scattering method is developed in Section 2 for the unmagnetised case, and extended to the magnetised case in Section 3. The response 4-tensors for the magnetised case are written down in Section 4, and approximations to them are discussed in Section 5. Relevant results are translated into 3-tensor notation in the Appendix.

There are three motivations for calculating the cubic response tensor. One is to treat four-wave interactions in a magnetised plasma, e.g. generalising the results of Melrose (1986*b*) to the magnetised case. Another is related to a suggestion by Nambu (personal communication 1985) that the inclusion of a magnetic field enhances turbulent bremsstrahlung, which we have argued does not exist (Melrose and Kuijpers 1984; Kuijpers and Melrose 1985). This point is discussed in Section 6. The third, and most immediate motivation, concerns an extension of a kinetic theory for parametric instabilities (Melrose 1986*c*) so that it provides a kinetic theory derivation of the Zakharov (1972) equations. This derivation is to be presented elsewhere; its generalisation to the magnetised case requires an appropriate approximate form for the cubic response tensor, derived here in Section 5 and written down in the Appendix.

2. Forward-scattering Method for an Unmagnetised Plasma

A covariant description of the orbit of a particle is of the form

$$x^{\mu} = X^{\mu}(\tau), \tag{1}$$

where $x = (t, \boldsymbol{x})$ is the 4-position and τ is the particle's proper time. The 4-velocity is $u(\tau) = \mathrm{d}X(\tau)/\mathrm{d}\tau$ and the 4-momentum is $p(\tau) = mu(\tau)$. Newton's equation of motion in covariant form is

$$\mathrm{d}p^{\mu}(\tau)/\mathrm{d}\tau = \mathscr{F}^{\mu}(\tau). \tag{2}$$

where $\mathscr{F}(\tau)$ is the 4-force. For a charge q perturbed by a fluctuating electromagnetic field described by the Fourier transform $A(k)$ of the 4-potential, (2) becomes

$$\frac{\mathrm{d}u^{\mu}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{i}\,q}{m}\int\frac{\mathrm{d}^4 k_1}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}k_1 X(\tau)}\,k_1\,u(\tau)\,G^{\mu\nu}(k_1, u(\tau))\,A_{\nu}(k_1), \tag{3}$$

with

$$G^{\mu\nu}(k, u) = g^{\mu\nu} - \frac{k^{\mu}u^{\nu}}{ku}, \tag{4}$$

where $ku = k^\alpha u_\alpha = \gamma(\omega - \boldsymbol{k}.\boldsymbol{v})$ denotes the invariant formed from k and u. Similarly we have $k^2 = \omega^2 - |\boldsymbol{k}|^2$ and $u^2 = 1$ (the units have $c = 1$).

The perturbations in the orbit may be found to any given order n in $A(k)$. A formal expansion in $A(k)$ is written as

$$X^\mu(\tau) = x_0^\mu + u_0^\mu \tau + \sum_{u=1}^{\infty} X^{(n)\mu}(\tau). \tag{5}$$

The term $u_0 \tau$ corresponds to constant rectilinear motion; x_0 and u_0 are constants. The derivative of (5) with respect to τ gives the corresponding expansion of $u(\tau)$. On substituting these two expansions in (3), and expanding the exponential in powers of the $X^{(n)}$, one obtains the following perturbation expansion:

$$\frac{\mathrm{d}u^{(1)\mu}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} X^{(1)\mu}(\tau) = \frac{\mathrm{i}\,q}{m} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \exp(-\mathrm{i}\, k_1 x_0 - \mathrm{i}\, k_1 u_0 \tau)$$

$$\times\, k_1 u_0\, G^{\mu\nu}(k_1, u_0)\, A_\nu(k_1), \tag{6}$$

$$\frac{\mathrm{d}u^{(2)\mu}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} X^{(2)\mu}(\tau) = \frac{\mathrm{i}\,q}{m} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \exp(-\mathrm{i}\, k_1 x_0 - \mathrm{i}\, k_1 u_0 \tau)$$

$$\times \left(-\mathrm{i}\, k_1 X^{(1)}(\tau) + u^{(1)}(\tau) \frac{\partial}{\partial u_0}\right) k_1 u_0\, G^{\mu\nu}(k_1, u_0)\, A_\nu(k_1), \tag{7}$$

$$\frac{\mathrm{d}u^{(3)}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} X^{(3)}(\tau) = \frac{\mathrm{i}\,q}{m} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \exp(-\mathrm{i}\, k_1 x_0 - \mathrm{i}\, k_1 u_0 \tau)$$

$$\times \left(-\mathrm{i}\, k_1 X^{(2)}(\tau) - \mathrm{i}\, k_1 X^{(1)}(\tau)\, u^{(1)}(\tau) \frac{\partial}{\partial u_0} + u^{(2)}(\tau) \frac{\partial}{\partial u_0}\right)$$

$$\times\, k_1 u_0\, G^{\mu\nu}(k_1, u_0)\, A_\nu(k_1), \tag{8}$$

and so on. One solves (6) for $u^{(1)}(\tau)$ and $X^{(1)}(\tau)$, substitutes in (7) and solves for $u^{(2)}(\tau)$ and $X^{(2)}(\tau)$, and so on.

The single-particle (sp) 4-current density is

$$J^\mu_{\mathrm{sp}}(x) = q \int \mathrm{d}\tau\, u^\mu(\tau)\, \delta^4(x - X(\tau)),$$

and its Fourier transform is

$$J^\mu_{\mathrm{sp}}(k) = q \int \mathrm{d}\tau\, u^\mu(\tau)\, \mathrm{e}^{\mathrm{i}kX(\tau)}. \tag{9}$$

The nth order current is obtained by expanding the $u(\tau)$, $X(\tau)$ and the exponential and collecting the terms of nth order in $A(k)$. One obtains

$$J^{\mu}_{\rm sp}(k) = \sum_{n=0}^{\infty} J^{(n)\mu}_{\rm sp}(k), \tag{10}$$

$$J^{(0)\mu}_{\rm sp}(k) = q u^{\mu}_0 \int d\tau \, \exp(\mathrm{i}\, kx_0 + \mathrm{i}\, ku_0\tau), \tag{11}$$

$$J^{(1)\mu}_{\rm sp}(k) = q \int d\tau \, \{ u^{(1)\mu}(\tau) + \mathrm{i}\, kX^{(1)}(\tau)\, u^{\mu}_0 \} \exp(\mathrm{i}\, kx_0 + \mathrm{i}\, ku_0\tau), \tag{12}$$

$$\begin{aligned} J^{(2)\mu}_{\rm sp}(k) = q \int d\tau \, [&u^{(2)\mu}(\tau) + \mathrm{i}\, kX^{(1)}(\tau)\, u^{(1)\mu}(\tau) \\ &+ \{\mathrm{i}\, kX^{(2)}(\tau) - \tfrac{1}{2}(kX^{(1)}(\tau))^2\}\, u^{\mu}_0] \exp(\mathrm{i}\, kx_0 + \mathrm{i}\, ku_0\tau), \end{aligned} \tag{13}$$

$$\begin{aligned} J^{(3)\mu}_{\rm sp}(k) = q \int d\tau \, [&u^{(3)\mu}(\tau) + \mathrm{i}\, kX^{(1)}(\tau)\, u^{(2)\mu}(\tau) \\ &+ \{\mathrm{i}\, kX^{(2)}(\tau) - \tfrac{1}{2}(kX^{(1)}(\tau))^2\}\, u^{(1)\mu}(\tau) \\ &+ \{\mathrm{i}\, kX^{(3)}(\tau) - kX^{(2)}(\tau)\, kX^{(1)}(\tau) - \tfrac{1}{6}\mathrm{i}(kX^{(1)}(\tau))^3\}\, u^{\mu}_0] \\ &\times \exp(\mathrm{i}\, kx_0 + \mathrm{i}\, ku_0\tau). \end{aligned} \tag{14}$$

We may write $J^{(n)}_{\rm sp}(k)$ in the form

$$\begin{aligned} J^{(n)\mu}_{\rm sp} = \int d\tau \int \frac{d^4 k_1}{(2\pi)^4} \cdots \frac{d^4 k_n}{(2\pi)^4}\, &\beta^{\mu\nu_1 \ldots \nu_n}(k, k_1, \ldots, k_n; u_0) \\ &\times A_{\nu_1}(k_1) \ldots A_{\nu_n}(k_n) \exp\{\mathrm{i}(k - k_1 - \ldots - k_n)(x_0 + u_0\tau)\}, \end{aligned} \tag{15}$$

which defines the β's.

The distribution function $F(u, x)$ in the 8-dimensional phase space constructed from the two 4-vectors u and x may be introduced as follows (Dewar 1977). The orbit of the particle defines a world line in this phase space. Consider a 7-dimensional surface orthogonal to the world line. Let the number of representative points of the distribution of particles be $d\mathcal{N}$ across the element $d^4u\, d^4x/d\tau$ of the 7-dimensional surface. The distribution function $F(u, x)$ is then defined by writing $d\mathcal{N}\, d\tau = F(u, x)\, d^4u\, d^4x$.

The average over the single-particle current (4) gives the induced current. This average is achieved formally by multiplying (14) by $d\mathcal{N}$ and integrating. One replaces $d\mathcal{N}\, d\tau$ by $F(u, x)\, d^4u\, d^4x$, and then makes the further replacement by $F(u_0, x_0)\, d^4u_0\, d^4x_0$ by appealing to Liouville's theorem. The resulting nth order induced current may be written in the form

$$J^{(n)\mu}(k) = \int d\lambda^{(n)}\, \alpha^{\mu\nu_1 \ldots \nu_n}(k, k_1, \ldots, k_n)\, A_{\nu_1}(k_1) \ldots A_{\nu_n}(k_n), \tag{16}$$

with

$$d\lambda^{(n)} = \frac{d^4 k_1}{(2\pi)^4} \cdots \frac{d^4 k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k - k_1 - \ldots - k_n). \tag{17}$$

By inspection one has

$$\alpha^{\mu\nu_1\dots\nu_n}(k, k_1, \dots, k_n) = \int \mathrm{d}^4 u\, F(u)\, \beta^{\mu\nu_1\dots\nu_n}(k, k_1, \dots, k_n; u)\,, \tag{18}$$

where we assume $F(u_0, x_0)$ to be independent of x_0, and drop the subscript on u_0.

The resulting expressions for the linear, quadratic and cubic response tensors were written down by Melrose (1983, 1986*b*) and by Melrose and Kuijpers (1984).

3. Perturbed Motion in a Magnetic Field

Consider a static magnetic field $\boldsymbol{B}$ in a frame, called the laboratory frame, in which there is no static electric field. This field may be described in a frame-independent form in terms of the Maxwell tensor $F_0^{\mu\nu}$. The invariant formed from this tensor is related to $B = |\boldsymbol{B}|$ in the laboratory frame by

$$B = (\tfrac{1}{2} F_0^{\mu\nu} F_{0\mu\nu})^{\frac{1}{2}}\,. \tag{19}$$

It is convenient to write (Melrose 1983)

$$F_0^{\mu\nu} = B f^{\mu\nu}\,, \tag{20}$$

$$g_\perp^{\mu\nu} = -f^{\mu}{}_{\alpha} f^{\alpha\nu}\,, \qquad g_\parallel^{\mu\nu} = g^{\mu\nu} - g_\perp^{\mu\nu}\,, \tag{21a, b}$$

where $g^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is the metric tensor. In the laboratory frame with $\boldsymbol{B}$ along the 3-axis, the matrix representations of these tensors are

$$f^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad g_\perp^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22a, b}$$

$$g_\parallel^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{22c}$$

The equation of motion for a charge q subject to both the static and a fluctuating field is

$$\frac{\mathrm{d}u^\mu(\tau)}{\mathrm{d}\tau} = \frac{q}{m} F_0^{\mu\nu}\, u_\nu(\tau) + \frac{\mathrm{i}\,q}{m} \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k_1 X(\tau)}\, k_1\, u(\tau)\, G^{\mu\nu}(k_1, u(\tau))\, A_\nu(k_1)\,. \tag{23}$$

After solving the unperturbed equation of motion, i.e. (23) for $A(k_1) = 0$, a perturbation solution of the form (5) is sought. The form (5) is replaced by

$$X^{\mu}(\tau) = x_0^{\mu} + t^{\mu\nu}(\tau)u_0 + \sum_{n=1}^{\infty} X^{(n)\mu}(\tau), \tag{24}$$

with

$$t^{\mu\nu}(\tau) = g_{\parallel}^{\mu\nu}\tau + g_{\perp}^{\mu\nu}\frac{\sin\Omega_0\tau}{\Omega_0} - \eta f^{\mu\nu}\frac{\cos\Omega_0\tau}{\Omega_0}, \tag{25}$$

and with

$$\Omega_0 = |q|B/m, \qquad \eta = q/|q|. \tag{26a, b}$$

To solve (23), we first introduce a Laplace transform, writing

$$u^{\mu}(\omega_0) = \int_0^{\infty} d\tau\, e^{i\omega_0\tau} u^{\mu}(\tau). \tag{27}$$

Denoting the final term in (23) by $S^{\mu}(\tau)$, one has

$$-i\,\omega_0\, u^{\mu}(\omega_0) - u_0^{\mu} = \eta\Omega_0 f^{\mu\nu} u_{\nu}(\omega_0) + \int_0^{\infty} d\tau\, e^{i\omega_0\tau} S^{\mu}(\tau). \tag{28}$$

One may solve (28) by introducing $\tau^{\mu\nu}(\omega_0)$ as the solution of

$$(-i\,\omega_0\, g^{\mu\nu} - \eta\Omega_0 f^{\mu\nu})\tau_{\nu\rho}(\omega_0) = -i\,\omega_0\, g^{\mu}{}_{\rho}, \tag{29}$$

i.e.

$$\tau^{\mu\nu}(\omega_0) = g_{\parallel}^{\mu\nu} + \frac{\omega_0^2}{\omega_0^2 - \Omega_0^2} g_{\perp}^{\mu\nu} + \frac{i\,\eta\omega_0\,\Omega_0}{\omega_0^2 - \Omega_0^2} f^{\mu\nu}. \tag{30}$$

Then (28) implies

$$u^{\mu}(\omega_0) = \frac{i}{\omega_0}\tau^{\mu\nu}(\omega_0)\left(u_{0\nu} + \int_0^{\infty} d\tau\, e^{i\omega_0\tau} S_{\nu}(\tau)\right). \tag{31}$$

Now inverting the Laplace transform one finds

$$u^{\mu}(\tau) = \dot{t}^{\mu\nu}(\tau)\, u_{0\nu} + \int_0^{\tau} d\tau'\, \dot{t}^{\mu\nu}(\tau - \tau')\, S_{\nu}(\tau'), \tag{32}$$

$$X^{\mu}(\tau) = x_0^{\mu} + t^{\mu\nu}(\tau)\, u_{0\nu} + \int_0^{\tau} d\tau'' \int_0^{\tau''} d\tau'\, \dot{t}^{\mu\nu}(\tau'' - \tau')\, S_{\nu}(\tau'), \tag{33}$$

where $\dot{t}(\tau)$ denotes $dt(\tau)/d\tau$, and where

$$\int_0^{\infty} d\tau\, e^{i\omega_0\tau}\, \dot{t}^{\mu\nu}(\tau) = \frac{i}{\omega_0}\tau^{\mu\nu}(\omega_0) \tag{34}$$

has been used. The perturbation expansion (24) is now straightforward using (33).

The unperturbed orbit is $X^{(0)}(\tau) = x_0 + t(\tau)\, u_0$, and the unperturbed 4-velocity is $u^{(0)}(\tau) = \dot{t}(\tau)\, u_0$. These appear in the following expressions for the perturbations in the orbit:

$$X^{(n)\mu}(\tau) = \frac{\mathrm{i}\,q}{m} \int_0^\tau \mathrm{d}\tau'' \int_0^{\tau'} \mathrm{d}\tau'\; t^\mu_a(\tau''-\tau') \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}k_1 X^{(0)}(\tau)}$$

$$\times H^{(n)a\nu}(k_1,\tau)\, A_\nu(k_1), \tag{35}$$

with

$$H^{(1)a\nu}(k,\tau) \doteq k u^{(0)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau)), \tag{36}$$

$$H^{(2)a\nu}(k,\tau) = -\mathrm{i}\, k X^{(1)}(\tau)\, k u^{(0)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau))$$

$$+ k u^{(1)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau)) - k^a u^{(1)}_\beta(\tau)\, G^{\beta\nu}(k, u^{(0)}(\tau)), \tag{37}$$

$$H^{(3)a\nu}(k,\tau) = [-\tfrac{1}{2}\{ k X^{(1)}(\tau)\}^2 - \mathrm{i}\, k X^{(2)}(\tau)]\, k u^{(0)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau))$$

$$-\mathrm{i}\, k X^{(1)}(\tau)\{ k u^{(1)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau))$$

$$- k^a u^{(1)}_\beta(\tau)\, G^{\beta\nu}(k, u^{(0)}(\tau))\} + k u^{(2)}(\tau)\, G^{a\nu}(k, u^{(0)}(\tau))$$

$$- k^a u^{(2)}_\beta(\tau)\, G^{\beta\nu}(k, u^{(0)}(\tau)). \tag{38}$$

The perturbations in the 4-velocity $u^{(n)}(\tau) = \mathrm{d}X^{(n)}(\tau)/\mathrm{d}\tau$ follow trivially from (35) with equations (36)–(38). The single-particle currents then follow from (12)–(14) by replacing u_0 by $t(\tau)\,u_0$.

To evaluate the integrals over proper time τ it is necessary to expand in Bessel functions. To facilitate this, we first introduce the components, in the laboratory frame

$$u_0 = \gamma(1,\ v_\perp \cos\phi,\ v_\perp \sin\phi,\ v_\parallel), \tag{39}$$

$$k = (\omega,\ k_\perp \cos\psi,\ k_\perp \sin\psi,\ k_\parallel). \tag{40}$$

Then the factor $\mathrm{i}\, k t(\tau) u_0$ in the exponents of (12)–(14), as modified, becomes

$$\mathrm{i}\, k t(\tau) u_0 = \mathrm{i}(k u)_\parallel\, \tau - k_\perp R \sin(\Omega_0 \tau + \eta\psi - \eta\phi), \tag{41}$$

with

$$(k u)_\parallel = g^{\alpha\beta}_\parallel k_\alpha u_{0\beta} = \gamma(\omega - k_\parallel v_\parallel), \tag{42}$$

$$R = \gamma v_\perp / \Omega_0. \tag{43}$$

The generating function for Bessel functions then gives

$$\mathrm{e}^{\mathrm{i}k t(\tau) u_0} = \sum_{s=-\infty}^{\infty} J_s(k_\perp R) \exp[\mathrm{i}\{(k u)_\parallel - s\Omega_0\}\tau - \mathrm{i}\, s\eta(\psi - \phi)]. \tag{44}$$

After some manipulations one finds

$$u^{(0)\mu}(\tau)\exp\{\mathrm{i}\,kX^{(0)}(\tau)\} = \mathrm{e}^{\mathrm{i}kX_0}\sum_{s=-\infty}^{\infty} U^{\mu}(s,k,u)$$

$$\times\exp\left[\mathrm{i}\{(ku)_{\parallel}-s\Omega\}\tau-\mathrm{i}\,s\eta(\psi-\phi)\right], \tag{45}$$

with

$$U^{\mu}(s,k,u) = \Bigg(\gamma J_s(k_{\perp}R),\ \frac{\gamma v_{\perp}}{2}\{\mathrm{e}^{-\mathrm{i}\eta\psi}J_{s-1}(k_{\perp}R)+\mathrm{e}^{\mathrm{i}\eta\psi}J_{s+1}(k_{\perp}R)\},$$

$$-\frac{\mathrm{i}\eta\gamma v_{\perp}}{2}\{\mathrm{e}^{-\mathrm{i}\eta\psi}J_{s-1}(k_{\perp}R)-\mathrm{e}^{\mathrm{i}\eta\psi}J_{s+1}(k_{\perp}R)\},\ \gamma v_{\parallel}\,J_s(k_{\perp}R)\Bigg). \tag{46}$$

Note the identity

$$k\,U(s,k,u) = \{(ku)_{\parallel}-s\Omega_0\}\,J_s(k_{\perp}R). \tag{47}$$

After expanding in Bessel functions the integral over τ'' and τ' in (35) with (36)–(38) may be reduced to the form (cf. equation 34)

$$\mathrm{e}^{\mathrm{i}\Omega\tau}\int_0^{\tau}\mathrm{d}\tau''\int_0^{\tau}\mathrm{d}\tau'\ \dot{t}^{\alpha\beta}(\tau''-\tau')\,\mathrm{e}^{\mathrm{i}\Omega\tau'} = -\frac{\tau^{\alpha\beta}(\Omega)}{\Omega^2}, \tag{48}$$

with

$$\Omega = (ku)_{\parallel}-s\Omega_0. \tag{49}$$

It is possible to arrange the factors involving tensorial indices in (36)–(38) such that they can be expressed in terms of $\tau^{\alpha\beta}(\Omega)$ and (cf. equation 4)

$$G^{\mu\nu}(s,k,u) = g^{\mu\nu}J_s(k_{\perp}R)-\frac{k^{\mu}\,U^{\nu}(s,k,u)}{(ku)_{\parallel}-s\Omega_0}. \tag{50}$$

Note the identity

$$k_{\nu}\,G^{\mu\nu}(s,k,u) = 0. \tag{51}$$

4. Response Tensors for a Magnetised Plasma

The calculation of the response tensors for a magnetised plasma is analogous to that for an unmagnetised plasma. An integral over the initial positions of the particles, i.e. the integral over x_0, gives zero except for $k-k_1-\ldots-k_n=0$, a condition which appears in a δ-function. In addition, for the magnetised case the distribution cannot depend on the azimuthal angle of the spiralling particles, and the integral over this angle ϕ gives zero except for $s=s_1+\ldots+s_n$.

In writing down the resulting expressions it is convenient to simplify the notation further. As in (49) let Ω denote $(ku)_{\parallel}-s\Omega_0$ and similarly let Ω_r denote $(k_r\,u)_{\parallel}-s_r\,\Omega_0$ with $r=1,2,\ldots$. Further, let the arguments s and u in $G^{\mu\nu}(s,k,u)$, as defined by

(50), be implicit so that one writes $G^{\mu\nu}(k)$. In this abbreviated notation the expression for the linear response tensor becomes

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m}\sum_{s=-\infty}^{\infty}\int d^4u\, F(u)\, G^{\alpha\mu}(k)\,\tau_{\alpha\beta}(\Omega)\, G^{*\beta\nu}(k). \tag{52}$$

This form may be obtained from that derived using Vlasov theory by a partial integration. This point is repeated in terms of the more familiar 3-tensor notation in the Appendix.

The response tensors should satisfy the charge-continuity condition

$$k_\mu\,\alpha^{\mu\nu}(k) = 0, \quad k_\mu\,\alpha^{\mu\nu\rho}(k, k_1, k_2) = 0, \quad k_\mu\,\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = 0, \tag{53a}$$

and the gauge-invariance conditions

$$k_\nu\,\alpha^{\mu\nu}(k) = 0, \qquad k_{1\nu}\,\alpha^{\mu\nu\rho}(k, k_1, k_2) = 0,$$

$$k_{2\rho}\,\alpha^{\mu\nu\rho}(k, k_1, k_2) = 0, \qquad k_{1\nu}\,\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = 0, \quad \text{etc.} \tag{53b}$$

The form (52) manifestly satisfies (53) in view of the identity (51). It is convenient to write the expressions for the quadratic and cubic responses so that the tensor indices $\mu\nu\rho$ and $\mu\nu\rho\sigma$, respectively, appear only in terms of the relevant G's, i.e.

$$G^{\mu}_{\alpha}(k)\, G^{*\nu}_{\beta}(k_1)\, G^{*\rho}_{\gamma}(k_2) \quad \text{and} \quad G^{\mu}_{\alpha}(k)\, G^{*\nu}_{\beta}(k_1)\, G^{*\rho}_{\gamma}(k_2)\, G^{*\sigma}_{\delta}(k_3)$$

respectively; then equations (53) are manifestly satisfied. The relevant forms are

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) = -\frac{q^3}{m^2}\int d^4u\, F(u) \sum_{\substack{s, s_1, s_2=-\infty \\ s=s_1+s_2}}^{\infty}$$

$$\times\, e^{-i\eta(s\psi - s_1\psi_1 - s_2\psi_2)}\, G^{\mu}_{\alpha}(k)\, G^{*\nu}_{\beta}(k_1)\, G^{*\rho}_{\gamma}(k_2)\, f^{\alpha\beta\gamma}(k, k_1, k_2), \tag{54}$$

$$\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = -\frac{q^4}{m^3}\int d^4u\, F(u) \sum_{\substack{s, s_1, s_2, s_3=-\infty \\ s=s_1+s_2+s_3}}^{\infty}$$

$$\times\, e^{-i\eta(s\psi - s_1\psi_1 - s_2\psi_2 - s_3\psi_3)}\, G^{\mu}_{\alpha}(k)\, G^{*\nu}_{\beta}(k_1)\, G^{*\rho}_{\gamma}(k_2)\, G^{*\sigma}_{\delta}(k_3)\, f^{\alpha\beta\gamma\delta}(k, k_1, k_2, k_3). \tag{55}$$

Unique results are obtained only if the symmetry properties

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) = \alpha^{\mu\rho\nu}(k, k_2, k_1), \tag{56}$$

$$\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = \alpha^{\mu\nu\sigma\rho}(k, k_1, k_3, k_2) = \alpha^{\mu\rho\nu\sigma}(k, k_2, k_1, k_3) \tag{57}$$

are imposed.

The explicit results obtained are

$$f^{\alpha\beta\gamma}(k,k_1,k_2)=\tfrac{1}{2}\bigg(k_\theta\,\frac{\tau^{\theta\beta}(\Omega_1)}{\Omega_1}\,\tau^{\alpha\gamma}(\Omega_2)+k_\theta\,\frac{\tau^{\theta\gamma}(\Omega_2)}{\Omega_2}\,\tau^{\alpha\beta}(\Omega_1)$$
$$+k_{1\theta}\,\frac{\tau^{\alpha\theta}(\Omega)}{\Omega}\,\tau^{\beta\gamma}(\Omega_2)+k_{2\theta}\,\frac{\tau^{\alpha\theta}(\Omega)}{\Omega}\,\tau^{\gamma\beta}(\Omega_1)$$
$$-k_{1\theta}\,\frac{\tau^{\theta\gamma}(\Omega_2)}{\Omega_2}\,\tau^{\alpha\beta}(\Omega)-k_{2\theta}\,\frac{\tau^{\theta\beta}(\Omega_1)}{\Omega_1}\,\tau^{\alpha\gamma}(\Omega)\bigg), \tag{58}$$

$$f^{\alpha\beta\gamma\delta}(k,k_1,k_2,k_3)=\tfrac{1}{6}\bigg\{\bigg(\frac{\tau^{\alpha\beta}(\Omega)k_{1\phi}-\tau^{\alpha\beta}(\Omega_1)k_\phi}{\Omega-\Omega_1}-\frac{\tau^{\alpha\eta}(\Omega)}{\Omega}\,k_{1\eta}\,g^{\beta}{}_{\phi}$$
$$-\frac{\tau^{\eta\beta}(\Omega_1)}{\Omega_1}\,k_\eta\,g^{\alpha}{}_{\phi}\bigg)\tau^{\phi\theta}(\Omega-\Omega_1)\bigg(-\frac{\tau^{\gamma\delta}(\Omega_3)k_{2\theta}+\tau^{\delta\gamma}(\Omega_2)k_{3\theta}}{\Omega_2+\Omega_3}$$
$$+\frac{\tau^{\eta\delta}(\Omega_3)}{\Omega_3}\,k_{2\eta}\,g^{\gamma}{}_{\theta}+\frac{\tau^{\eta\gamma}(\Omega_2)}{\Omega_2}\,k_{3\eta}\,g^{\delta}{}_{\theta}\bigg)$$
$$-k_{1\theta}\,k_{1\eta}\bigg(\frac{\tau^{\alpha\theta}(\Omega)}{\Omega_2}\,\tau^{\beta\gamma}(\Omega_2)\,\frac{\tau^{\eta\delta}(\Omega_3)}{\Omega_3}+\frac{\tau^{\alpha\theta}(\Omega)}{\Omega}\,\frac{\tau^{\eta\gamma}(\Omega_2)}{\Omega_2}\,\tau^{\beta\delta}(\Omega_3)$$
$$-\tau^{\alpha\beta}(\Omega)\,\frac{\tau^{\eta\gamma}(\Omega_2)}{\Omega_2}\,\frac{\tau^{\theta\delta}(\Omega_3)}{\Omega_3}\bigg)+k_\theta\,k_\eta\,\tau^{\alpha\beta}(\Omega_1)\,\frac{\tau^{\theta\gamma}(\Omega_2)}{\Omega_2}\,\frac{\tau^{\eta\delta}(\Omega_3)}{\Omega_3}$$
$$+(1,\nu)\leftrightarrow(2,\rho)+(1,\nu)\leftrightarrow(3,\sigma)\bigg\}, \tag{59}$$

where $(1,\nu)\leftrightarrow(2,\rho)$ and $(1,\nu)\leftrightarrow(3,\sigma)$ indicate additional terms obtained from those shown by making the indicated substitutions. Note that in (58) one has $k=k_1+k_2$ and $\Omega=\Omega_1+\Omega_2$ and in (59) one has $k=k_1+k_2+k_3$ and $\Omega=\Omega_1+\Omega_2+\Omega_3$.

The symmetry properties (56) and (57) have been imposed, and by inspection the following symmetry properties are satisfied (cf. Melrose 1972):

$$\alpha^{\mu\nu\rho}(k,k_1,k_2)=\alpha^{\nu\mu\rho}(-k_1,-k,k_2), \tag{60}$$

$$\alpha^{\mu\nu\rho\sigma}(k,k_1,k_2,k_3)=\alpha^{\nu\mu\rho\sigma}(-k_1,-k,k_2,k_3). \tag{61}$$

In the proof of these one needs to note that on making the changes $k\rightarrow -k$, $k_1\rightarrow -k_1$ one may also reverse the sign of s and s_1 in the sums and so ensure that one has $\Omega\rightarrow-\Omega$, $\Omega_1\rightarrow-\Omega_1$ in (54) and (55), and then one uses

$$\tau^{\alpha\beta}(-\Omega)=\tau^{\beta\alpha}(\Omega). \tag{62}$$

As stated in the Introduction, explicit results for the quadratic response tensor have been derived previously, and (54) with (58) was written down by Melrose (1983). However, there has been no previous detailed calculation of the cubic response tensor for a magnetised plasma, i.e. no form of (55) with (59) has been presented hitherto. It is possible to translate the result into 3-tensor notation, but for the cubic response in general the resulting expression is very cumbersome indeed, and only relevant approximate forms are written down in the Appendix.

5. Approximate Forms

For most practical purposes approximate forms of the tensors are required, and the most important are the cold-plasma and longitudinal approximations. [These approximations were discussed by Melrose (1986*b*) for the unmagnetised case.]

Cold-plasma Approximation

The cold-plasma approximation involves neglecting all thermal motions. This is reasonable provided all relevant fluctuations have phase speed much greater than the thermal speed (of the particular species of particle in question, usually electrons). For the quadratic response tensor this requires $|\omega|/|\boldsymbol{k}|$, $|\omega_1|/|\boldsymbol{k}_1|$, $|\omega_2|/|\boldsymbol{k}_2| \gg V$, where V is the thermal speed. For the cubic response tensor, not only these inequalities and $|\omega_3|/|\boldsymbol{k}_3| \gg V$ need to be satisfied, but also all the inequalities $|\omega\pm\omega_1|/|\boldsymbol{k}\pm\boldsymbol{k}_1|$, $|\omega\pm\omega_2|/|\boldsymbol{k}\pm\boldsymbol{k}_2|, \ldots, |\omega_2\pm\omega_3|/|\boldsymbol{k}_2\pm\boldsymbol{k}_3| \gg V$ need to be satisfied. That is, all the waves involved, and all the beats between them need to have phase speeds $\gg V$.

When these inequalities are satisfied, one may approximate $F(u)$ in (52), (54) and (55) by $F(u) = n\delta^4(u-\bar{u})$, where $\bar{u}$ is the 4-velocity of the rest frame and n is the number density in this frame. In the rest frame one has $v_\parallel = v_\perp = 0$, $\gamma = 1$, $R = 0$ in (46), and then only $s = 0$ is nonzero in $U(s, k, u)$. In any frame one has $U(s{=}0, k, \bar{u}) = \bar{u}$, $U(s{\neq}0, k, \bar{u}) = 0$, and $G^{\mu\nu}(s{=}0, k, \bar{u})$ reduces to $G^{\mu\nu}(k, \bar{u})$, as defined in the unmagnetised case (cf. equation 4). Thus (52), (54) and (55) simplify in that the sums over the s's are omitted, the integral over $d^4 u\, F(u)$ is replaced by the number density n, the G's are re-interpreted as the unmagnetised forms with $u = \bar{u}$, and the Ω's are interpreted as $k\bar{u}$'s.

The resulting forms may be obtained using cold-plasma theory. This involves setting up a fluid description of the particles (here in covariant form) and solving for the nonlinear currents. The cold-plasma approximation for the quadratic response tensor was obtained in 3-tensor notation in the form written down in the Appendix by Melrose and Sy (1972*b*) (cf. also Trakhtengerts 1970; Giles 1974; Stenflo 1973).

If not all the fields may be treated using the cold-plasma approximation, one may still simplify by using the following prescription for those fields which do satisfy the high-phase-speed condition. Consider the field described by μ and k. In (54) and (55) μ appears in terms of $G^{\alpha\mu}(s, k, u)$, as defined by (50). In the rest frame it is obvious that only $s = 0$ contributes, and this therefore applies in an arbitrary frame. Moreover, in the rest frame one has $u = \bar{u} = (1, \boldsymbol{0})$, and hence $\Omega = k\bar{u}$ and $G^{\alpha\mu}(s{=}0, k, u) = G^{\alpha\mu}(k, \bar{u})$. The factors $G^{\alpha\mu}(k, \bar{u})$ and $\tau^{\alpha\beta}(k\bar{u})$ no longer depend on the variable of integration, and may be taken outside the integral. This simplifying approximation may be used for any high-phase-speed disturbance [cf. the derivations of (66) and (67) below].

Longitudinal Approximation

Suppose the field associated with the index μ is longitudinal. (The concept of a longitudinal field is frame-dependent, and is defined such that the field is longitudinal, in the usual sense, in the rest frame.) One is free to describe a longitudinal field using the Coulomb gauge, in which case one has $A^\mu(k) = \big(\Phi(k), \boldsymbol{0}\big)$ where $\Phi(k)$ is the electrostatic potential. It follows that all relevant information for longitudinal fields must be contained in the $\mu = 0$ component. In practice the $\mu = 0$ components $\alpha^{0\nu\rho}(k, k_1, k_2)$ and $\alpha^{0\nu\rho\sigma}(k, k_1, k_2, k_3)$ are found simply by replacing $G^{\alpha\mu}$ by

$$G^{\alpha 0}(k) = \frac{J_s(k_\perp R)}{\Omega}(\Omega - \gamma\omega,\ -\gamma \mathbf{k}) \tag{63}$$

in (54) and (55). The integrals are then to be performed in the rest frame.

The resulting expressions for $\alpha^{0\nu\rho}$ and $\alpha^{0\nu\rho\sigma}$ may be used to construct the full tensors, valid in an arbitrary frame, by multiplying them by

$$g^{\mu}(k, \bar{u}) = \frac{k\bar{u}}{(k\bar{u})^2 - k^2}\, k_\alpha\, G^{\alpha\mu}(k, \bar{u}). \tag{64}$$

The argument for this is as follows. The full tensors must reduce to the calculated forms $\alpha^{0\nu\rho}$ and $\alpha^{0\nu\rho\sigma}$ in the rest frame $\bar{u} = (1, \mathbf{0})$, and must satisfy the charge-continuity condition (or the gauge-invariance condition for any index other than μ). The only 4-vector which satisfies the latter condition, which is constructed only from k and $\bar{u}$, and which reduces to $(1, \mathbf{0})$ in the rest frame is $g^{\mu}(k, \bar{u})$.

Low-frequency Disturbance

Suppose that k describes a low-frequency disturbance in the rest frame. Then Ω is small in the sense that the dominant term should be that corresponding to the largest power of Ω in the denominator. In practice low-frequency fields are usually also approximately longitudinal.

Consider the linear response to a low-frequency field. Assuming that the field is also longitudinal and that the particles are nonrelativistic, (52) gives

$$\alpha_{\mathrm{L}}^{\mu\nu}(k) \approx -\frac{q^2}{m}\left(\sum_s \int \mathrm{d}^4 u\, F(u) \frac{k^\theta \tau_{\theta\eta}(\Omega) k^\eta}{\Omega^2}\, J_s^2(k_\perp R)\right)$$
$$\times \left(\frac{k u_0}{k^2 - (k u_0)^2}\right)^2 k_\alpha\, G^{\alpha\mu}(k, u_0) k_\beta\, G^{\beta\nu}(k, u_0), \tag{65}$$

where the subscript L on $\alpha^{\mu\nu}$ is introduced to denote this approximation.

Now consider the quadratic response tensor when k and k_1 may be treated using the cold-plasma approximation, and with k_2 being a low frequency longitudinal disturbance.

The dominant terms in the expression (58) are the two with Ω_2 in the denominator, and only these two terms are retained. As the disturbances at k and k_1 are treated using the cold-plasma approximation, only $s = 0$ and $s_1 = 0$ contribute, and with $|k_2 u_0| \ll |k u_0| \approx |k_1 u_0|$ by hypothesis, one has $\Omega_1 \approx \Omega$. Then taking the cold-plasma forms outside the integral in (54) leaves

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) \approx \frac{q^3}{2m^2}\, G^{\alpha\mu}(k_1 u_0) \tau_{\alpha\beta}(k u_0)\, G^{\beta\nu}(k_1, u_0)$$
$$\times \left(\sum_{s_2=-\infty}^{\infty} \int \mathrm{d}^4 u\, F(u) \frac{k_2^\gamma \tau_{\gamma\delta}(\Omega_2)}{\Omega_2^2}\, k_2^\delta\, J_{s_2}^2(k_{2\perp} R)\right)$$
$$\times \frac{k_2 u_0}{(k_2 u_0)^2 - k_2^2}\, k_{2\theta}\, G^{\theta\rho}(k_2, u_0), \tag{66}$$

with the G's and the τ's defined by (4) and (30) respectively, and where the integral is to be evaluated in the rest frame. Comparison with (65) shows that the integral may be re-expressed in terms of the longitudinal part of the linear response tensor. The form (66) is familiar in terms of its counterpart in 3-tensor notation for an unmagnetised plasma (cf. the Appendix).

Low-frequency Beat

Consider the case of the cubic response tensor where all four fields (described by k, k_1, k_2 and k_3) have high phase speeds, and the beat between two of them is of low frequency, say that at $k-k_1 = k_2+k_3$. Then the dominant terms in (59) are those with $(\Omega-\Omega_1)^2 = (\Omega_2+\Omega_3)^2$ in the denominator. Proceeding as in the derivation of (66), in this case one finds

$$\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) \approx -\frac{q^4}{6m^3} G^{\alpha\mu}(k, u_0)\,\tau_{\alpha\beta}(ku_0)\, G^{\beta\nu}(k, u_0)\, G^{\gamma\rho}(k_2, u_0)$$

$$\times\,\tau_{\gamma\delta}(\Omega_2)\, G^{\delta\sigma}(k_3, u_0) \sum_{s-s_1=-\infty}^{\infty} \int \mathrm{d}^4 u\, F(u) \frac{(k-k_1)^\theta \tau_{\theta\eta}(\Omega-\Omega_1)(k-k_1)^\eta}{(\Omega-\Omega_1)^2}$$

$$\times\, J^2_{s-s_1}((k-k_1)_\perp R)\,. \tag{67}$$

Again the integral is of the same form as that in the approximation (65) to the linear response tensor.

The form (67) is that required in the kinetic theory derivation of the Zakharov equation for a magnetised plasma (cf. discussion in the Introduction). A 3-tensor version of (67) is given by equation (A11) in the Appendix.

6. Non-existence of Turbulent Bremsstrahlung in a Magnetic Field

Turbulent bremsstrahlung (Tsytovich *et al.* 1975) has proved to be a controversial effect; see e.g. the controversy cited by Tsytovich *et al.* (1981). More recently Melrose and Kuijpers (1984) and Kuijpers and Melrose (1985) have argued on general grounds that both the original form of the proposed mechanism and an alternative form (called 'induced bremsstrahlung') by Nambu (1981) do not exist. Nambu (1986 and personal communication) has argued that the effect does exist and is enhanced by the presence of a magnetic field. The general arguments for the non-existence of this effect can be summarised and extended to the magnetised case as follows.

The supposed effect arises from a contribution to the imaginary part of the nonlinear correction to the linear response tensor due to resonant ion sound waves. The two terms which contribute to this nonlinear correction have been written down by Melrose and Kuijpers (1984, equations 27a, b) in the notation used here. The important feature to note is that these terms have the form of integrals over $\boldsymbol{k}_1$ with integrands involving either (the Tsytovich *et al.* form) $\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3)$ evaluated at $k_2 = k$ and $k_3 = -k_1$ or (the Nambu form) $\alpha^{\mu\nu\theta}(k, k_1, k-k_1)\, D_{\theta\eta}(k-k_1)\, \alpha^{\eta\rho\sigma}(k-k_1, k_2, k_3)$ similarly evaluated. The ν and ρ indices are projected onto polarisation vectors for the ion sound waves and k_1 and $k_3 = -k_1$ satisfy the dispersion relation for ion sound waves. The imaginary parts arise from the resonant denominators for the ion sound waves, i.e. from $\Omega_1 = 0$ and $\Omega_3 = 0$ here. There are the two contributions from $\Omega_1 = 0$ and $\Omega_3 = 0$ (for either form of turbulent bremsstrahlung) and the argument for non-existence is that the two contributions cancel exactly.

This cancellation is less controversial for the Nambu (1981, 1983, 1986) form. In brief the contribution from the resonance at $\Omega_1 = 0$ in $\alpha^{\mu\nu\theta}(k, k_1, k-k_1)$ is proportional to $-\mathrm{i}\pi\delta(\Omega_1)$, and the contribution from the resonance at $\Omega_3 = -\Omega_1$ in (note here that $k_2 = k$ and $k_3 = -k_1$)

$$\begin{aligned}\alpha^{\eta\rho\sigma}(k-k_1, k_2, k_3) &= \alpha^{\eta\rho\sigma}(k-k_1, k, -k_1) = \alpha^{\rho\sigma\eta}(-k, -k_1, -k+k_1)\\ &= \{\alpha^{\rho\sigma\eta}(k, k_1, k-k_1)\}^*\end{aligned}$$

has the opposite sign, leading to the exact cancellation. This null result has been found by all authors who have considered the problem, with the exception of Nambu himself [cf. the discussion of this point by Kuijpers and Melrose (1985)].

Melrose and Kuijpers (1984) and Kuijpers and Melrose (1985) effectively argued that an analogous exact cancellation occurs for the Tsytovich *et al.* form. The essential point concerns the sequence in which one (I) takes the imaginary parts and (S) sets $k_3 = -k_1$. For the resonance at $\Omega_3 = 0$, the term $1/\Omega_3$ is treated as follows according to the two alternative possibilities (denoting the relevant steps by I and S as indicated):

$$\frac{1}{\Omega_3} \xrightarrow{\mathrm{I}} \mathrm{Im}\left(\frac{1}{\Omega_3 + \mathrm{i}0}\right) = -\mathrm{i}\pi\delta(\Omega_3) \xrightarrow{\mathrm{S}} -\mathrm{i}\pi\delta(\Omega_1),$$

$$\frac{1}{\Omega_3} \xrightarrow{\mathrm{S}} -\frac{1}{\Omega_1} \xrightarrow{\mathrm{I}} -\mathrm{Im}\left(\frac{1}{\Omega_1 + \mathrm{i}0}\right) = \mathrm{i}\pi\delta(\Omega_1).$$

In effect Melrose and Kuijpers argued for the former procedure, in which case exact cancellation occurs, and Tsytovich *et al.* (1975) in their equation (7) used the latter procedure, in which case the two resonant contributions are equal and add. [The resonant parts can differ at most by a sign due to the symmetry property $\alpha^{\mu\nu\rho\sigma}(k, k_1, k_2, k_3) = \alpha^{\mu\sigma\rho\nu}(k, k_3, k_2, k_1)$ for the nonresonant part.]

These arguments are not dependent on whether or not the plasma is magnetised. Thus, contrary to Nambu's (1986) claim, the inclusion of the magnetic field is irrelevant to the argument as to whether or not turbulent bremsstrahlung exists.

7. Conclusions

The objective of the work reported here has been the derivation of an explicit form for the cubic response tensor for an arbitrary magnetised plasma. The general form can be written relatively concisely in 4-tensor form [cf. equation (55) with (59)]. However, translating this into 3-tensor notation leads to an excessively cumbersome form. Only when simplifying approximations are made is it practicable to write down explicit 3-tensor forms, and this is done in the Appendix. As mentioned in the Introduction this cubic response tensor is required for at least three purposes, only one of which is discussed in any detail here, specifically turbulent bremsstrahlung (Section 6).

The method of calculation here involves two differences from the conventional approach. One is the use of a covariant formalism. As indicated above, the conciseness which 4-tensor notation allows greatly simplifies the details of the analysis. The other difference is the use of the forward-scattering method. The conventional approach is based on use of the Vlasov equation in which the perturbations in the distribution function are found at a fixed point $(\boldsymbol{x}, \boldsymbol{p})$ in phase space as a function of time t, and are then Fourier transformed. In the forward-scattering method the distribution function is kept fixed, and the perturbations in $\boldsymbol{x}$ and $\boldsymbol{p}$ are found as a function of t using a perturbation expansion of the equations of motion. These two approaches in the 6-dimensional phase space are formally equivalent in the same sense as the Lagrangian and Eulerian viewpoints in 3-dimensional space in fluid mechanics are equivalent.

One obvious difference in the forms of the tensors which arise in the two approaches relates to derivatives (with respect to momentum) of the distribution function. In the Vlasov approach the nth order response tensor involves up to nth order derivatives of the distribution function, and in the forward-scattering approach no derivatives appear. The two results are related by partial integration, although some care is required with singular terms (e.g. Melrose 1986*b*). Some care is also required in taking the resonant parts. As shown by Melrose and Kuijpers (1984), the resonant parts involve first and only first derivatives of the distribution function. The resonant parts are obtained by imposing the causal condition in either case, and neither approach has any advantage over the other for this purpose. An exception is for the linear response for which the Vlasov approach leads to the required first derivatives directly.

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Appendix. 3-Tensor Forms

In conventional 3-tensor notation, used in this Appendix, the indices are latin, and run over 1 to 3 and are all lowered. There is a source of confusion in that the 3-tensor, say $\alpha_{ij}(\omega, \boldsymbol{k})$, relating the 3-current $J_i(\omega, \boldsymbol{k})$ to the vector potential $A_j(\omega, \boldsymbol{k})$ in the temporal gauge $[\Phi(\omega, \boldsymbol{k}) = 0]$, can be confused with the $\mu = i$, $\nu = j$ component of the 4-tensor $\alpha_{\mu\nu}(k)$; the two differ by a sign. To minimise the possibility of confusion, here the tensors are written as conductivity 3-tensors relating $\boldsymbol{J}(\omega, \boldsymbol{k})$ to the electric field $\boldsymbol{E}(\omega, \boldsymbol{k}) = \mathrm{i}\,\omega \boldsymbol{A}(\omega, \boldsymbol{k})$, where $\boldsymbol{A}(\omega, \boldsymbol{k})$ is in the temporal gauge. The weak-turbulence expansion then becomes

$$J_i(\omega, \boldsymbol{k}) = \sigma_{ij}(\omega, \boldsymbol{k})\, E_j(\omega, \boldsymbol{k}) + \sum_{n=2}^{\infty} \int \mathrm{d}\lambda^{(n)}$$

$$\times\, S_{ij_1 \ldots j_n}(\omega, \boldsymbol{k}; \omega_1, \boldsymbol{k}_1; \ldots; \omega_n, \boldsymbol{k}_n)\, E_{j_1}(\omega_1, \boldsymbol{k}_1) \ldots E_{j_n}(\omega_n, \boldsymbol{k}_n). \tag{A1}$$

It is convenient to write

$$\tau_{ij}(\omega_s) = \begin{bmatrix} \dfrac{\omega_s^2}{\omega_s^2 - \Omega^2} & \dfrac{\mathrm{i}\,\eta\omega_s\,\Omega}{\omega_s^2 - \Omega^2} & 0 \\[2ex] -\dfrac{\mathrm{i}\,\eta\omega_s\,\Omega}{\omega_s^2 - \Omega^2} & \dfrac{\omega_s^2}{\omega_s^2 - \Omega^2} & 0 \\[2ex] 0 & 0 & 1 \end{bmatrix}, \tag{A2}$$

with [note that the meaning of Ω here is different from that in equation (49)]

$$\omega_s = \omega - s\Omega - k_\parallel\, v_\parallel\,, \qquad \Omega = \Omega_0/\gamma\,. \tag{A3}$$

Note that τ_{ij}, as defined by (A2), differs from the $\mu = i$, $\nu = j$ component of $\tau_{\mu\nu}$, as defined by (30), by a sign. It is also convenient to introduce $\boldsymbol{V}(s, \boldsymbol{k}, \boldsymbol{v})$ by writing (46) in the form

$$U^\mu(s, k, u) = \big(\gamma J_s(k_\perp R),\ \gamma\, \boldsymbol{V}(s, \boldsymbol{k}, \boldsymbol{v})\big)\,. \tag{A4}$$

The expression for the linear conductivity 3-tensor implied by (52) is

$$\sigma_{ij}(\omega, \boldsymbol{k}) = \frac{\mathrm{i}\,q^2}{m\omega} \sum_{s=-\infty}^{\infty} \int \frac{\mathrm{d}^3 p}{\gamma}\, f(\boldsymbol{p}) \bigg\{ J_s^2(k_\perp R)\, \tau_{ij}(\omega_s)$$

$$+ \frac{J_s(k_\perp R)}{\omega_s} \Big(\tau_{im}(\omega_s)\, k_m\, V_j^*(s, \boldsymbol{k}, \boldsymbol{v}) + V_i(s, \boldsymbol{k}, \boldsymbol{v}) k_m\, \tau_{mj}(\omega_s) \Big)$$

$$+ \frac{1}{\omega_s^2} \Big(k_l\, k_m\, \tau_{lm}(\omega_s) - \frac{\omega^2}{c^2} \Big) V_i(s, \boldsymbol{k}, \boldsymbol{v})\, V_j^*(s, \boldsymbol{k}, \boldsymbol{v}) \bigg\}, \tag{A5}$$

where ordinary units are now used with c the speed of light. This form follows by applying the forward-scattering method to an expression quoted by Melrose and Sy (1972*a*) for the scattering current. The more familiar method based on the Vlasov equation leads to the expression

$$\sigma_{ij}(\omega, \boldsymbol{k}) = -\frac{\mathrm{i}\, q^2}{\omega} \int \mathrm{d}^3 p \left\{ \frac{v_\parallel}{v_\perp} \left(v_\perp \frac{\partial}{\partial p_\parallel} - v_\parallel \frac{\partial}{\partial p_\perp} \right) f(\boldsymbol{p})\, b_i\, b_j \right.$$

$$\left. + \sum_{s=-\infty}^{\infty} \frac{V_i(s, \boldsymbol{k}, \boldsymbol{v})\, V_j^*(s, \boldsymbol{k}, \boldsymbol{v})}{\omega - s\Omega - k_\parallel v_\parallel} \left(\frac{\omega - k_\parallel v_\parallel}{\Omega} \frac{\partial}{\partial p_\perp} + k_\parallel \frac{\partial}{\partial p_\parallel} \right) f(\boldsymbol{p}) \right\}, \qquad \text{(A6)}$$

where $\boldsymbol{b}$ is a unit vector along the ambient magnetic field. It is possible to derive (A5) from (A6) by partially integrating and rearranging the resulting expression. This proves tedious; *inter alia* it involves using the recursion formulas and the sum rules for the Bessel functions and re-arranging the sum over s.

It is impracticable to write out the 3-tensor form for the quadratic response tensor (54) with (58). Formally it involves 96 terms [each contraction over a 4-index involves two 3-tensor terms and the contractions over α, β, γ and θ lead to a factor 8 times the 6 terms in (58)]. The useful forms are the cold-plasma approximation (Melrose and Sy 1972*b*)

$$S_{ijl}(\omega, \boldsymbol{k}; \omega_1, \boldsymbol{k}_1; \omega_2, \boldsymbol{k}_2) = -\frac{q^3 n}{2m^2 \omega_1 \omega_2} \left(\frac{k_r}{\omega_1} \tau_{rj}(\omega)\, \tau_{il}(\omega_2) + \frac{k_r}{\omega_2} \tau_{rl}(\omega_2)\, \tau_{ij}(\omega_1) \right.$$

$$+ \frac{k_{1r}}{\omega} \tau_{ir}(\omega)\, \tau_{jl}(\omega_2) + \frac{k_{2r}}{\omega} \tau_{ir}(\omega)\, \tau_{lj}(\omega_1)$$

$$\left. - \frac{k_{2r}}{\omega_1} \tau_{rj}(\omega)\, \tau_{il}(\omega) - \frac{k_{1r}}{\omega_2} \tau_{rl}(\omega_2)\, \tau_{ij}(\omega) \right), \qquad \text{(A7)}$$

and the approximate form (66), which translates into (Melrose and Sy 1972*b*)

$$S_{ijl}(\omega, \boldsymbol{k}; \omega_1, \boldsymbol{k}_1; \omega_2, \boldsymbol{k}_2) = \frac{\epsilon_0\, q}{m\omega} \tau_{ij}(\omega)\, k_{2l}\, \chi_{\mathrm{L}}(\omega_2, \boldsymbol{k}_2), \qquad \text{(A8)}$$

with the form (65) written as a susceptibility

$$\chi_{\mathrm{L}}(\omega, \boldsymbol{k}) = -\frac{q^2}{\epsilon_0\, m} \sum_{s=-\infty}^{\infty} \int \mathrm{d}^3 p\, f(\boldsymbol{p}) \frac{J_s^2(k_\perp R)}{\omega_s^2} \left(1 - \frac{(\omega - \omega_s)^2}{|\boldsymbol{k}|^2 c^2} \right). \qquad \text{(A9)}$$

The cold-plasma approximation to the cubic response tensor implied by (55) with (58) is

$$S_{ijlm}(\omega, \boldsymbol{k}; \omega_1, \boldsymbol{k}_1; \omega_2, \boldsymbol{k}_2; \omega_3, \boldsymbol{k}_3) = -\frac{\mathrm{i}\, q^4 n}{6m^3 \omega_1\, \omega_2\, \omega_3}$$

$$\times \Bigg\{\Bigg(\frac{\tau_{ij}(\omega) k_{1r} - \tau_{ij}(\omega_1) k_r}{\omega - \omega_1} - \frac{k_{1a}\tau_{ia}(\omega)}{\omega}\,\delta_{rj}$$

$$- k_a \frac{\tau_{aj}(\omega_1)}{\omega_1}\,\delta_{ri}\Bigg)\tau_{rs}(\omega-\omega_1)\Bigg(-\frac{\tau_{lm}(\omega_3) k_{2s} + \tau_{ml}(\omega_2) k_{3s}}{\omega_2+\omega_3}$$

$$+ k_{2a}\frac{\tau_{am}(\omega_3)}{\omega_3}\,\delta_{sl} + k_{3a}\frac{\tau_{al}(\omega_2)}{\omega_2}\,\delta_{sm}\Bigg)$$

$$- k_{1r}\, k_{1s}\Bigg(\frac{\tau_{ir}(\omega)}{\omega}\,\tau_{jl}(\omega_2)\,\frac{\tau_{sm}(\omega_3)}{\omega_3} + \frac{\tau_{ir}(\omega)}{\omega}\,\frac{\tau_{sl}(\omega_2)}{\omega_2}\,\tau_{jm}(\omega_3)$$

$$- \tau_{ij}(\omega)\,\frac{\tau_{rl}(\omega_2)}{\omega_2}\,\frac{\tau_{sm}(\omega_3)}{\omega_3}\Bigg)$$

$$+ k_r\, k_s\, \tau_{ij}(\omega_1)\,\frac{\tau_{rl}(\omega_2)}{\omega_2}\,\frac{\tau_{sm}(\omega_3)}{\omega_3} + (1,j)\leftrightarrow(2,l) + (1,j)\leftrightarrow(3,m)\Bigg\}, \qquad \text{(A10)}$$

and the form corresponding to (67) is

$$S_{ijlm}(\omega, \boldsymbol{k}; \omega_1, \boldsymbol{k}_1; \omega_2, \boldsymbol{k}_2; \omega_3, \boldsymbol{k}_3) = -\frac{\mathrm{i}\,\epsilon_0\, q^2}{m^2 \omega \omega_2^2}\,\tau_{ij}(\omega)\,\tau_{lm}(\omega_2)$$

$$\times |\boldsymbol{k} - \boldsymbol{k}_1|^2 \chi_{\mathrm{L}}(\omega - \omega_1, \boldsymbol{k} - \boldsymbol{k}_1), \qquad \text{(A11)}$$

where we set $\omega_1 = \omega$ and $\omega_3 = -\omega_2$ except in the difference $\omega - \omega_1 - \omega_2 + \omega_3$.

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